Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

PROBLEM 135 - TRIANGLE MARATHON 101 - 200

MARIN CHIRCIU

1. In ΔABC

$$a\sqrt{b}+b\sqrt{c}+c\sqrt{a}\le 3R\sqrt{2p}$$

Proposed by Daniel Sitaru - Romania

Remark

2. In ΔABC

$$a\sqrt{b}+b\sqrt{c}+c\sqrt{a}\le 2(R+r)\sqrt{2p}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the CBS inequality, we have:

$$\left(\sum a\sqrt{b}\right)^2=\left(\sum\sqrt{a}\sqrt{ab}\right)^2\le\sum a\cdot\sum ab=2p\cdot(p^2+r^2+4Rr)\overbrace{\le}^{Gerretsen}$$

$$\le 2p\cdot(4R^2+4Rr+3r^2+r^2+4Rr)=$$

$$=2p\cdot 4(R+r)^2,\ \textit{wherefrom}\ \sum a\sqrt{b}\le 2(R+r)\sqrt{2p}$$

The equality holds if and only if the triangle is equilateral.

□

Remark

We can write the double inequality:

3. In ΔABC

$$a\sqrt{b}+b\sqrt{c}+c\sqrt{a}\le 2(R+r)\sqrt{2p}\le 3R\sqrt{2p}$$

Proof.

Taking into account Euler's inequality we obtain $2(R+r)\sqrt{2p}\le 3R\sqrt{2p}$.

The equality holds if and only if the triangle is equilateral.

□

Remark.

In the same note we can propose:

4. In ΔABC

$$a\sqrt{b+c}+b\sqrt{c+a}+c\sqrt{a+b}\le 4(R+r)\sqrt{p}\le 6R\sqrt{p}$$

Proof.

Using CBS inequality we have:

$$\left(\sum a\sqrt{b+c}\right)^2 = \left(\sum \sqrt{a}\sqrt{a(b+c)}\right)^2 \sum a\cdot\sum 2bc = 4p\cdot(p^2+r^2+4Rr) \overbrace{\leq}^{Gerretsen}$$

$$4p\cdot(4R^2+4Rr+3r^2+r^2+4Rr) = 4p\cdot 4(R+r)^2, \text{ wherefrom } \sum a\sqrt{b+c} \leq 4(R+r)\sqrt{p}.$$

Taking into account Euler's inequality we obtain $4(R+r)\sqrt{p} \leq 6R\sqrt{p}$.

The equality holds if and only if the triangle is equilateral.

□

Remark

The inequality can be strengthened

5. In ΔABC

$$\boldsymbol{a\sqrt{b+nc}+b\sqrt{c+na}+c\sqrt{a+nb} \leq 2(R+r)\sqrt{2(n+1)p} \leq 3R\sqrt{2(n+1)p}, \text{ where } n \geq 0.}$$

Proposed by Marin Chirciu - Romania

Proof.

Using CBS inequality we have:

$$\left(\sum a\sqrt{b+nc}\right)^2 = \left(\sum \sqrt{a}\sqrt{a(b+nc)}\right)^2 \leq \sum a\cdot\sum (n+1)bc = 2(n+1)p\cdot(p^2+r^2+4Rr) \overbrace{\leq}^{Gerretsen}$$

$$= 2(n+1)p\cdot(4R^2+4Rr+3r^2+r^2+4Rr) = 2(n+1)p\cdot 4(R+r)^2, \text{ wherefrom}$$

$$\sum a\sqrt{b+nc} \leq 2(R+r)\sqrt{2(n+1)p}.$$

Taking into account Euler's inequality we obtain $2(R+r)\sqrt{2(n+1)p} \leq 3R\sqrt{2(n+1)p}$.

□

Remark

For $n=0$ *in* ***5.*** *we obtain* ***3.****, and for* $n=1$ *in* ***5.*** *we obtain* ***4.***

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.

E-mail address: dansitaru63@yahoo.com