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MORLEY'S TRIANGLE

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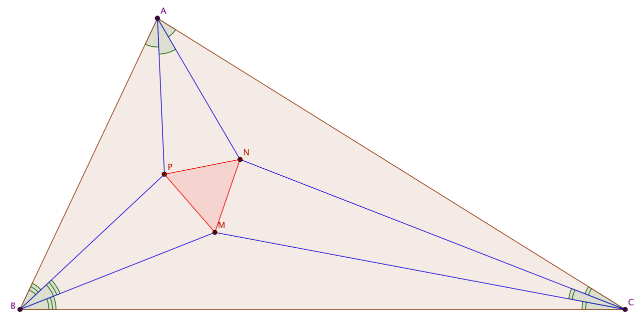

Proof.

$$S' = \left(8R\sin\frac{A}{3}\sin\frac{B}{3}\sin\frac{C}{3}\right)^2\frac{\sqrt{3}}{4}$$

$$b = 2R\sin B = 2R\sin\frac{B}{3}\left(3-4\sin^2\frac{B}{3}\right) =$$

$$= 2R\sin\frac{B}{3}\left(3\cos^2\frac{B}{3}-\sin^2\frac{B}{3}\right) =$$

$$= 8R\sin\frac{B}{3}\left(\frac{3}{4}\cos^2\frac{B}{3}-\frac{1}{4}\sin^2\frac{B}{3}\right) =$$

$$= 8R\sin\frac{B}{3}\left(\sin^2\frac{\pi}{3}\cos^2\frac{B}{3}-\sin^2\frac{B}{3}\cos^2\frac{\pi}{3}\right) =$$

$$= 8R\sin\frac{B}{3}\left(\frac{\pi}{3}\cos\frac{B}{3}-\sin\frac{B}{3}\cos\frac{\pi}{3}\right)\left(\sin\frac{\pi}{3}\cos\frac{B}{3}+\sin\frac{B}{3}\cos\frac{\pi}{3}\right)$$

$$b = 8R\sin\frac{B}{3}\sin\left(\frac{\pi}{3}-\frac{B}{3}\right)\sin\left(\frac{\pi}{3}+\frac{B}{3}\right)$$

Analogous:

$$c = 8R\sin\frac{C}{3}\sin\left(\frac{\pi}{3}-\frac{C}{3}\right)\sin\left(\frac{\pi}{3}+\frac{C}{3}\right)$$

In ΔANC*:*

$$\frac{AN}{\sin\frac{C}{3}} = \frac{b}{\sin(\pi-\frac{A}{3}-\frac{C}{3})}$$

$$AN = \frac{b\sin\frac{C}{3}}{\sin\frac{A+C}{3}} = \frac{b\sin\frac{C}{3}}{\sin\frac{\pi-B}{3}} =$$

$$= \frac{2R\sin\frac{B}{3}\sin(\frac{\pi}{3}-\frac{B}{3})\sin(\frac{\pi}{3}+\frac{B}{3})\sin\frac{C}{3}}{\sin(\frac{\pi}{3}-\frac{B}{3})}$$

$$AN = 2R\sin\frac{B}{3}\sin\left(\frac{\pi}{3}+\frac{B}{3}\right)\sin\frac{C}{3}$$

Analogous:

$$AP = 8R\sin\frac{B}{3}\sin\frac{C}{3}\sin\left(\frac{\pi}{3}+\frac{C}{3}\right)$$

In ΔMNP:

$$NP^2 = AN^2 + AP^2 - 2ANAP\cos\frac{A}{3} =$$

$$= 64R^2\sin^2\frac{B}{3}\sin^2\frac{C}{2}\left(\sin^2\left(\frac{\pi}{3}+\frac{B}{3}\right)+\sin^2\left(\frac{\pi}{3}+\frac{C}{3}\right)-\right.$$

$$\left.-2\sin\left(\frac{\pi}{3}+\frac{B}{3}\right)\sin\left(\frac{\pi}{3}+\frac{C}{3}\right)\cos\frac{A}{3}\right)$$

We prove that:

$$\sin^2\left(\frac{\pi}{3}+\frac{B}{3}\right)+\sin^2\left(\frac{\pi}{3}+\frac{C}{3}\right)-2\sin\left(\frac{\pi}{3}+\frac{B}{3}\right)\sin\left(\frac{\pi}{3}+\frac{C}{3}\right)\cos\frac{A}{3}$$

$$= \sin^2\frac{A}{3}$$

$$\frac{1-\cos\left(\frac{2\pi}{3}+\frac{2B}{3}\right)}{2}+\frac{1-\cos\left(\frac{2\pi}{3}+\frac{2c}{3}\right)}{2}-\sin^2\frac{A}{3} =$$

$$= 2\sin\left(\frac{\pi}{3}+\frac{B}{3}\right)\sin\left(\frac{\pi}{3}+\frac{C}{3}\right)\cos\frac{A}{3}$$

$$\cos^2\frac{A}{3}-\frac{1}{2}\cdot 2\cos\left(\frac{2\pi}{3}+\frac{B+C}{3}\right)\cos\frac{B-C}{3} =$$

$$= 2\sin\left(\frac{\pi}{3}+\frac{B}{3}\right)\sin\left(\frac{\pi}{3}+\frac{C}{3}\right)\cos\frac{A}{3}$$

$$\cos^2\frac{A}{3}+\cos\frac{A}{3}\cos\frac{B-C}{3} = 2\sin\frac{\pi+B}{3}\sin\frac{\pi+C}{3}\cos\frac{A}{3}$$

$$\cos\frac{A}{3}+\cos\frac{B-C}{3} = 2\sin\frac{\pi+B}{3}\sin\frac{\pi+C}{3}$$

$$2\cos\frac{A+B-C}{6}\cos\frac{A-B+C}{6} = 2\sin\frac{\pi+B}{3}\sin\frac{\pi+C}{3}$$

$$\cos\frac{\pi-2C}{6}\cos\frac{\pi-2B}{6} = \sin\frac{\pi+B}{3}\sin\frac{\pi+C}{3}$$

$$\sin\left(\frac{\pi}{2}-\frac{\pi-2C}{6}\right)\sin\left(\frac{\pi}{2}-\frac{\pi-2B}{6}\right) = \sin\frac{\pi+B}{3}\sin\frac{\pi+C}{3}$$

$$\sin\frac{\pi+C}{3}\sin\frac{\pi+B}{3} = \sin\frac{\pi+B}{3}\sin\frac{\pi+C}{3}$$

$$NP^2 = 64R^2\sin^2\frac{A}{3}\sin^2\frac{B}{3}\sin^2\frac{C}{3}$$

$$S' = S[MNP] = \frac{NP^2\sqrt{3}}{4} =$$

$$= 8R^2\sqrt{3}\sin^2\frac{A}{3}\sin^2\frac{B}{3}\sin^2\frac{C}{3}$$

□

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