The Symphony No. 2^n

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Learning to lead. Learning to create. Learning to grow, develop. Learning to care.

(The Motto of Tran Dai Nghia High School, HCM City, Vietnam)

ABSTRACT

In this article, I would like to present my solution for the definite integral:

$$\int \frac{dx}{1+x^{2^n}} \quad (n \in \mathbb{N}, n \geq 2)$$

To solve this generalized definite integral, I suggest a predicted result, and then prove that my suggestion is true. In fact, I had spent 2 years (from December 2010 to November 2012) working on several problems to find the right prediction. This article is presented in 4 sections – *Molto Allegro*, *Andante*, *Menuetto*, and *Allegro assai*.

I. Molto Allegro

I shall prove that:

$$\begin{aligned} I_{2^n} &= \int \frac{dx}{1+x^{2^n}} \\ &= \frac{1}{2^n} \sum_{k=0}^{2^{n-2}-1} \left\{ \cos\frac{(2k+1)\pi}{2^n} \left[2\tan^{-1} \frac{x - \frac{1}{x}}{2\cos\frac{(2k+1)\pi}{2^n}} \right. \right. \\ &\qquad \left. \left. + \ln\left(\frac{x^2 + 2x\cos\frac{(2k+1)\pi}{2^n} + 1}{x^2 - 2x\cos\frac{(2k+1)\pi}{2^n} + 1} \right) \right] \right\} + C \quad (n \in \mathbb{N}, n \geq 2) \end{aligned}$$

Denote:

$$I_4^+(\lambda) = \int \frac{x^2+1}{x^4+\lambda x^2+1} dx$$

$$I_4^-(\lambda) = \int \frac{x^2-1}{x^4+\lambda x^2+1} dx$$

Of which λ is a given real number.

With $\lambda = -2\cos\theta$, I get:

$$I_4^+(-2\cos\theta) = \int \frac{x^2+1}{x^4-2x^2\cos\theta+1}dx$$

$$I_4^+(-2\cos\theta) = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}-2\cos\theta}dx$$

$$I_4^+(-2\cos\theta) = \int \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2-2\cos\theta}$$

$$I_4^+(-2\cos\theta) = \int \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+4\left(\sin\frac{\theta}{2}\right)^2}$$

$$I_4^+(-2\cos\theta) = \frac{1}{2}\csc\frac{\theta}{2}\tan^{-1}\frac{x-\frac{1}{x}}{2\sin\frac{\theta}{2}}+C$$

And

$$I_4^-(-2\cos\theta) = \int \frac{x^2-1}{x^4-2x^2\cos\theta+1}dx$$

$$I_4^-(-2\cos\theta) = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}-2\cos\theta}dx$$

$$I_4^-(-2\cos\theta) = \int \frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-2-2\cos\theta}$$

$$I_4^-(-2\cos\theta) = \int \frac{d\left(x+\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+4\left(\cos\frac{\theta}{2}\right)^2}$$

$$I_4^-(-2\cos\theta) = \frac{1}{4}\sec\frac{\theta}{2}\ln\left|\frac{x+\frac{1}{x}-2\cos\frac{\theta}{2}}{x+\frac{1}{x}+2\cos\frac{\theta}{2}}\right|+C$$

$$I_4^-(-2\cos\theta) = \frac{1}{4}\sec\frac{\theta}{2}\ln\left(\frac{x^2-2x\cos\frac{\theta}{2}+1}{x^2+2x\cos\frac{\theta}{2}+1}\right)+C$$

Proving the predicted result is true for the generalized definite integral I_{2^n} (n is an integer greater than or equal to 2), I form the function $F_{2^n}(x)$ as follow:

$$F_{2^n}(x) = \frac{1}{2^n}\sum_{k=0}^{2^{n-2}-1}\left\{\cos\frac{(2k+1)\pi}{2^n}\left[2\tan^{-1}\frac{x-\frac{1}{x}}{2\cos\frac{(2k+1)\pi}{2^n}} + \ln\left(\frac{x^2+2x\cos\frac{(2k+1)\pi}{2^n}+1}{x^2-2x\cos\frac{(2k+1)\pi}{2^n}+1}\right)\right]\right\} + C$$

Of which

$$\alpha_k = \frac{(2k+1)\pi}{2^n} \quad \left(k = \overline{0; 2^{n-2}-1}\right)$$

Where α_k are all odd multiples of $\frac{\pi}{2^n}$ and not greater than $\frac{\pi}{2}$. By $\cos\alpha_k = \sin\left(\frac{\pi}{2}-\alpha_k\right)$, I may comment that the expression of $F_{2^n}(x)$ remains unchanged if "cos" letters are replaced by "sin" letters. Thanks to above results of $I_4^+(-2\cos\theta)$ and $I_4^-(-2\cos\theta)$, the function $F_{2^n}(x)$ could be transformed as:

$$F_{2^n}(x) = \frac{1}{2^n}\sum_{k=0}^{2^{n-2}-1}\left[2\cos\frac{(2k+1)\pi}{2^n}\tan^{-1}\frac{x-\frac{1}{x}}{2\cos\frac{(2k+1)\pi}{2^n}} + \cos\frac{(2k+1)\pi}{2^n}\ln\left(\frac{x^2+2x\cos\frac{(2k+1)\pi}{2^n}+1}{x^2-2x\cos\frac{(2k+1)\pi}{2^n}+1}\right)\right] + C$$

$$F_{2^n}(x) = \frac{1}{2^n}\sum_{k=0}^{2^{n-2}-1}\left[2\sin\frac{(2k+1)\pi}{2^n}\tan^{-1}\frac{x-\frac{1}{x}}{2\sin\frac{(2k+1)\pi}{2^n}} + \cos\frac{(2k+1)\pi}{2^n}\ln\left|\frac{x+\frac{1}{x}+2\cos\frac{(2k+1)\pi}{2^n}}{x+\frac{1}{x}-2\cos\frac{(2k+1)\pi}{2^n}}\right|\right] + C$$

$$F_{2^n}(x) = \frac{1}{2^{n-2}}\sum_{k=0}^{2^{n-2}-1}\left[\frac{1}{2}\left(\sin\frac{(2k+1)\pi}{2^n}\right)^2\csc\frac{(2k+1)\pi}{2^n}\tan^{-1}\frac{x-\frac{1}{x}}{2\sin\frac{(2k+1)\pi}{2^n}} + \frac{1}{4}\left(\cos\frac{(2k+1)\pi}{2^n}\right)^2\sec\frac{(2k+1)\pi}{2^n}\ln\left|\frac{x+\frac{1}{x}+2\cos\frac{(2k+1)\pi}{2^n}}{x+\frac{1}{x}-2\cos\frac{(2k+1)\pi}{2^n}}\right|\right] + C$$

Section I-*Molto Allegro* of *the Symphony No.* 2^n may end here. Next, Section II-*Andante* would bring us the derivative calculation of $F_{2^n}(x)$, which proves that the predicted suggestion is true.

II. Andante

With

$$F_{2^n}(x)=\frac{1}{2^{n-2}}\sum_{k=0}^{2^{n-2}-1}\left[\frac{1}{2}\left(\sin\frac{(2k+1)\pi}{2^n}\right)^2\csc\frac{(2k+1)\pi}{2^n}\tan^{-1}\frac{x-\frac{1}{x}}{2\sin\frac{(2k+1)\pi}{2^n}}\right.$$
$$\left.+\frac{1}{4}\left(\cos\frac{(2k+1)\pi}{2^n}\right)^2\sec\frac{(2k+1)\pi}{2^n}\ln\left|\frac{x+\frac{1}{x}+2\cos\frac{(2k+1)\pi}{2^n}}{x+\frac{1}{x}-2\cos\frac{(2k+1)\pi}{2^n}}\right|\right]+C$$

I may obtain:

$$F'_{2^n}(x)=\frac{1}{2^{n-2}}\sum_{k=0}^{2^{n-2}-1}\left[\left(\sin\frac{(2k+1)\pi}{2^n}\right)^2\frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+4\left(\sin\frac{(2k+1)\pi}{2^n}\right)^2}\right.$$
$$\left.+\left(\cos\frac{(2k+1)\pi}{2^n}\right)^2\frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2-4\left(\cos\frac{(2k+1)\pi}{2^n}\right)^2}\right]$$

$$F'_{2^n}(x)=\frac{1}{2^{n-2}}\sum_{k=0}^{2^{n-2}-1}\left[\left(\sin\frac{(2k+1)\pi}{2^n}\right)^2\frac{(x^2+1)}{x^4+1-2x^2\cos\frac{(2k+1)\pi}{2^{n-1}}}\right.$$
$$\left.+\left(\cos\frac{(2k+1)\pi}{2^n}\right)^2\frac{(1-x^2)}{x^4+1-2x^2\cos\frac{(2k+1)\pi}{2^{n-1}}}\right]$$

$$F'_{2^n}(x)=\frac{1}{2^{n-2}}\sum_{k=0}^{2^{n-2}-1}\frac{1-x^2\cos\frac{(2k+1)\pi}{2^{n-1}}}{x^4+1-2x^2\cos\frac{(2k+1)\pi}{2^{n-1}}}$$

Let

$$\beta_k=\frac{(2k+1)\pi}{2^{n-1}}\quad\left(k=\overline{0;2^{n-2}-1}\right)$$

Then

$$\beta_{2^{n-2}-1-k}=\frac{[2(2^{n-2}-1-k)+1]\pi}{2^{n-1}}$$
$$\beta_{2^{n-2}-1-k}=\frac{(2^{n-1}-1-2k)\pi}{2^{n-1}}$$
$$\beta_{2^{n-2}-1-k}=\pi-\frac{(2k+1)\pi}{2^{n-1}}$$
$$\beta_{2^{n-2}-1-k}=\pi-\beta_k$$

Which implies:

$$\cos(\beta_{2^{n-2}-1-k})=-\cos\beta_k$$

It is important to note that, in 2^{n-2} possible values of k, choosing k and then determining $2^{n-2}-1-k$ does not depend on the position of k in a set of 2^{n-2} values. Accordingly, this set could be separated into two halves, the front contains 2^{n-3} elements from $k=0$ to $k=2^{n-3}-1$, and the back contains 2^{n-3} remaining elements from $k=2^{n-3}$ to $k=2^{n-2}-1$. This argument is similar to the fact that $C_n^k = C_n^{n-k}$. For convenience, I may call this note *the combination argument*.

Following up the above transformation:

$$F'_{2^n}(x) = \frac{1}{2^{n-2}} \sum_{k=0}^{2^{n-3}-1} \left[\frac{1 - x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}}{x^4 + 1 - 2x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}} + \frac{1 + x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}}{x^4 + 1 + 2x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}} \right]$$

$$F'_{2^n}(x) = \frac{1}{2^{n-2}} \sum_{k=0}^{2^{n-3}-1} \left[\frac{1}{x^4 + 1 - 2x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}} + \frac{1}{x^4 + 1 + 2x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}} \right.$$
$$\left. + x^2 \cos\frac{(2k+1)\pi}{2^{n-1}} \left(\frac{1}{x^4 + 1 + 2x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}} - \frac{1}{x^4 + 1 - 2x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}} \right) \right]$$

$$F'_{2^n}(x) = \frac{1}{2^{n-2}} \sum_{k=0}^{2^{n-3}-1} \frac{2(x^4+1) - 4x^4 \left(\cos\frac{(2k+1)\pi}{2^{n-1}}\right)^2}{x^8 + 1 - 2x^4 \cos\frac{(2k+1)\pi}{2^{n-2}}}$$

$$F'_{2^n}(x) = \frac{1}{2^{n-3}} \sum_{k=0}^{2^{n-3}-1} \frac{1 + x^4 \left[1 - 2\left(\cos\frac{(2k+1)\pi}{2^{n-1}}\right)^2\right]}{x^8 + 1 - 2x^4 \cos\frac{(2k+1)\pi}{2^{n-2}}}$$

$$F'_{2^n}(x) = \frac{1}{2^{n-3}} \sum_{k=0}^{2^{n-3}-1} \frac{1 - x^4 \cos\frac{(2k+1)\pi}{2^{n-2}}}{x^8 + 1 - 2x^4 \cos\frac{(2k+1)\pi}{2^{n-2}}}$$

Here come the arcs of

$$\gamma_k = \frac{(2k+1)\pi}{2^{n-2}} \quad \left(k = \overline{0; 2^{n-3}-1}\right)$$

I am not able to transform the above to the umpteenth time! Looking back, I wonder whether there is a link between α_k, β_k, and γ_k. Section III-*Menuetto* would reveal this.

III. Menuetto

Put

$$\Psi_m(x) = \frac{1}{2^{n-m}} \sum_{k=0}^{2^{n-m}-1} \frac{1 - x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}}{x^{2^m} + 1 - 2x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}}$$

Forming $\Psi_m(x)$, I may rewrite:

$$F'_{2^n}(x) = \Psi_2(x) = \frac{1}{2^{n-2}} \sum_{k=0}^{2^{n-2}-1} \frac{1 - x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}}{x^4 + 1 - 2x^2 \cos\frac{(2k+1)\pi}{2^{n-1}}}$$

As well as

$$F'_{2^n}(x) = \Psi_3(x) = \frac{1}{2^{n-3}} \sum_{k=0}^{2^{n-3}-1} \frac{1 - x^4 \cos\frac{(2k+1)\pi}{2^{n-2}}}{x^8 + 1 - 2x^4 \cos\frac{(2k+1)\pi}{2^{n-2}}}$$

It is highly important to note that:

$$F'_{2^n}(x) = \Psi_2(x) = \Psi_3(x)$$

Before using this notification, it is necessary to ensure that:

$$\begin{cases} m \in \mathbb{N} \\ m \geq 2 \\ 2^{n-m} - 1 \geq 0 \end{cases} \Leftrightarrow \begin{cases} m \in \mathbb{N} \\ m \geq 2 \\ 2^{n-m} \geq 1 \end{cases} \Leftrightarrow \begin{cases} m \in \mathbb{N} \\ m \geq 2 \\ n - m \geq 0 \end{cases} \Leftrightarrow \begin{cases} m \in \mathbb{N} \\ m \geq 2 \\ m \leq n \end{cases}$$

Due to

$$\Psi_2(x) = \Psi_3(x)$$

There should be a question on the link between $\Psi_m(x)$ and $\Psi_{m+1}(x)$. If this link actually exists, it is the key to clinch the generalized definite integral.

Express:

$$\Psi_{m+1}(x) = \frac{1}{2^{n-m-1}} \sum_{k=0}^{2^{n-m-1}-1} \frac{1 - x^{2^m} \cos\frac{(2k+1)\pi}{2^{n-m}}}{x^{2^{m+1}} + 1 - 2x^{2^m} \cos\frac{(2k+1)\pi}{2^{n-m}}}$$

Now, I am going to prove that

$$\Psi_m(x) = \Psi_{m+1}(x)$$

Of which

$$\begin{cases} m \in \mathbb{N} \\ m \geq 2 \\ m < n \end{cases}$$

Denote:

$$\phi_k = \frac{(2k+1)\pi}{2^{n-m+1}} \quad (k = \overline{0; 2^{n-m}-1})$$

Whereby:

$$\phi_{2^{n-m}-1-k} = \frac{[2(2^{n-m}-1-k)+1]\pi}{2^{n-m+1}}$$
$$\phi_{2^{n-m}-1-k} = \frac{(2^{n-m+1}-1-2k)\pi}{2^{n-m+1}}$$
$$\phi_{2^{n-m}-1-k} = \pi - \frac{(2k+1)\pi}{2^{n-m+1}}$$
$$\phi_{2^{n-m}-1-k} = \pi - \phi_k$$

As a result:

$$\cos(\phi_{2^{n-m}-1-k}) = -\cos\phi_k$$

Using the above mentioned *combination argument*, I am able to transform:

$$\Psi_m(x) = \frac{1}{2^{n-m}} \sum_{k=0}^{2^{n-m}-1} \frac{1 - x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}}{x^{2^m} + 1 - 2x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}}$$

$$\Psi_m(x) = \frac{1}{2^{n-m}} \sum_{k=0}^{2^{n-m-1}-1} \left[\frac{1 - x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}}{x^{2^m} + 1 - 2x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}} + \frac{1 + x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}}{x^{2^m} + 1 + 2x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}} \right]$$

$$\Psi_m(x) = \frac{1}{2^{n-m}} \sum_{k=0}^{2^{n-m-1}-1} \left[\frac{1}{x^{2^m} + 1 - 2x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}} + \frac{1}{x^{2^m} + 1 + 2x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}} + x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}} \left(\frac{1}{x^{2^m} + 1 + 2x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}} - \frac{1}{x^{2^m} - 1 + 2x^{2^{m-1}} \cos\frac{(2k+1)\pi}{2^{n-m+1}}} \right) \right]$$

$$\Psi_m(x) = \frac{1}{2^{n-m}} \sum_{k=0}^{2^{n-m-1}-1} \frac{2\left(x^{2^{m+1}}+1\right) - 4x^{2^m}\left(\cos\frac{(2k+1)\pi}{2^{n-m+1}}\right)^2}{x^{2^{m+1}}+1-2x^{2^m}\cos\frac{(2k+1)\pi}{2^{n-m}}}$$

$$\Psi_m(x) = \frac{1}{2^{n-m-1}} \sum_{k=0}^{2^{n-m-1}-1} \frac{1+x^{2^m}\left[1-2\left(\cos\frac{(2k+1)\pi}{2^{n-m+1}}\right)^2\right]}{x^{2^{m+1}}+1-2x^{2^m}\cos\frac{(2k+1)\pi}{2^{n-m}}}$$

$$\Psi_m(x) = \frac{1}{2^{n-m-1}} \sum_{k=0}^{2^{n-m-1}-1} \frac{1-x^{2^m}\cos\frac{(2k+1)\pi}{2^{n-m}}}{x^{2^{m+1}}+1-2x^{2^m}\cos\frac{(2k+1)\pi}{2^{n-m}}}$$

$$\Psi_m(x) = \Psi_{m+1}(x)$$

I have proved that

$$\Psi_m(x) = \Psi_{m+1}(x) \quad \forall m: \begin{cases} m \in \mathbb{N} \\ m \geq 2 \\ m < n \end{cases}$$

This results in:

$$F'_{2^n}(x) = \Psi_2(x) = \Psi_3(x) = \cdots = \Psi_{n-1}(x) = \Psi_n(x) \quad \forall n: \begin{cases} n \in \mathbb{N} \\ n \geq 2 \end{cases}$$

Therefore, QED is obtained as per the expression of $\Psi_n(x)$. Indeed:

$$\Psi_n(x) = \frac{1}{2^{n-n}} \sum_{k=0}^{2^{n-n}-1} \frac{1-x^{2^{n-1}}\cos\frac{(2k+1)\pi}{2^{n-n+1}}}{x^{2^n}+1-2x^{2^{n-1}}\cos\frac{(2k+1)\pi}{2^{n-n+1}}}$$

$$\Psi_n(x) = \sum_{k=0}^{0} \frac{1-x^{2^{n-1}}\cos\frac{(2k+1)\pi}{2}}{x^{2^n}+1-2x^{2^{n-1}}\cos\frac{(2k+1)\pi}{2}}$$

$$\Psi_n(x) = \frac{1-x^{2^{n-1}}\cos\frac{\pi}{2}}{x^{2^n}+1-2x^{2^{n-1}}\cos\frac{\pi}{2}}$$

$$\Psi_n(x) = \frac{1}{x^{2^n}+1}$$

As

$$F'_{2^n}(x) = \Psi_n(x) = \frac{1}{x^{2^n}+1} \quad \forall n: \begin{cases} n \in \mathbb{N} \\ n \geq 2 \end{cases}$$

Which means:

$$F_{2^n}(x) = \int \frac{dx}{x^{2^n}+1} \quad \forall n: \begin{cases} n \in \mathbb{N} \\ n \geq 2 \end{cases}$$

The proof is granted upon the completion of Section III-*Menuetto.* The conclusion is presented in Section IV-*Allegro assai*, *the Symphony No.* 2^n's final movement.

IV. Allegro assai

I have proved that

$$F_{2^n}(x) = \int \frac{dx}{x^{2^n}+1} \quad \forall n: \begin{cases} n \in \mathbb{N} \\ n \geq 2 \end{cases}$$

Which means

$$\begin{aligned} I_{2^n} &= \int \frac{dx}{1+x^{2^n}} \\ &= \frac{1}{2^n} \sum_{k=0}^{2^{n-2}-1} \left\{ \cos\frac{(2k+1)\pi}{2^n} \left[2\tan^{-1} \frac{x - \frac{1}{x}}{2\cos\frac{(2k+1)\pi}{2^n}} \right.\right. \\ &\qquad \left.\left. + \ln\left(\frac{x^2 + 2x\cos\frac{(2k+1)\pi}{2^n} + 1}{x^2 - 2x\cos\frac{(2k+1)\pi}{2^n} + 1} \right) \right] \right\} + C \quad (n \in \mathbb{N}, n \geq 2) \end{aligned}$$

This is such a beautiful expression, which could also be written as

$$\begin{aligned} I_{2^n} &= \int \frac{dx}{1+x^{2^n}} \\ I_{2^n} &= \frac{1}{2^n} \sum_{k=0}^{2^{n-2}-1} \left\{ \cos\frac{(2k+1)\pi}{2^n} \left[2\tan^{-1} \frac{x - \frac{1}{x}}{2\cos\frac{(2k+1)\pi}{2^n}} \right.\right. \\ &\qquad \left.\left. + \ln\left(\frac{x^2 + 2x\cos\frac{(2k+1)\pi}{2^n} + 1}{x^2 - 2x\cos\frac{(2k+1)\pi}{2^n} + 1} \right) \right] \right\} + C \\ I_{2^n} &= \frac{1}{2^n} \sum_{k=0}^{2^{n-2}-1} \left[2\cos\frac{(2k+1)\pi}{2^n} \tan^{-1} \frac{x - \frac{1}{x}}{2\cos\frac{(2k+1)\pi}{2^n}} \right. \\ &\qquad \left. + \cos\frac{(2k+1)\pi}{2^n} \ln\left(\frac{x^2 + 2x\cos\frac{(2k+1)\pi}{2^n} + 1}{x^2 - 2x\cos\frac{(2k+1)\pi}{2^n} + 1} \right) \right] + C \end{aligned}$$

$$I_{2^n} = \frac{1}{2^n} \sum_{k=0}^{2^{n-2}-1} \left[2 \sin \frac{(2k+1)\pi}{2^n} \tan^{-1} \frac{x - \frac{1}{x}}{2 \sin \frac{(2k+1)\pi}{2^n}} + \cos \frac{(2k+1)\pi}{2^n} \ln \left| \frac{x + \frac{1}{x} + 2 \cos \frac{(2k+1)\pi}{2^n}}{x + \frac{1}{x} - 2 \cos \frac{(2k+1)\pi}{2^n}} \right| \right] + C$$

Furthermore, I may denote:

$$\begin{cases} f_{(\theta)}(x) = 2 \tan^{-1} \dfrac{x - \frac{1}{x}}{2 \sin \theta} \\ g_{(\theta)}(x) = \ln \left| \dfrac{x + \frac{1}{x} + 2 \cos \theta}{x + \frac{1}{x} - 2 \cos \theta} \right| \end{cases}$$

Then the result of I_{2^n} could be expressed as:

$$I_{2^n} = \int \frac{dx}{1 + x^{2^n}}$$

$$I_{2^n} = \frac{1}{2^n} \sum_{k=0}^{2^{n-2}-1} \left[\sin \frac{(2k+1)\pi}{2^n} f_{\left(\frac{(2k+1)\pi}{2^n}\right)}(x) + \cos \frac{(2k+1)\pi}{2^n} g_{\left(\frac{(2k+1)\pi}{2^n}\right)}(x) \right] + C$$

Of which, n is an integer greater than or equal to 2.

FINE.

As the conclusion of *the Symphony No.* 2^n, I am glad to tell, what I present in this article is only the destination of an odyssey that every moment is worthy. It would be my pleasure to share all that moments in another article soon.