

## SOME TECHNIQUES OF USING THE STOLZ –CESÀRO’S THEOREM

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ABSTRACT. The article refers to the techniques of using Stolz Cesro theorem.

The Stolz –Cesàro theorem is named according to the mathematicians Ernesto Cesàro (Italia) and Otto Stolz (Austrian) who stated and proved it for the first time. This theorem plays an important role for solving the convergence of a class of sequences. Stolz –Cesàro theorem is stated and understood easily. However, it is quite difficult to use in solving problems. This article will provide some techniques through exercises.

We use problems which were discussed in Vietnam Mathematical Forum [4] and Math Stackexchange [5] for our exercises.

### Introduction

#### The Stolz –Cesàro theorem: General situation

Let  $\{a_n\}_{n \geq 1}$  be an any sequence of real numbers and suppose that  $\{b_n\}_{n \geq 1}$  is a sequence of positive numbers such that  $b_n$  is strictly monotone increasing to  $\infty$ . Then

$$\liminf_{n \rightarrow \infty} \frac{a_n}{b_n} \geq \liminf_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$
$$\limsup_{n \rightarrow \infty} \frac{a_n}{b_n} \geq \limsup_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

Proof:

Readers can read the proof of this theorem in [2].

**Corollary:** Let  $\{a_n\}_{n \geq 1}$  and  $\{b_n\}_{n \geq 1}$  be two sequences of real numbers. Assume that  $\{(b_n)_{n \geq 1}\}$  is strictly monotone and is a divergent consequence and the following limit exists:

$$\lim_{n \rightarrow +\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l$$

Then, the limit  $\lim_{n \rightarrow +\infty} \frac{a_n}{b_n}$  also exists and it is equal to  $l$ . This corollary is an important result and we often use it for solving problems.

**L’Hospital’s rule:** Given functions  $f$  and  $g$  which are differential on an open interval  $I$ . They except possibly at a point  $c$  contained in  $I$ , if

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0 \text{ or } \infty, g'(x) \neq 0$$

for all  $x$  in  $I$  with  $x \neq c$  and

$$\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

exists, then:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

It is an important rule to calculate the limit of fractions, we can find a lot of websites, books or articles to prove it, we provide you a website [3].

These are all of knowledge that you need to know and read this article. Now we are going to go some applications of the Stolz –Cesàro theorem.

### Some Applications of the the Stolz –Cesàro theorem

#### Example 1

If  $\{x_n\}_{n \geq 1}, n \geq 0, \lim_{n \rightarrow \infty} \frac{x_n}{n} = +\infty$ . Then

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \sum_{k=1}^n \frac{1}{\sqrt{x_k}} = 0$$

*Proof.*  $b_n = \sqrt{n}, a_n = \sum_{k=1}^n \frac{1}{\sqrt{x_k}}$

$\Rightarrow b_n$  is strictly increasing, unbounded and

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{x_{n+1}}}}{\sqrt{n+1} - \sqrt{n}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{x_{n+1}} + \sqrt{n}} \\ &= \lim_{n \rightarrow \infty} \left( \sqrt{\frac{n+1}{x_{n+1}}} + \sqrt{\frac{n}{x_{n+1}}} \right) \end{aligned}$$

However, we have:  $\lim_{n \rightarrow \infty} \frac{x_n}{n} = +\infty \Rightarrow \begin{cases} \lim_{n \rightarrow \infty} \frac{n}{x_n} = 0 \\ \lim_{n \rightarrow \infty} \left( \frac{n+1}{x_{n+1}} - \frac{1}{n} \right) = 0. \end{cases}$

$\Rightarrow L = 0$

So we have  $\lim_{n \rightarrow \infty} \left( \frac{1}{\sqrt{n}} \sum_{k=1}^n \frac{1}{\sqrt{x_k}} \right) = 0$  if  $\lim_{n \rightarrow \infty} \frac{x_n}{n} = +\infty$  □

#### Example 2

Let  $p \in \mathbb{R}, p \neq 1$ , we have.

$$\lim_{n \rightarrow \infty} \frac{1^p + 2^p + 3^p + \dots + n^p}{n^{p+1}} = \frac{1}{p+1}.$$

*Proof.* Choose  $a_n = \sum_{i=1}^n i^p, b_n = n^{p+1}$ . Then, we have :

$$\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n \rightarrow \infty} \frac{(n+1)^p}{(n+1)^{p+1} - n^{p+1}}$$

We invert the fraction and compute as follows:

$$\frac{(n+1)^{p+1} - n^{p+1}}{(n+1)^p} = \frac{1 - \left(\frac{1}{n+1}\right)^{p+1}}{\frac{1}{n+1}} \quad (*)$$

Let  $q = \frac{1}{n+1}$  ( $0 < q < 1$ ) and

$$f(x) = \frac{1 - (1-x)^{p+1}}{x}$$

We can calculate the limit of  $f$  when  $x$  tends to 0.

$$\begin{aligned} \lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \frac{1 - (1-x)^{p+1}}{x} \\ &= \lim_{x \rightarrow 0} \frac{(p+1)(1-x)^p}{1} \quad (L'Hopital \text{ rule}) \\ &= p+1 \end{aligned}$$

Then, replacing  $x$  by  $\frac{1}{n+1}$  we have:

$$\lim_{n \rightarrow \infty} \frac{1 - \left(\frac{1}{n+1}\right)^{p+1}}{\frac{1}{n+1}} = p+1 \quad \text{or} \quad \lim_{n \rightarrow \infty} \frac{(n+1)^{p+1} - n^{p+1}}{(n+1)^{p+1}} = p+1$$

It implies:

$$\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \frac{1}{p+1}$$

then

$$\lim_{n \rightarrow \infty} \frac{1^p + 2^p + 3^p + \dots + n^p}{n^{p+1}} = \frac{1}{p+1}$$

□

### Example 3

Calculate:

$$\lim \frac{n+1}{1 + \sqrt{2} + \sqrt{3} + \dots + \sqrt{n}}$$

Proof:

Letting  $\begin{cases} a_n = n+1 \\ b_n = \sum_{i=1}^n \sqrt{i} \end{cases}$ , we have

$$\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1}} = 0.$$

So we have:  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n+1}{1 + \sqrt{2} + \sqrt{3} + \sqrt{4} + \dots + \sqrt{n}} = 0.$

**Remarks:** A different solution for this problem is as follows. It may be quite difficult to know how we can think like this !

Note that

$$\sqrt{1} + \sqrt{2} + \dots + \sqrt{n} = \sqrt{n} \sum_{k=1}^n \sqrt{\frac{k}{n}}$$

Using the definition of the Riemann integral we have:

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n \sqrt{\frac{k}{n}} = \int_0^1 \sqrt{x} dx = \frac{2}{3}$$

Then

$$\lim_{n \rightarrow +\infty} \frac{n+1}{1 + \sqrt{2} + \dots + \sqrt{n}} = \lim_{n \rightarrow +\infty} \frac{n+1}{n\sqrt{n}} \frac{1}{\frac{1}{n} \sum_{k=1}^n \sqrt{\frac{k}{n}}} = 0$$

#### Example 4

If  $\{u_n\}_{n \geq 1}$  is a sequence of positive real numbers and if  $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = u$  then  $\lim_{n \rightarrow \infty} \sqrt[n]{u_n} = u$ .

Proof:

Letting  $\lim_{n \rightarrow \infty} a_n = \ln u_n, b_n = n$ , we get:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} &= \lim_{n \rightarrow \infty} \frac{\ln \frac{u_{n+1}}{u_n}}{1} = \lim_{n \rightarrow \infty} (\ln \frac{u_{n+1}}{u_n}) = \ln u \\ \Rightarrow \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\ln u_n}{n} = \lim_{n \rightarrow \infty} \lim \sqrt[n]{u_n} = \lim u \\ \Rightarrow \sqrt[n]{u_n} &= u. \quad QED \end{aligned}$$

#### Example 5

Let  $(x_n)_{n \geq 1}, \begin{cases} x_1 = 1 \\ x_{n+1} = \sqrt{x_1 + x_2 + \dots + x_n} \end{cases}$ . Find  $\lim_{n \rightarrow \infty} \frac{x_n}{n}$ .

Proof:

We note that  $x_{n+1}^2 = x_n^2 + x_n$  and all of terms of  $\{x_n\}_{n \geq 1}$  are positive. Moreover, we can prove that  $\{x_n\}_{n \geq 1}$  is strictly increasing. If it is bounded then limit  $\{x_n\}_{n \geq 1}$  exists and denotes by  $L$ .

Then,  $L^2 = L^2 + L \Leftrightarrow L = 0$ . It is contradictious.

So  $\lim_{n \rightarrow +\infty} x_n = +\infty$ .

Then we have:

$$\lim_{n \rightarrow +\infty} \frac{x_{n+1}}{x_n} = \lim_{n \rightarrow +\infty} \sqrt{1 + \frac{1}{x_n}} = 1$$

Putting  $\begin{cases} a_n = x_n, \\ b_n = n \end{cases}$  then we have:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{x_n}{n} &= \lim_{n \rightarrow +\infty} \frac{x_{n+1} - x_n}{n+1 - n} = \lim_{n \rightarrow +\infty} (x_{n+1} - x_n) \\ &= \lim_{n \rightarrow +\infty} \frac{x_{n+1}^2 - x_n^2}{x_{n+1} + x_n} = \lim_{n \rightarrow +\infty} \frac{x_n}{x_{n+1} + x_n} \\ &= \lim_{n \rightarrow +\infty} \frac{1}{1 + \frac{x_{n+1}}{x_n}} = \frac{1}{2} \end{aligned}$$

**Example 6**

Let  $\{x_n\}_{n \geq 1}$  be a sequence of real numbers with  $x_1 = 1$  and

$$x_{n+1} = \frac{x_n^2 + 2}{n + 1}$$

for any  $n \geq 1$ . Find:

$$\lim_{n \rightarrow +\infty} \frac{\sum_{k=1}^n x_k}{\sum_{k=1}^n \frac{1}{k}}$$

Proof:

By induction we have  $0 < x_n \leq 2$  for any  $n \geq 2$ .

Then we have:

$$0 < x_{n+1} \leq \frac{2^2 + 2}{n + 1} \rightarrow 0 \quad \text{when } n \rightarrow +\infty$$

Applying Stolz-Cesàro we have:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{\sum_{k=1}^n x_k}{\sum_{k=1}^n \frac{1}{k}} &= \lim_{n \rightarrow +\infty} \frac{x_n}{\frac{1}{n}} = \lim_{n \rightarrow +\infty} n \frac{x_{n-1}^2 + 2}{n} \\ &= \lim_{n \rightarrow +\infty} x_{n-1}^2 + 2 = 2 \end{aligned}$$

So we have the limit:

$$\lim_{n \rightarrow +\infty} \frac{\sum_{k=1}^n x_k}{\sum_{k=1}^n \frac{1}{k}} = 2$$

**Example 7**

Let  $\{x_n\} \in \mathbb{R}$  with  $\lim_{n \rightarrow \infty} x_n \sum_{k=1}^n x_k^2 = 1$ . Prove that  $\lim_{n \rightarrow \infty} \sqrt[3]{3n}x_n = 1$ .

Proof:

Let  $S_n = \sum_{k=1}^n x_k^2$  then  $x_n S_n \rightarrow 1$ .

We can prove easily that  $S_n$  is strictly increasing.

If  $S_n$  is convergent to  $L$  then  $x_n \rightarrow \frac{1}{L} > 0$ .

Then we have

$$x_n^2 \rightarrow \frac{1}{L^2}$$

$$S_n = \sum_{k=1}^n x_k^2 \rightarrow +\infty \quad (\text{Because terms of } x_k^2 \text{ are finite})$$

It is contradictious.

Then we have  $\lim_{n \rightarrow \infty} S_n = +\infty \Rightarrow \lim_{n \rightarrow \infty} x_n = 0$ .

$$S_n^3 - S_{n-1}^3 = x_n^2 (S_n^2 + S_n S_{n-1} + S_{n-1}^2) \rightarrow 3$$

Because  $\begin{cases} x_n S_n \rightarrow 1 \\ x_n S_{n-1} \rightarrow 1 \end{cases}$ , using Stolz - Cesàro we have

$$\lim_{n \rightarrow \infty} \frac{S_n^3}{n} = \lim_{n \rightarrow \infty} \frac{S_n^3 - S_{n-1}^3}{n - (n-1)} = 3$$

We deduce that

$$\lim_{n \rightarrow \infty} \frac{S_n}{\sqrt[3]{3n}} = \lim_{n \rightarrow \infty} \frac{\sqrt[3]{3n}}{S_n} x_n S_n = 1.$$

#### REFERENCES

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