Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open acces

SOLUTION TO PROBLEM SP.055. FROM ROMANIAN MATHEMATICAL MAGAZINE NUMBER 4, SPRING 2017

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SP.055. Let m_a, m_b, m_c be the lengths of medians of a triangle ABC with inradius r. Prove that

$$\frac{m_a + m_b + m_c}{\sin^2 A + \sin^2 B + \sin^2 C} \geq 4r.$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Proof.

With sine theorem we write the inequality:

$$\frac{m_a + m_b + m_c}{a^2 + b^2 + c^2} \geq \frac{r}{R^2} \Leftrightarrow \sum m_a \geq \frac{r}{R^2} \cdot \sum a^2.$$

We use the known inequality $m_a \geq \frac{b^2 + c^2}{4R}$ *it follows:*

$\sum m_a \geq \sum \frac{b^2 + c^2}{4R} = \frac{2\sum a^2}{4R} = \sum \frac{\sum a^2}{2R} \geq \frac{r}{R^2} \cdot \sum a^2$, *where the last inequality is equivalent with* $R \geq 2r$, *namely Euler's inequality.*

The equality holds if and only if the triangle is equilateral

□

The inequality can be strengthened:

1. Let m_a, m_b, m_c be the lengths of medians of a triangle ABC with inradius r. Prove that

$$\frac{m_a + m_b + m_c}{\sin^2 A + \sin^2 B + \sin^2 C} \geq 2R.$$

Proof.

With sine theorem we write the inequality:

$$\frac{m_a + m_b + m_c}{a^2 + b^2 + c^2} \geq \frac{1}{2R} \Leftrightarrow \sum m_a \geq \frac{1}{2R} \cdot \sum a^2.$$

Using the known inequality $m_a \geq \frac{b^2 + c^2}{4R}$ *it follows:*

$$\sum m_a \geq \sum \frac{b^2 + c^2}{4R} = \frac{2\sum a^2}{4R} = \frac{1}{2R} \cdot \sum a^2.$$

Equality holds if and only if the triangle is equilateral.

□

*Inequality **1.** is stronger then **SP.055.***

2. Let a, b, c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{m_a + m_b + m_c}{\sin^2 A + \sin^2 B + \sin^2 C} \geq 2R \geq 4r.$$

Proof.

*We use Inequality **1.** and Euler's inequality $R \geq 2r$.*

The equality holds if and only if the triangle is equilateral.

□

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