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SOLUTION TO THE PROBLEM 82 MATH ADEVENTURES ON CUTTHEKNOTMATH 51-100 BY ALEXANDER BOGOMOLNY AND DANIEL SITARU ROMANIAN MATHEMATICAL MAGAZINE

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1. Prove that in triangle ABC, with angles $A; B; C$ side lengths $a; b; c$ the following inequality holds:

$$\frac{a(b+c)}{bc\cdot\cos^2\frac{A}{2}}+\frac{b(c+a)}{ca\cdot\cos^2\frac{B}{2}}+\frac{c(a+b)}{ab\cdot\cos^2\frac{C}{2}}\geq 8$$

Proposed by Daniel Sitaru - Romania

Proof.

We have

$$\sum\frac{a(b+c)}{bc\cdot\cos^2\frac{A}{2}}=\sum\frac{a(b+c)}{bc\cdot\frac{p(p-a)}{bc}}=\sum\frac{a(b+c)}{p(p-a)}=\sum\frac{a(2p-a)(p-b)(p-c)}{p(p-a)(p-b)(p-c)}=\frac{4R}{r}\geq 8,$$

where the last inequality follows from Euler's inequality $R\geq 2r$.

The equality holds if and only if $a=b=c$.

□

Next, are proposed inequalities for sums having the form $\sum\frac{a(b+c)}{bc\cdot f(A)}$, *where f is one of the trigonometric functions.*

2. Prove that in any triangle ABC, with angles $A; B; C$ side lengths $a; b; c$ the following inequality holds:

$$\frac{a(b+c)}{bc\cdot\sin^2\frac{A}{2}}+\frac{b(c+a)}{ca\cdot\sin^2\frac{B}{2}}+\frac{c(a+b)}{ab\cdot\sin^2\frac{C}{2}}\geq 12.$$

Proposed by Marin Chirciu - Romania

Proof.

We have

$$\sum\frac{a(b+c)}{bc\cdot\sin^2\frac{A}{2}}=\sum\frac{a(b+c)}{bc\cdot\frac{(p-b)(p-c)}{bc}}=\sum\frac{a(b+c)}{(p-b)(p-c)}=\sum\frac{a(2p-a)(p-a)}{(p-a)(p-b)(p-c)}=\frac{12R}{r}\geq 24$$

where the last inequality follows from Euler's inequality $R\geq 2r$.

The equality holds if and only if $a=b=c$.

□

3. Prove that in triangle ABC, with angles $A;B;C$ side lengths $a;b;c$ the following inequality holds:

$$\frac{a(b+c)}{bc\cdot\sin^2 A}+\frac{b(c+a)}{ca\cdot\sin^2 B}+\frac{c(a+b)}{ab\cdot\sin^2 C}\geq 8.$$

Proof 1.

We have

$$\sum\frac{a(b+c)}{bc\cdot\sin^2 A}=\sum\frac{a(b+c)}{bc\cdot\frac{a^2}{4R^2}}=\frac{4R^2}{abc}\sum(b+c)=\frac{4R^2}{4pRr}\cdot=\frac{4R}{r}\geq 8,$$

where the last inequality follows from Euler's inequality $R\geq 2r$.

The equality holds if and only if $a=b=c$.

□

Proof 2.

*We add the inequalities **1.** and **2.***

□

4. Prove that in triangle ABC, with angles $A;B;C$ side lengths $a;b;c$ the following inequality holds:

$$\frac{a(b+c)}{bc\cdot\sin A}+\frac{b(c+a)}{cd\cdot\sin B}+\frac{c(a+b)}{ab\cdot\sin C}\geq 4\sqrt{3}.$$

Proof.

We have

$$\sum\frac{a(b+c)}{bc\cdot\sin A}=\sum\frac{a(b+c)}{bc\cdot\frac{a}{2R}}=\frac{2R}{abc}\sum a(b+c)=\frac{2R}{4pRr}\cdot 2\sum bc=\frac{p^2+r^2+4Rr}{pr}\geq 4\sqrt{3},$$

where the last inequality follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$,

Doucet's inequality $p\sqrt{3}\leq 4R+r$ and Euler's inequality $R\geq 2r$.

The equality holds if and only if $a=b=c$.

□

5. Prove that in triangle ABC, with angles $A;B;C$ side lengths $a;b;c$ the following inequality holds:

$$\frac{a(b+c)}{bc\cdot\sin^3 A}+\frac{b(c+a)}{ca\cdot\sin^3 B}+\frac{c(a+b)}{ab\cdot\sin^3 C}\geq\frac{16}{\sqrt{3}}$$

Proof.

We have

$$\sum\frac{a(b+c)}{bc\cdot\sin^3 A}=\sum\frac{a(b+c)}{bc\cdot\frac{a^3}{8R^3}}=\frac{8R^3}{abc}\sum\frac{b+c}{a}=\frac{8R^3}{4pRr}\cdot\frac{p^2+r^2-2Rr}{2Rr}=\frac{R(p^2+r^2-2Rr)}{pr^2}\geq\frac{16}{\sqrt{3}},$$

where the last inequality follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$,

Doucet's inequality $p\sqrt{3}\leq 4R+r$ and Euler's inequality $R\geq 2r$.

The equality holds if and only if $a=b=c$.

□

6. Prove that in triangle ABC, with angles $A; B; C$ side lengths $a; b; c$ the following inequality holds:

$$\frac{a(b+c)}{bc\cdot\sin^4 A}+\frac{b(c+a)}{ca\cdot\sin^4 B}+\frac{c(a+b)}{ab\cdot\sin^4 C}\geq\frac{32}{3}.$$

Proposed by Marin Chirciu - Romania

Proof.

We have

$$\sum\frac{a(b+c)}{bc\cdot\sin^4 A}=\sum\frac{a(b+c)}{bc\cdot\frac{a^4}{16R^4}}=\frac{16R^4}{abc}\sum\frac{b+c}{a^2}=$$

$$=\frac{16R^4}{4pRr}\cdot\frac{p^4+p^2(2r^2-10Rr)+r^2(4R+r)(2R+r)}{8pR^2r^2}=$$

$$=\frac{R}{2p^2r^3}\left[p^4+p^2(2r^2-10Rr)+r^2(4R+r)(2R+r)\right]\geq\frac{32}{3},$$

where the last inequality holds if

$$3R[p^4+p^2(2r^2-10Rr)+r^2(4R+r)(2R+r)]\geq 64p^2r^3\Leftrightarrow$$

$$p^2(3Rp^2-30R^2r+6Rr^2-64r^3)+3Rr^2(8R^2+6Rr+r^2)\geq 0.$$

We distinguish the cases:

1. If $3Rp^2-30R^2r+6Rr^2-6r^3\geq 0$*, the inequality is equivalent.*

2. If $3Rp^2-30R^2r+6Rr^2-64r^3<0$*, we rewrite the inequality:* $p^2(30R^2r-6Rr^2+64r^3-3Rp^2)\leq 3Rr^2(8R^2+6Rr+r^2)$*, which follows from Gerretsen's inequality* $16Rr-5r^2\leq p^2\leq 4R^2+4Rr+3r^2$*. It remains to prove that:*

$$(4R^2+4Rr+3r^2)[30R^2r-6Rr^2+64r^3-3R(16Rr-5r^2)]\leq 3Rr^2(8R^2+6Rr+r^2)\Leftrightarrow$$

$$\Leftrightarrow 18R^4+15R^3r-55R^2r^2-70Rr^3-48r^4\geq 0\Leftrightarrow$$

$$\Leftrightarrow (R-2r)(18R^3+51R^2r+47Rr^2+24r^3)\geq 0,$$ *obviously from Euler's inequality* $R\geq 2r$.

The equality holds if and only if $a=b=c$.

□

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