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6 SOLUTIONS AND 6 EXTENSIONS FOR AN INEQUALITY

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ABSTRACT. In this paper are founded several ways to develop a given inequality.

Let $a, b > 0$. Prove that:

$$\mathbf{4 \leq \left(\frac{2ab}{a+b} + \sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{a+b}{2ab} + \sqrt{\frac{2}{a^2+b^2}}\right) \leq \frac{(a+b)^2}{ab}}$$

Daniel Sitaru, AoPS, 23 December 2016

Solution 1 by Soumava Chakarborty - Kolkata - India.

$$LHS = 3 + \frac{4\sqrt{a}}{a+b} + \frac{a+b}{\sqrt{ab}} + \frac{4ab}{(a+b)^2} + \frac{(a+b)^2}{4ab}$$

$$\leq 3 + 2 + \frac{a+b}{\sqrt{ab}} + 1 + \frac{(a+b)^2}{4ab} = 6 + \frac{a+b}{\sqrt{ab}} + \frac{(a+b)^2}{4ab}.$$

Suffice it to show that: $6 + \frac{a+b}{\sqrt{ab}} + \frac{(a+b)^2}{4ab} \leq 5 + 2\left(\frac{a}{b} + \frac{b}{a}\right)$

which is equivalent to

$$1 + \frac{a+b}{\sqrt{ab}} + \frac{(a+b)^2}{4ab} \leq \frac{2(a^2+b^2)}{ab} \tag{1}$$

*Now, by the **GM - HM inequality**,* $\sqrt{ab} \geq \frac{2ab}{a+b}$, *hence* $\frac{a+b}{\sqrt{ab}} \leq \frac{(a+b)^2}{2ab}$.

It follows that:

$$\frac{a+b}{\sqrt{ab}} + 1 + \frac{(a+b)^2}{4ab} \leq 1 + \frac{3(a+b)^2}{4ab}. \tag{2}$$

(1) *and* (2) *show that the problem will be solved if we manage to prove*

$$\frac{3(a+b)^2 + 4ab}{4ab} \leq \frac{2(a^2+b^2)}{ab}.$$

This is equivalent to $3a^2+3b^2+10ab \leq 8a^2+8b^2$, *or* $5a^2+5b^2-1ab \geq 0$, *which is simply*

$$5(a-b) \geq 0$$

□

Solution 2 by Kevin Soto Palacios - Huarmey - Peru.

We employ the obvious $\dfrac{2\sqrt{ab}}{a+b} \le 1$ *and* $\dfrac{4ab}{(a+b)^2} \le 1$ *to obtain*

$$LHS \le 6 + \frac{3(a+b)^2}{4ab}$$

Suffice it to prove that $6 + \dfrac{3(a+b)^2}{4ab} \le 5 + 2\left(\dfrac{a}{b} + \dfrac{b}{a}\right)$.

Now note that

$$1 + \frac{3(a+b)^2}{4ab} \le \frac{5}{2} + \frac{3}{4}\left(\frac{a}{b} + \frac{b}{a}\right) \le 2\left(\frac{a}{b} + \frac{b}{a}\right),$$

Because, $\dfrac{5}{2} \le \dfrac{5}{4}\left(\dfrac{a}{b} + \dfrac{b}{a}\right)$

$\square$

Solution 3 by Soumava Chakraborty - Kolkata - India.

Using the AM - GM inequality, $\dfrac{a+b}{2} \ge \sqrt{ab}$ *and* $\dfrac{2}{a+b} \le \dfrac{1}{\sqrt{ab}}$

$$LHS \le \left(\sqrt{ab} + \sqrt{ab} + \frac{a+b}{2}\right)\left(\frac{a+b}{2ab} + \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{ab}}\right) =$$

$$= \left(2\sqrt{ab} + \frac{a+b}{2}\right)\left(\frac{a+b}{2ab} + \frac{2}{\sqrt{ab}}\right) = \frac{a+b}{\sqrt{ab}} + 4 + \frac{(a+b)^2}{4ab} + \frac{a+b}{\sqrt{ab}} =$$

$$= 4 + \frac{(a+b)^2}{4ab} + \frac{2(a+b)}{\sqrt{ab}}$$

Thus, suffice it to prove that

$$4 + \frac{(a+b)^2}{4ab} + \frac{2(a+b)}{\sqrt{ab}} \le 5 + 2\left(\frac{a}{b} + \frac{b}{a}\right)$$

which is equivalent to

$$\frac{(a+b)^2}{4ab} + \frac{2(a+b)}{\sqrt{ab}} \le 1 + 2\frac{a^2+b^2}{ab} = \frac{2a^2+2b^2+ab}{ab}$$

Now, from $\dfrac{1}{\sqrt{ab}} \le \dfrac{a+b}{2ab}$ *we get* $\dfrac{2(a+b)}{\sqrt{ab}} \le \dfrac{(a+b)^2}{ab}$, *implying*

$$\frac{(a+b)^2}{4ab} + \frac{2(a+b)}{\sqrt{ab}} \le \frac{(a+b)^2}{4ab} + \frac{(a+b)^2}{a} = \frac{5(a+b)^2}{4ab}$$

Thus, suffice it to prove $\dfrac{5}{4} \cdot \dfrac{(a+b)^2}{ab} \le \dfrac{2a^2+2b^2+ab}{ab}$, *i.e.,*

$$5a^2 + 5b^2 + 10ab \le 8a^2 + 8b^2 + 4ab$$

which reduces to $3(a-b)^2 \ge 0$.

$\square$

Solution 4 by Abdallah El Farissi - Bechar - Algerie.

Let

$$A = \Big(\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2}\Big)\Big(\frac{a+b}{2ab} + \frac{1}{\sqrt{ab}} + \frac{2}{a+b}\Big) =$$

$$= \Big(\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2}\Big)\Big(\frac{2}{a+b} + \frac{1}{\sqrt{ab}} + \frac{a+b}{2ab}\Big) =$$

$$= \frac{1}{ab}\Big(\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2}\Big)^2 \le \frac{1}{ab}(\sqrt{ab} + a + b)^2$$

$$= \frac{1}{ab}\Big(ab + 2\sqrt{ab}(a+b) + (a+b)^2\Big) \le \frac{1}{ab}\Big(ab + 2(a+b)^2\Big) =$$

$$= \frac{1}{ab}\Big(5ab + 2(a^2+b^2)\Big) = 5 + 2\Big(\frac{a}{b} + \frac{b}{a}\Big).$$

□

Solution 5 by Soumava Chakraborty - Kolkata - India.

$$\textit{Define } = \frac{2ab}{a+b}, y = \sqrt{ab}, z = \frac{a+b}{2}. \textit{ We have } x \le y \le z \textit{ and } y^2 = xz.$$

$$LHS = \left(\sum_{cycl} x\right)\left(\sum_{cycl} \frac{1}{x}\right) = \frac{(\sum_{cycl} x)(xy+yz+zx)}{xyz}$$

$$= \frac{(\sum_{cycl} x)(xy+yz+y^2)}{xyz} = \frac{(x+y+z)^2}{xz} = \frac{(x+y+z)^2}{y^2}$$

$$RHS = \frac{2a^2+2b^2+5ab}{ab} = \frac{2(a+b)^2+ab}{ab} = 1 + \frac{8z^2}{y^2}$$

The required inequality is equivalent to $\frac{(x+y+z)^2}{y^2} \le 1 + \frac{8z^2}{y^2}$. *which is*

$(x+y+z)^2 - y^2 \le 8z^2$, *or* $(x+2y+z)(x+z) \le 8z^2$.

Since $x \le y \le z, x+2y+z \le 4z$ *and* $x+z \le 2z$, *which shows that, indeed,*

$(x+2y+z)(x+z) \le 8z^2$.

□

Solution 6 by Daniel Sitaru - Romania.

We know that, for positive a, b, c,

$$\frac{2ab}{a+b} \le \sqrt{ab} \le \frac{a+b}{2}$$

We'll use ***Schweitzer's inequality:***

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) \le \frac{(m+M)^2 n^2}{4mM},$$

where $x_1, \ldots, x_n \in [m, M], m > 0$.

with $n = 3, m = x_1 = \frac{2ab}{a+b}, x_2 = \sqrt{ab}$, *and* $x_3 = \frac{a+b}{2} = M$, *we directly get*

$$A = \Big(\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2}\Big)\Big(\frac{a+b}{2ab} + \frac{1}{\sqrt{ab}} + \frac{2}{a+b}\Big) \le 9\frac{(\frac{2ab}{a+b} + \frac{a+b}{2})^2}{4ab}$$

$$= \frac{9}{4ab}\left[\left(\frac{2ab}{a+b}\right)^2 + 2ab + \left(\frac{a+b}{2}\right)^2\right] \le \frac{9}{4ab}\left[(\sqrt{ab})^2 + 2ab + \left(\frac{a+b}{2}\right)^2\right]$$

$$= \frac{9}{4ab}\left[3ab + \left(\frac{a+b}{2}\right)^2\right] = \frac{9}{16ab}[14ab + (a^2+b^2)] = \frac{63}{8} + \frac{9}{16}\left(\frac{a}{b} + \frac{b}{a}\right).$$

Now, $1 \le \frac{1}{2}\left(\frac{a}{b} + \frac{b}{a}\right)$. *It then follows that* $\frac{23}{8} \le \frac{23}{16}\left(\frac{a}{b} + \frac{b}{a}\right)$ *and, subsequently,*

$$\frac{63}{8} + \frac{9}{16}\left(\frac{a}{b} + \frac{b}{a}\right) \le \frac{40}{8} + \frac{32}{16}\left(\frac{a}{b} + \frac{b}{a}\right) = 5 + 2\left(\frac{a}{b} + \frac{b}{a}\right).$$

This double inequality can be developed.

□

Let $a, b > 0$. Prove that:

Extension 1:

$$4 \le \left(\frac{2ab}{a+b} + \sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{a+b}{2ab} + \sqrt{\frac{2}{a^2+b^2}}\right) \le 4 - 2n + n\left(\frac{a}{b} + \frac{b}{a}\right), \text{ with } n \ge \frac{1}{2}.$$

Proof (Marin Chirciu).

We denote $\frac{2a}{a+b} = x, \sqrt{\frac{a^2+b^2}{2}} = y$ *and* $E = \left(\frac{2ab}{a+b} + \sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{a+b}{2ab} + \sqrt{\frac{2}{a^2+b^2}}\right).$

We have $E = (x+y)\left(\frac{1}{x} + \frac{1}{y}\right) = 2 + \frac{x^2+y^2}{xy}$. *We look for an inequality having the form* $E \le k + n\left(\frac{a}{b} + \frac{b}{a}\right).$

We obtain $E \le k + n\left(\frac{a}{b} + \frac{b}{a}\right) \Leftrightarrow 2 + \frac{x^2+y^2}{xy} = 2 + \frac{\left(\frac{2ab}{a+b}\right) + \frac{a^2+b^2}{2}}{\frac{2ab}{a+b} \cdot \frac{a^2+b^2}{2}} \le 2 + \frac{\left(\frac{2ab}{a+b}\right)^2 + \frac{a^2+b^2}{2}}{\frac{2ab}{a+b} \cdot \frac{a+b}{2}} = 2 + \frac{\left(\frac{2ab}{a+b}\right) + \frac{a^2+b^2}{2}}{ab} \le$

$k + n\left(\frac{a}{b} + \frac{b}{a}\right)$, *where the penultimate inequality follows from:* $\sqrt{\frac{a^2+b^2}{2}} \ge \frac{a+b}{2}$

and the last is equivalent with: $2ab + \left(\frac{2ab}{a+b}\right)^2 + \frac{a^2+b^2}{2} \le kab + n(a^2+b^2)$

$\Leftrightarrow 4ab(a+b)^2 + 8a^2b^2 + (a^2+b^2)(a+b)^2 \le 2kab(a+b)^2 + 2n(a^2+b^2)(a+b)^2 \Leftrightarrow$

(1) $(2n-1)a^4 + (2k+4n-6)a^3b + (4k+4n-18)a^2b^2 + (2k+4n-6)ab^3 + (2n-1)b^4 \ge 0$

In Horner's scheme we put the condition that (1) *to be double root and we obtain:* $8k + 16n - 32 = 0, 16k + 32n - 64 = 0$, *wherefrom* $k + 2n = 4$. *It follows that* (1) *can be written:*

$$(a-b)^2 \Leftrightarrow [(2n-1)a^2 + (2k+8n-8)ab + (8k+18n-33)b^2] \ge 0 \Leftrightarrow$$

$$(a-b)^2 \Leftrightarrow [(2n-1)a^2 + 4nab + (2n-1)b^2] \ge 0.$$

Putting the condition that the right parenthesis to be positive we obtain

$$2n - 1 \ge 0$$

The equality holds if and only if $a = b$. *Obviously* $E = (x+y)\left(\frac{1}{x} + \frac{1}{y}\right) \ge 4.$

Remark: We denote the arithmetic means, the geometric means, the harmonic means, respectively the square means with: $M_a = \frac{a+b}{2}, M_g = \sqrt{ab}, M_h = \frac{2ab}{a+b}, M_p = \sqrt{\frac{a^2+b^2}{2}}.$

The inequality from extension 1 is:

$$4 \leq (M_h + M_p)\Big(\frac{1}{M_h} + \frac{1}{M_p}\Big) \leq 4 - 2n + n\Big(\frac{a}{b} + \frac{b}{a}\Big), \textit{ where } n \geq \frac{1}{2}.$$

We will prove that: $4 \leq (x+y)\Big(\frac{1}{x} + \frac{1}{y}\Big) \leq 4 - 2n + n\Big(\frac{a}{b} + \frac{b}{a}\Big)$, *where* $n \geq \frac{1}{4}$ *or*

$$n \geq \frac{1}{2}, \textit{ for } \{x, y\} \subset \{M_a, M_g, M_h, M_p\}.$$

□

Extension 2:

$$\boldsymbol{4 \leq \Big(\frac{a+b}{2} + \sqrt{ab}\Big)\Big(\frac{2}{a+b} + \frac{1}{\sqrt{ab}}\Big) \leq 4 - 2n + n\Big(\frac{a}{b} + \frac{b}{a}\Big), \textbf{ with } n \geq \frac{1}{4}.}$$

Proof (Marin Chirciu).

We denote $\frac{a+b}{2} = x, \sqrt{ab} = y$ *and* $E = \Big(\frac{a+b}{2} + \sqrt{ab}\Big)\Big(\frac{2}{a+b} + \frac{1}{\sqrt{ab}}\Big)$.

We have $E = (x+y)\Big(\frac{1}{x}+\frac{1}{y}\Big) = 2+\frac{x^2+y^2}{xy}$. *We look for an inequality having the form*

$$E \leq k + n\Big(\frac{a}{b} + \frac{b}{a}\Big).$$

We obtain $E = 2+\frac{x^2+y^2}{xy} = 2+\frac{(\frac{a+b}{2})^2 + ab}{\frac{a+b}{2}\cdot\sqrt{ab}} \leq 2+\frac{(\frac{a+b}{2})^2 + ab}{\frac{a+b}{2}\cdot\frac{2ab}{a+b}} = 2+\frac{(\frac{a+b}{2})^2 + ab}{ab} =$

$= 3+\frac{(a+b)^2}{4ab} \leq k+n\Big(\frac{a}{b}+\frac{b}{a}\Big)$, *where the penultimate form* $\sqrt{ab} \geq \frac{2ab}{a+b}$, *and the last one is equivalent with:* $\frac{(a+b)^2}{4ab} \leq (k-3)+\frac{n(a^2+b^2)}{ab} \Leftrightarrow (4n-1)(a^2+b^2) \geq (4k-14)ab \Leftrightarrow$

$\Leftrightarrow (4n-1)a^2+(4k-14)ab+(4n-1)b^2 \geq 0$, (1). *Putting the condition* $\Delta = 0$ *we obtain* $k+2n = 4$ *and* (1) *can be written:* $\Leftrightarrow (4n-1)(a^2+b^2)-(8n-2)ab \geq 0 \Leftrightarrow (4n-1)(a-b)^2 \geq 0$, *true for*

$n \geq \frac{1}{4}$. *The equality holds if and only if* $a = b$.

□

Extension 3:

$$\boldsymbol{4 \leq \Big(\frac{a+b}{2} + \frac{2ab}{a+b}\Big)\Big(\frac{2}{a+b} + \frac{a+b}{2ab}\Big) \leq 4 - 2n + n\Big(\frac{a}{b} + \frac{b}{a}\Big), \textbf{ with } n \geq \frac{1}{4}.}$$

Proof (Marin Chirciu).

We denote $\frac{a+b}{2} = x, \frac{2ab}{a+b} = y$ *and* $E = \Big(\frac{a+b}{2} + \frac{2ab}{a+b}\Big)\Big(\frac{2}{a+b} + \frac{a+b}{2}\Big)$.

We have $E = (x+y)\Big(\frac{1}{x}+\frac{1}{y}\Big) = 2+\frac{x^2+y^2}{xy}$. *We look for an inequality having the form* $E \leq k+n\Big(\frac{a}{b}+\frac{b}{a}\Big)$

We obtain $E = 2+\frac{x^2+y^2}{xy} = 2+\frac{(\frac{a+b}{2})^2 + (\frac{2ab}{a+b})^2}{\frac{a+b}{2}\cdot\frac{2ab}{a+b}} = 2+\frac{(\frac{a+b}{2})^2 + (\frac{2ab}{a+b})^2}{ab} \leq k+n\Big(\frac{a}{b}+\frac{b}{a}\Big)$,

where the last one is equivalent with: $\Big(\frac{a+b}{2}\Big)^2+\Big(\frac{2ab}{a+b}\Big)^2 \leq (k-2)ab+n(a^2+b^2) \Leftrightarrow$

$$(a+b)^4 + 16a^2b^2 \leq (4k-8)ab(a+b)^2 + 4n(a^2+b^2)(a+b)^2 \Leftrightarrow$$

$$(4n-1)a^4+(4k+8n-12)a^3b+(8k+8n-38)a^2b^2+$$
$$+(4k+8n-12)ab^3+(4n-1)b^2\geq 0 \tag{1}$$

In Horner's scheme we put the condition that (1) *to be double root and we obtain:*

$$16k+32n-64=0, 32k+64n-128=0, \textit{ wherefrom } k+2n=4$$

It follows that (1) *can be written*

$$(a-b)^2[(4n-3)a^2+(4k+16n-14)ab+(16k+36n-65)b^2]\geq 0 \Leftrightarrow$$
$$(a-b)^2[(4n-1)a^2+(2n+2)ab+(4n-1)b^2]\geq 0$$

Putting the condition that the right parenthesis to be positive we obtain $4n-1\geq 0$

The equality holds if and only if $a=b$.

□

Extension 4:

$$\mathbf{4\leq\left(\frac{a+b}{2}+\sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{2}{a+b}+\sqrt{\frac{2}{a^2+b^2}}\right)\leq 4-2n+n\Big(\frac{a}{b}+\frac{b}{a}\Big), \text{ with } n\geq\frac{1}{4}}.$$

Proof (Marin Chirciu).

We denote $\frac{a+b}{2}=x, \sqrt{\frac{a^2+b^2}{2}}=y$ *and* $E=\Big(\frac{a+b}{2}+\sqrt{\frac{a^2+b^2}{2}}\Big)\Big(\frac{2}{a+b}+\sqrt{\frac{2}{a^2+b^2}}\Big)$.

We have $=(x+y)\Big(\frac{1}{x}+\frac{1}{y}\Big)=2+\frac{x^2+y^2}{xy}$. *We look for an inequality having the form* $E\leq k+n\Big(\frac{a}{b}+\frac{b}{a}\Big)$.

We obtain $E=2+\frac{x^2+y^2}{xy}=2+\frac{(\frac{a+b}{2})^2+\frac{a^2+b^2}{2}}{\frac{a+b}{2}\cdot\sqrt{\frac{a^2+b^2}{2}}}\leq 2+\frac{(\frac{a+b}{2})^2+\frac{a^2+b^2}{2}}{\frac{a+b}{2}\cdot\frac{a+b}{2}}=3+\frac{2(a^2+b^2)}{(a+b)^2}\leq k+n\Big(\frac{a}{b}+\frac{b}{a}\Big)$

where the penultimate inequality follows from:

$\sqrt{\frac{a^2+b^2}{2}}\geq\frac{a+b}{2}$, *and the last one is equivalent to* $\frac{2(a^2+b^2)}{(a+b)^2}\leq(k-3)+\frac{n(a^2+b^2)}{ab}\Leftrightarrow$

$$na^2+(k+2n-5)a^3b+(2k+2n-6)a^2b^2+(k+2n-5)ab^3+nb^4\geq 0 \tag{1}$$

In Horner's scheme we put the condition that (1) *to be double root and we obtain:*

$$4k+8n-16=0, 8k+16n-32=0, \textit{ wherefrom } k+2n=4$$

It follows that (1) *can be written*

$$(a-b)^2[na^2+(k+4n-5)ab+(4k+9n-16)b^2]\geq 0\Leftrightarrow$$
$$(a-b)^2[na^2+(2n-1)ab+nb^2]\geq 0, \textit{ true for } n\geq\frac{1}{4}$$

rPutting the condition that the right parenthesis to be positive we obtain $4n-1\geq 0$.

The equality holds if and only if $a=b$.

□

Extension 5:

$$\mathbf{4\leq\Big(\sqrt{ab}+\frac{2ab}{a+b}\Big)\Big(\frac{1}{\sqrt{ab}}+\frac{a+b}{2ab}\Big)\leq 4-2n+n\Big(\frac{a}{b}+\frac{b}{a}\Big), \text{ with } n\geq\frac{1}{4}}.$$

Proof (Marin Chirciu).

$$\textit{We denote } \sqrt{ab} = x,\ \frac{2ab}{a+b} = y \textit{ and } E = \left(\sqrt{ab} + \frac{2ab}{a+b}\right)\left(\frac{1}{\sqrt{ab}} + \frac{a+b}{2ab}\right)$$

$$\textit{We have } E = (x+y)\left(\frac{1}{x} + \frac{1}{y}\right) = 2 + \frac{x^2+y^2}{xy}$$

$$\textit{We look for an inequality having the form } E \le k + n\left(\frac{a}{b} + \frac{b}{a}\right).$$

We obtain: $E = 2+\frac{x^2+y^2}{xy} = 2+\frac{ab+(\frac{2ab}{a+b})^2}{\sqrt{ab}\cdot\frac{2ab}{a+b}} \le 2+\frac{ab+(\frac{2ab}{a+b})^2}{\frac{2ab}{a+b}\cdot\frac{2ab}{a+b}} = 3+\frac{ab}{(\frac{2ab}{a+b})^2} = 3+\frac{(a+b)^2}{4ab} =$

$= 3+\frac{(a+b)^2}{4ab} \le k+n\left(\frac{a}{b}+\frac{b}{a}\right)$, *where the penultimate inequality follows from* $\sqrt{ab} \ge \frac{2ab}{a+b}$

and the last one is equivalent with $\frac{(a+b)^2}{4ab} \le (k-3)+\frac{n(a^2+b^2)}{ab} \Leftrightarrow (4n-1)(a^2+b^2) \ge (4k-14)ab \Leftrightarrow$

$$\Leftrightarrow (4n-1)a^2 + (4k-14)ab + (4n-1)b^2 \ge 0 \tag{1}$$

Putting the condition $\Delta = 0$ *we obtain* $k + 2n = 4$ *and* (1) *can be written:*

$(4n-1)(a^2+b^2) - (8n-2)ab \ge 0 \Leftrightarrow (4n-1)(a-b)^2 \ge 0$, *true for*

$n \ge \frac{1}{4}$. *The equality holds if and only if* $a = b$.

□

Extension 6:

$$\mathbf{4 \le \left(\sqrt{ab}+\sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{1}{\sqrt{ab}}+\sqrt{\frac{2}{a^2+b^2}}\right) \le 4-2n+n\left(\frac{a}{b}+\frac{b}{a}\right), \text{ with } n \ge \frac{1}{2}}$$

Proof (Marin Chirciu).

We denote $\sqrt{ab} = x,\ \sqrt{\frac{a^2+b^2}{2}} = y$ *and* $E = \left(\sqrt{ab}+\sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{1}{\sqrt{ab}}+\sqrt{\frac{2}{a^2+b^2}}\right)$.

We have $E = (x+y)\left(\frac{1}{x}+\frac{1}{y}\right) = 2 + \frac{x^2+y^2}{xy}$. *We look for an inequality*

$$\textit{We look for an inequality having the form } E \le k + n\left(\frac{a}{b} + \frac{b}{a}\right)$$

We obtain:

$$E = 2+\frac{x^2+y^2}{xy} = 2+\frac{ab+\frac{a^2+b^2}{2}}{\sqrt{ab}\cdot\sqrt{\frac{a^2+b^2}{2}}} \le 2+\frac{ab+\frac{a^2+b^2}{2}}{\frac{2ab}{a+b}\cdot\frac{a+b}{2}} = 2+\frac{ab+\frac{a^2+b^2}{2}}{ab} = 3+\frac{a^2+b^2}{2ab} \le k+n\left(\frac{a}{b}+\frac{b}{a}\right)$$

where the penultimate inequality follows from $\sqrt{\frac{a^2+b^2}{2}} \ge \frac{a+b}{2}$ *and* $\sqrt{ab} \ge \frac{2ab}{a+b}$

and the last one is equivalent with:

$$\frac{(a^2+b^2)}{2ab} \le (k-3) + \frac{n(a^2+b^2)}{ab} \Leftrightarrow (2n-1)(a^2+b^2) + (2k-6)ab \ge 0 \Leftrightarrow$$

$$\Leftrightarrow (2n-1)a^2 + (2k-6)ab + (2n-1)b^2 \ge 0 \tag{1}$$

Putting the condition $\Delta = 0$ *and we obtain* $k+2n = 4$ *the inequality* (1) *can be written*

$$(2n-1)a - b^2 \geq 0, \text{ true for } n \geq 12$$

The equality holds if and only if $a = b$.

$\square$

References

[1] http://www.cut-the-knot.org.
[2] http://www.ssmrmh.ro

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