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CONTENT

TECHNIQUES FOR INTEGRAL CALCULUS

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Abstract: *In this paper was presented few general techniques for integral calculus and applications.*

1. Introduction.

Proposition 1.1: If $f:[a,b]\to R, g:[a,b]\to R^*, u:[a,b]\to R$ continuous functions and

$$f(x)+f(s-x)=u(x), g(x)=g(s-x), \forall x\in[a,b], s=a+b, then:$$

$$\int_a^b \frac{f(x)}{g(x)}dx=\frac{1}{2}\int_a^b \frac{u(x)}{g(x)}dx$$

Proof: Use substitution $x=s-t$, we have:

$$\int_a^b \frac{f(x)}{g(x)}dx=\int_a^b \frac{f(s-t)}{g(s-t)}(-dt)=\int_a^b \frac{u(t)-f(t)}{g(t)}dt=\int_a^b \frac{u(t)}{g(t)}dt-\int_a^b \frac{f(t)}{g(t)}dt$$

$$\int_a^b \frac{f(x)}{g(x)}dx=\frac{1}{2}\int_a^b \frac{u(x)}{g(x)}dx$$

Application 1.1 Find:

$$\Omega=\int_0^\pi \frac{(x+1)\sin x}{3+\cos^2 x}dx$$

Solution: $\Omega=\int_0^\pi \frac{(x+1)\sin x}{3+\cos^2 x}dx=\int_0^\pi \frac{x\sin x}{3+\cos^2 x}dx+\int_0^\pi \frac{\sin x}{3+\cos^2 x}dx=I_1+I_2$

$$I_1=\int_0^\pi \frac{x\sin x}{3+\cos^2 x}dx=\int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{3+\cos^2 x}dx=$$

$$=\pi\int_0^\pi \frac{\sin x}{3+\cos^2 x}dx-\int_0^\pi \frac{x\sin x}{3+\cos^2 x}dx\Rightarrow I_1=\pi I_2-I_1\Rightarrow I_1=\frac{\pi}{2}I_2$$

$$I_2=\int_0^\pi \frac{\sin x}{3+\cos^2 x}dx=-\int_0^\pi \frac{(\cos x)'}{3+\cos^2 x}dx=-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\cos x}{\sqrt{3}}\right)\Big|_0^\pi=\frac{2\pi}{3\sqrt{3}}$$

Application 1.2 Find:

$$\boldsymbol{\Omega = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{x\tan^{-1}x}{1+e^{\tan x}}dx}$$

Solution: $\Omega = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{x\tan^{-1}x}{1+e^{\tan x}}dx = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{(-x)\tan^{-1}(-x)}{1+e^{\tan(-x)}}dx = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{xe^{\tan x}\cdot\tan^{-1}x}{1+e^{\tan x}}dx$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{(1+e^{\tan x}-1)x\cdot\tan^{-1}x}{1+e^{\tan x}}dx = \int_{-\sqrt{3}}^{\sqrt{3}} x\cdot\tan^{-1}x\,dx - I$$

$$2I = \int_{-\sqrt{3}}^{\sqrt{3}} x\cdot\tan^{-1}x\,dx = 2\int_0^{\sqrt{3}} x\cdot\tan^{-1}x\,dx$$

$$\Omega = \int_0^{\sqrt{3}} x\cdot\tan^{-1}x\,dx = \left(\frac{x^2}{2}-\tan^{-1}x\right)\Bigg|_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}}\frac{x^2}{1+x^2}dx = \frac{\pi}{3}-\frac{\sqrt{3}}{2}$$

Application 1.3 Find:

$$\boldsymbol{\Omega = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} x\cdot\log\left(1+e^{x\sqrt{1-x^2}}\right)dx}$$

Solution: $\Omega = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} x\cdot\log\left(1+e^{x\sqrt{1-x^2}}\right)dx \overset{x=-y}{=} -\int_{-\sqrt{2}/2}^{\sqrt{2}/2} y\cdot\log\left(1+e^{-y\sqrt{1-y^2}}\right)dy =$

$$= -\int_{-\sqrt{2}/2}^{\sqrt{2}/2} y\cdot\log\left(\frac{1+e^{y\sqrt{1-y^2}}}{e^{y\sqrt{1-y^2}}}\right)dy = -\Omega + \int_{-\sqrt{2}/2}^{\sqrt{2}/2} y\cdot\log\left(e^{y\sqrt{1-y^2}}\right)dy = -\Omega + \int_{-\sqrt{2}/2}^{\sqrt{2}/2} y^2\sqrt{1-y^2}dy$$

$$2\Omega = 2\int_0^{\sqrt{2}/2} y^2\sqrt{1-y^2}dy \overset{y=\sin t}{=} 2\int_0^{\pi/4} \sin^2 t\sqrt{1-\sin^2 t}\cdot\cos t\,dt$$

$$I = \int_0^{\pi/4} \sin^2 t\cos^2 t\,dt = \frac{1}{4}\int_0^{\pi/4}\sin^2 2t\,dt = \frac{1}{4}\int_0^{\pi/4}\frac{1-\cos 4t}{2}dt = \frac{1}{8}\left(x-\frac{1}{4}\sin 4t\right)_0^{\frac{\pi}{4}} = \frac{\pi}{32}$$

Application 1.4 Find:

$$\boldsymbol{\Omega = \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan x)}{2+\sin 2x+\cos 2x}dx}$$

Solution: $\mathrm{I}=\int_0^{\frac{\pi}{4}}\frac{\log(1+\mathrm{tgx})}{2+\sin2x+\cos2x}\mathrm{dx}=\int_0^{\frac{\pi}{4}}\frac{\log\left(1+\mathrm{tg}\left(\frac{\pi}{4}-\mathrm{y}\right)\right)}{2+\sin\left(\frac{\pi}{2}-2\mathrm{y}\right)+\cos\left(\frac{\pi}{2}-2\mathrm{y}\right)}\mathrm{dy}=\int_0^{\frac{\pi}{4}}\frac{\log2-\log(1+tgy)}{2+sin2y+cos2y}dy$

$$2I=\log2\int_0^{\frac{\pi}{4}}\frac{1}{2+\frac{2\tan x}{1+\tan^2x}+\frac{1-\tan^2x}{1+\tan^2x}}dx=\log2\int_0^1\frac{1}{t^2+2t+3}dt\Rightarrow I=\frac{\log 2}{2\sqrt{2}}\left(\tan^{-1}\sqrt{2}-\frac{\pi}{4}\right)$$

Application 1.5 Find:

$$\boldsymbol{\Omega}=\int_0^{2\pi}\frac{\boldsymbol{x}+\tan(\sin\boldsymbol{x})}{2+\cos\boldsymbol{x}}\boldsymbol{dx}$$

Solution 1: $I=\int_0^{2\pi}\frac{x+\tan(\sin x)}{2+\cos x}dx=\int_0^{2\pi}\frac{2\pi-y-\tan(\sin x)}{2+\cos y}dy=2\pi\int_0^{2\pi}\frac{1}{2+\cos y}dy-\Omega$

$$\frac{1}{\pi}I=\int_0^{2\pi}\frac{1}{2+\cos x}dx=\int_0^{\pi}\frac{1}{2+\cos x}dx+\int_{\pi}^{2\pi}\frac{1}{2+\cos x}dx=$$

$$\int_0^{\pi}\frac{1}{2+\cos x}dx+\int_0^{\pi}\frac{1}{2-\cos x}dx=\int_0^{\frac{\pi}{2}}\frac{1}{2+\cos x}dx+\int_0^{\frac{\pi}{2}}\frac{1}{2-\cos x}dx+\int_0^{\frac{\pi}{2}}\frac{1}{2+\sin x}dx+$$

$$\int_0^{\frac{\pi}{2}}\frac{1}{2-\sin x}dx=2\int_0^1\frac{1}{t^2+3}dt+\frac{2}{3}\int_0^1\frac{1}{t^2+\frac{1}{3}}dt+\int_0^1\frac{1}{(t+1)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt+$$

$$\int_0^1\frac{1}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt=\frac{2\pi\sqrt{3}}{3}\Rightarrow I=\frac{2\pi^2\sqrt{3}}{3}$$

Solution 2: $I=\int_0^{2\pi}\frac{x+\tan(\sin x)}{2+\cos x}dx=\int_{-\pi}^{\pi}\frac{\pi+y-\tan(\sin y)}{2-\cos y}dy=\int_{-\pi}^{\pi}\frac{\pi}{2-\cos y}dy+$

$$+\int_{-\pi}^{\pi}\frac{y-\tan(\sin y)}{2-\cos y}dy=\int_0^{\pi}\frac{2\pi}{2-\cos y}dy=2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2+\sin z}dz=$$

$$=2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2+\frac{2\tan\left(\frac{z}{2}\right)}{1+\tan^2\left(\frac{z}{2}\right)}}dz=2\pi\int_{-1}^{1}\frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt=\frac{2\pi^2\sqrt{3}}{3}$$

2. General result.

Proposition 2.1 : If $f:\mathbb{R}\to\mathbb{R}$ is continuous function with property

$$\boldsymbol{\alpha f(x+b)+\beta f(c-x)=g(x+b), \forall x \in \mathbb{R}; (1)}$$

where $\alpha, \beta \in \mathbb{R}^*, \alpha+\beta \neq 0, b, c \in \mathbb{R}$, and $g: \mathbb{R} \to \mathbb{R}$ continuous function with primitives G.

Then for all $m, n \in \mathbb{R}$, such that $m+n=b+c$,

$$\boldsymbol{I_{m,n}=\int_m^n f(x)\,dx=\frac{G(n)-G(m)}{\alpha+\beta}; (2)}$$

Proof. Replacing x with $x-b$, we get $\alpha f(x)+\beta f(b+c-x)=g(x)$, and hence,

$f(x)=\frac{1}{\alpha}g(x)-\frac{\beta}{\alpha}f(b+c-x)$ Integrating on the interval $[m,n]$, we get:

$$I_{m,n}=\int_m^n f(x)\,dx=\frac{1}{\alpha}\int_m^n g(x)\,dx-\frac{\beta}{\alpha}\int_m^n f(b+c-x)\,dx=$$

$$=\frac{1}{\alpha}[G(n)-G(m)]-\frac{\beta}{\alpha}\int_m^n f(b+c-x)\,dx \overset{b+c-x=t}{=} \frac{1}{\alpha}[G(n)-G(m)]+\frac{\beta}{\alpha}\int_n^m f(t)\,dt$$

$$I_{m,n}\left(1+\frac{\beta}{\alpha}\right)=\frac{1}{\alpha}[G(n)-G(m)], \text{then } I_{m,n}=\int_m^n f(x)\,dx=\frac{G(n)-G(m)}{\alpha+\beta}; (2)$$

Application 2.1. If $h: \mathbb{R} \to \mathbb{R}$ is odd function, θ −periodic and with f' continuous, then find:

$$\boldsymbol{\Omega=\int_0^{2\theta}\frac{xh'(x)}{1+h^2(x)}dx}$$

Solution: Let $f(x)=\frac{xh'(x)}{1+h^2(x)}$, then $f(x+2\theta)+f(-x)=2\theta\cdot\frac{f(x)}{1+f^2(x)}$ and for $\alpha=\beta=1$,

$$b=2\theta, c=0, g(x)=2\theta\cdot\frac{h'(x)}{1+h^2(x)}$$

Now, g has the primitives $G: \mathbb{R} \to \mathbb{R}, G(x)=2\theta\cdot\tan^{-1}x$. For $m+n=2\theta$ and using Proposition 2, we have:

$$I_{m,n}=\int_m^n f(x)\,dx=\theta[\tan^{-1}h(n)-\tan^{-1}h(m)]$$

Application 2.2 Find:

$$\boldsymbol{\Omega=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx}$$

Solution: Let $(x)=\frac{x\sin x}{1+\cos^2 x}$, then $f(x)+f(\pi-x)=\frac{\pi\sin x}{1+\cos^2 x}$ and using Proposition 2.1, we get:

$$I_{m,n}=\int_m^n f(x)\,dx=\frac{\pi}{2}[\tan^{-1}(\cos m)-\tan^{-1}(\cos n)]; m+n=\pi$$

For $m=0, n=\pi \Rightarrow \Omega=\frac{\pi^2}{4}$.

Application 2.3. **Find:**

$$\boldsymbol{\Omega = \int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx}$$

Solution: Let $f(x) = log\,(1+\tan x)$, then

$$f\left(\frac{\pi}{4}-x\right) = \log\left(1+\frac{1-\tan x}{1+\tan x}\right) = \log 2 - \log(1+\tan x)\,, \qquad f(x)+f\left(\frac{\pi}{4}-x\right) = \log 2$$

Using Theorem 2, we get:

$$I_{m,n} = \int_m^n \log(1+\tan x)\,dx = \frac{n-m}{2}\log 2\,; m+n = \frac{\pi}{4}$$

For $m = 0, n = \frac{\pi}{4} \Rightarrow \Omega = \frac{\pi}{8}\log 2$

Application 2.4 If $\boldsymbol{h}$: $[\mathbf{0},\mathbf{1}] \to \mathbb{R}$ is continuous function, then prove:

$$\boldsymbol{\int_0^{\pi} x\cdot h(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} h(\sin x)\,dx}$$

Solution: Let $f(x) = x\cdot f(\sin x) - \frac{\pi}{2}h(\sin x)$, then $f(x)+f(\pi-x) = 0$. Using Theorem 2, we get:

$\int_m^n h(x)\,dx = 0; m+n = \pi$. For $m = 0, n = \pi$, we get the problem.

Application 2.3 Prove that:

$$\boldsymbol{\int_0^1 \frac{dx}{\sqrt{x^4-4x^3+6x^2-4x+2}} = \int_0^1 \frac{dx}{\sqrt{1+x^4}}}$$

Solution: Let $f: [0,1] \to \mathbb{R}, f(x) = \frac{1}{\sqrt{x^4-4x^3+6x^2-4x+2}} - \frac{1}{\sqrt{1+x^4}}$, then $f(x)+f(1-x) = 0$. Using Theorem 2, we get:

$\int_m^n f(x)\,dx = 0; m+n = 1$. For $m = 0, n = 1$, we get the problem.

3. Extension result.

Proposition 3.1 If $\boldsymbol{f}$: $[\boldsymbol{a-\theta, a+\theta}] \to \mathbb{R}$ is continuous function with property

$$\boldsymbol{\alpha f(a+x)+\beta f(a-x) = \gamma, \forall x \in [-\theta,\theta]; \alpha,\beta \in \mathbb{R}^*, \gamma \in \mathbb{R}}\textbf{, then:}$$

$$\textbf{(i)} \boldsymbol{\int_{a-\theta}^{a+\theta} f(x)\,dx = \frac{2\gamma}{\alpha+\beta}\cdot\theta; \alpha+\beta \neq 0}$$

$$\textbf{(ii)} \boldsymbol{\int_{a-\theta}^{a+\theta} f(x)\,dx = \frac{\gamma}{\alpha}\cdot\theta + \frac{\alpha-\beta}{\alpha}\int_a^{a+\theta} f(x)\,dx}$$

Proof: If $f: [a-\theta, a+\theta] \to \mathbb{R}$ is continuous function and $\varphi,\psi: [-\theta,\theta] \to [a-\theta, a+\theta]$,

$\psi(t)=a+t; \varphi(t)=a-t$, then

$$\text{(i)} \int_{a-\theta}^{a+\theta} f(x)\,dx=\int_{-\varphi(-\theta)}^{\varphi(\theta)} f(x)\,dx=\int_{-\theta}^{\theta} f(\varphi(t))\,\varphi'(t)dt=\int_{-\theta}^{\theta} f(a+t)\,dt=$$

$$=\int_{-\theta}^{\theta}\left(\frac{\gamma}{\alpha}-\frac{\beta}{\alpha} f(a-t)\right)dt=\frac{2\gamma}{\alpha}\cdot\theta-\frac{\beta}{\alpha}\int_{-\theta}^{\theta} f(a-t)\,dt=$$

$$=\frac{2\gamma}{\alpha}\cdot\theta+\frac{\beta}{\alpha}\int_{-\theta}^{\theta} f(\psi(t))\,\psi'(t)dt=\frac{2\gamma}{\alpha}\cdot\theta+\frac{\beta}{\alpha}\int_{a-\theta}^{a+\theta} f(x)\,dx$$

$$\int_{a-\theta}^{a+\theta} f(x)\,dx+\frac{\beta}{\alpha}\int_{a-\theta}^{a+\theta} f(x)\,dx=\frac{2\gamma}{\beta}\cdot\theta \Rightarrow \int_{a-\theta}^{a+\theta} f(x)\,dx=\frac{2\gamma}{\alpha+\beta}\cdot\theta; \alpha+\beta\neq 0$$

$$\text{(ii} \int_{a-\theta}^{a+\theta} f(x)\,dx=\int_{a-\theta}^{a} f(x)\,dx+\int_{a}^{a+\theta} f(x)\,dx$$

$$\int_{a-\theta}^{a} f(x)\,dx=\int_{\varphi(-\theta)}^{\varphi(0)} f(x)\,dx=\int_{-\theta}^{0} f(\varphi(t))\varphi'(t)\,dt=\int_{-\theta}^{0} f(a+t)\,dt=$$

$$=\int_{-\theta}^{0}\left(\frac{\gamma}{\alpha}-\frac{\beta}{\alpha} f(a-t)\right)dt=\frac{\gamma}{\alpha}\cdot\theta-\frac{\beta}{\alpha}\int_{-\theta}^{0} f(a-t)\,dt=$$

$$=\frac{\gamma}{\alpha}\cdot\theta+\frac{\beta}{\alpha}\int_{\psi(-\theta)}^{\psi(0)} f(\psi(t))\psi'(t)\,dt=\frac{\gamma}{\alpha}\cdot\theta+\frac{\beta}{\alpha}\int_{a+\theta}^{a} f(x)\,dx$$

So, we have:

$$\int_{a-\theta}^{a+\theta} f(x)\,dx=\frac{\gamma}{\alpha}\cdot\theta+\frac{\beta}{\alpha}\int_{a+\theta}^{a} f(x)\,dx+\int_{a}^{a+\theta} f(x)\,dx=\frac{\gamma}{\alpha}\cdot\theta+\frac{\alpha-\beta}{\alpha}\int_{a}^{a+\theta} f(x)\,dx$$

Definition. Function $\boldsymbol{f}:[\boldsymbol{a}-\boldsymbol{\theta},\boldsymbol{a}+\boldsymbol{\theta}]\to\mathbb{R}$ is $\boldsymbol{a}$ −even function, ($\boldsymbol{a}$ −odd function) if $\boldsymbol{f}(\boldsymbol{a}+\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{a}-\boldsymbol{x}); \forall \boldsymbol{x}\leq|\boldsymbol{\theta}|, (\boldsymbol{f}(\boldsymbol{a}+\boldsymbol{x})=-\boldsymbol{f}(\boldsymbol{a}-\boldsymbol{x}); \forall|\boldsymbol{x}|\leq\boldsymbol{\theta})$.

Application 3.1 Find:

$$\boldsymbol{\Omega_n}=\int_{\mathbf{0}}^{\mathbf{1}}\frac{\mathbf{4}\boldsymbol{x}^{\mathbf{3}}-\mathbf{6}\boldsymbol{x}^{\mathbf{2}}+\mathbf{8}\boldsymbol{x}-\mathbf{3}}{(\boldsymbol{x}^{\mathbf{2}}-\boldsymbol{x}+\mathbf{1})^{\boldsymbol{n}}}\boldsymbol{dx}; \boldsymbol{n}\in\mathbb{N}$$

Solution: $g(x)=x^2-x+1$ is $\frac{1}{2}-even$ and $h(x)=4x^3-6x^2+8x-3$ is $\frac{1}{2}-odd$, so

$f(x)=\frac{h(x)}{g^n(x)}$ is $\frac{1}{2}-odd$ and using Theorem 3, we get:

$$\Omega_n=\int_0^1\frac{4x^3-6x^2+8x-3}{(x^2-x+1)^n}dx=0.$$

Application 3.2 Find:

$$\boldsymbol{\Omega_n = \int_0^1 (2x-1)^{2n+1} e^{x-x^2}\, dx; n \in \mathbb{N}}$$

Solution: $g(x) = (2x-1)^{2n+1}$ is $\frac{1}{2} - odd$ function and $h(x) = e^{x-x^2}$ is $\frac{1}{2} - even$ function, then $f(x) = g(x) \cdot h(x)$ is $\frac{1}{2} - odd$ function. Using Theorem 3, we get:

$$\Omega_n = \int_0^1 (2x-1)^{2n+1} e^{x-x^2}\, dx = 0.$$

Corollary 3.1. : For any function $\boldsymbol{f}: [\boldsymbol{a}-\boldsymbol{\theta}, \boldsymbol{a}+\boldsymbol{\theta}] \to \mathbb{R}$ exist an function $\boldsymbol{f_1}, \boldsymbol{a} - \boldsymbol{even}$ and $\boldsymbol{f_2}, \boldsymbol{a} - \boldsymbol{odd}$ such that $\boldsymbol{f(x) = f_1(x) + f_2(x)}; \forall \boldsymbol{x} \in [\boldsymbol{a}-\boldsymbol{\theta}, \boldsymbol{a}+\boldsymbol{\theta}]$.

Corollary 3.2 : If $\boldsymbol{f, g}: [\boldsymbol{a}-\boldsymbol{\theta}, \boldsymbol{a}+\boldsymbol{\theta}] \to \mathbb{R}$ integrable functions and $\boldsymbol{f}$ is $\boldsymbol{a} - \boldsymbol{odd}$, then

$$\boldsymbol{\int_{a-\theta}^{a+\theta} f(x)g(x)\, dx = \int_a^{a+\theta} f(x)\big(g(x) + g(2a-x)\big)\, dx}$$

Application 3.3 Let $\boldsymbol{f}: [-\boldsymbol{1}, \boldsymbol{1}] \to \mathbb{R}$ continuous with property $\boldsymbol{f(x) + f(-x) = \pi}$;

$\forall \boldsymbol{x} \in [-\boldsymbol{1}, \boldsymbol{1}]$. Find:

$$\boldsymbol{\Omega_n = \int_0^{(2n+1)\pi} f(\cos x)\, dx; \forall n \in \mathbb{N}}$$

Solution: We have:

$$\Omega_n = \Omega_{n-1} + \int_{(2n-1)\pi}^{(2n+1)\pi} f(\cos x)\, dx,\ \ g(x) = f(\cos x) \text{ is } 2n\pi - odd, \text{ then:}$$

$$I = \int_{(2n-1)\pi}^{(2n+1)\pi} f(\cos x)\, dx = 2\int_{2n\pi}^{2n\pi+\pi} f(\cos x)\, dx = 2\int_0^{\pi} f(\cos(t+2n\pi))\, dt =$$

$$= 2\int_0^{\pi} f(\cos t)\, dt = -2\int_1^{-1} \frac{f(u)}{\sqrt{1-u^2}} du = 2\int_{-1}^{1} \frac{f(u)}{\sqrt{1-u^2}} du$$

$$g(u) = \frac{1}{\sqrt{1-u^2}} \text{ is } 0 - even \Rightarrow I = 2\int_{-1}^{1} \frac{f(u)}{\sqrt{1-u^2}} du = 2\int_0^1 \frac{f(u)+f(-u)}{\sqrt{1-u^2}} du = 2\pi \int_0^1 \frac{du}{\sqrt{1-u^2}} = \pi^2.$$

$$\text{So}, \Omega_n = \Omega_{n-1} + \pi^2 \Rightarrow \Omega_n = (n+1)\pi^2$$

Application 3.4 Find:

$$\boldsymbol{\Omega_n = \int_0^{\frac{\pi}{4}} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx}$$

Solution: $f(x) = \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}$ is $\frac{\pi}{2} - even$, then using Theorem 3, we get:

$$\Omega_n=\int_0^{\frac{\pi}{4}}\frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx=\int_0^{\pi}xf(x)\,dx=\int_{\frac{\pi}{2}}^{\pi}f(x)(x+\pi-x)\,dx=$$

$$\left(\because g(x)=1\text{ is}\frac{3\pi}{4}-even\text{ and }f\left(\frac{3\pi}{2}-x\right)=\frac{\cos^{2n}x}{\sin^{2n}x+\cos^{2n}x}\right)$$

$$=\pi\int_{\frac{\pi}{2}}^{\pi}f(x)\,dx=\pi\int_{\frac{3\pi}{4}}^{\pi}\left(f(x)+f\left(\frac{3\pi}{2}-x\right)\right)dx=\pi\int_{\frac{3\pi}{4}}^{\pi}dx=\frac{\pi^2}{4}$$

Application 3.5 Find:

$$\boldsymbol{\Omega=\int_0^{\frac{\pi}{4}}\frac{\log(1+\tan x)}{\sin 2x+\cos 2x}dx}$$

Solution: $f(x)=\frac{1}{\sin 2x+\cos 2x}=\frac{1}{\sqrt{2}\cos\left(2x-\frac{\pi}{4}\right)}\Rightarrow f\text{ is}\frac{\pi}{8}-even.$

$$\Omega=\int_0^{\frac{\pi}{4}}\frac{\log(1+\tan x)}{\sin 2x+\cos 2x}dx=\int_0^{\frac{\pi}{4}}f(x)\log(1+\tan x)\,dx=$$

$$=\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}f(x)\left(\log(1+\tan x)+\log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)\right)dx=$$

$$=\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}f(x)\left(\log(1+\tan x)+\log\left(\frac{2}{1+\tan x}\right)\right)dx=\frac{\log 2}{\sqrt{2}}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}\frac{dx}{\cos\left(2x-\frac{\pi}{4}\right)}$$

Proposition 3.2 Let $a,b\in\mathbb{R},a<b$ and $f:[0,b-a]\to\mathbb{R}$ continuous function, then

$$\boldsymbol{\int_a^b\frac{f(x-a)}{f(x-a)+f(b-x)}dx=\frac{b-a}{2}}$$

Proof: We have:

$$\Omega=\int_a^b\frac{f(x-a)}{f(x-a)+f(b-x)}dx=$$

$$=\int_a^{\frac{a+b}{2}}\frac{f(x-a)}{f(x-a)+f(b-x)}dx+\int_{\frac{a+b}{2}}^{b}\frac{f(x-a)}{f(x-a)+f(b-x)}dx=$$

$$=\int_a^{\frac{a+b}{2}}\frac{f(x-a)}{f(x-a)+f(b-x)}dx-\int_{\frac{a+b}{2}}^{b}\frac{f(b-t)}{f(b-t)+f(t-a)}dt=\int_a^{\frac{a+b}{2}}\frac{f(x-a)+f(b-x)}{f(x-a)+f(b-x)}dx=\frac{b-a}{2}$$

Application 3.6 Find:

$$\boldsymbol{\Omega=\int_n^{n+1}\frac{\tan^{-1}(x-n)}{\tan^{-1}(x-n)+\tan^{-1}(n+1-x)}dx,n\in\mathbb{N}}$$

Solution: Using Proposition 3.2, we have:

$$\Omega=\int_n^{n+1}\frac{\tan^{-1}(x-n)}{\tan^{-1}(x-n)+\tan^{-1}(n+1-x)}dx=\frac{1}{2}$$

Application 3.7 For $\boldsymbol{n}\in\mathbb{N},\boldsymbol{n}>1$ find:

$$\boldsymbol{\Omega=\int_{n-1}^{n+1}\frac{\tan^{-1}(x-n+1)}{\tan^{-1}\left(\frac{2}{(x-n)^2}\right)}dx}$$

Solution: Using the relation: $\tan^{-1}\alpha+\tan^{-1}\beta=\tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$

we have: $\tan^{-1}\big(x-(n-1)\big)+\tan^{-1}\big((n+1)-x\big)=\tan^{-1}\left(\frac{2}{(x-n)^2}\right)$

$$\Omega=\int_{n-1}^{n+1}\frac{\tan^{-1}(x-n+1)}{\tan^{-1}\left(\frac{2}{(x-n)^2}\right)}dx\overset{(t=2n-x)}{=}\int_{n-1}^{n+1}\frac{\tan^{-1}(n+1-t)}{\tan^{-1}\left(\frac{2}{(x-n)^2}\right)}dt$$

Hence: $2\Omega=\Omega=\int_{n-1}^{n+1}\frac{\tan^{-1}(x-n+1)+\tan^{-1}(n+1-x)}{\tan^{-1}\left(\frac{2}{(n-1)^2}\right)}dx=2$

4. Improper integrals with parameter.

Application 1.4 Find:

$$\boldsymbol{\Omega=\int_0^{\frac{\pi}{2}}\log\left(\frac{1-t\,cos\,x}{1+t\,cos\,x}\right)\frac{dx}{\cos x}};\ |\boldsymbol{t}|<1$$

Solution: Let us denote:

$$F(t)=\int_0^{\frac{\pi}{2}}\log\left(\frac{1-t\,cos\,x}{1+t\,cos\,x}\right)\frac{dx}{\cos x};|t|<1;F(0)=0$$

$$F'(t)=\int_0^{\frac{\pi}{2}}\frac{2}{t^2\cos^2x-1}dx=-2\int_0^{\infty}\frac{1}{1-t^2-u^2}du=-\frac{2}{\sqrt{1-t^2}}\cdot\frac{\pi}{2}$$

$$F(t)=-\pi\sin^{-1}t+C,\text{but }F(0)=0,\text{then }C=0.$$

Application 4.2 Find:

$$\boldsymbol{\Omega=\int_0^{\frac{\pi}{2}}\log\left(\frac{a+b\sin x}{a-b\sin x}\right)\frac{dx}{\sin x}};\ \boldsymbol{0\le b}<a$$

Solution: $\Omega=\int_0^{\frac{\pi}{2}}\log\left(\frac{a+b\,sin\,x}{a-b\,sin\,x}\right)\frac{dx}{\sin x}=-\int_0^{\frac{\pi}{2}}\log\left(\frac{a-b\,sin\,x}{a+b\,sin\,x}\right)\frac{dx}{\sin x}=$

$$=-\int_0^{\frac{\pi}{2}}\log\left(\frac{1-\frac{b}{a}cos\left(\frac{\pi}{2}-x\right)}{1+\frac{b}{a}cos\left(\frac{\pi}{2}-x\right)}\right)\frac{dx}{\cos\left(\frac{\pi}{2}-x\right)}\overset{\left(\frac{\pi}{2}-x\to x;\frac{a}{b}=t\right)}{=}$$

$$= -\int_0^{\frac{\pi}{2}} \log\left(\frac{1 - t\,cos\,x}{1 + t\,cos\,x}\right)\frac{dx}{\cos x} = \pi\sin^{-1} t = \pi\sin^{-1}\left(\frac{b}{a}\right)$$

Application 4.3 Find:

$$\boldsymbol{\Omega = \int_0^{\pi} \log(1 + t\,cos\,x)\frac{dx}{\cos x}};\ |\boldsymbol{t}| < 1$$

Solution: We have:

$$\Omega = -\int_0^{\pi} \log(1 + t\,cos\,x)\frac{dx}{\cos x} = \int_0^{\frac{\pi}{2}} \log(1 + t\cos x)\,dx + \int_{\frac{\pi}{2}}^{\pi} \log(1 + t\,cos\,x)\frac{dx}{\cos x} =$$

$$= \int_0^{\frac{\pi}{2}} \log(1 + t\,cos\,x)\frac{dx}{\cos x} - \int_{\frac{\pi}{2}}^{\pi} \log(1 - t\,cos(\pi - x))\ \frac{dx}{\cos(\pi - x)} = \int_0^{\frac{\pi}{2}} \log\left(\frac{1 + t\,cos\,x}{1 - t\,cos\,x}\right)\frac{dx}{\cos x} = \pi\sin^{-1} t$$

Application 4.4 Find:

$$\boldsymbol{\Omega = \int_0^{\frac{\pi}{2}} \frac{\tan^{-1}(t\tan x)}{\tan x}\,dx},\ |\boldsymbol{t}| < 1$$

Solution: Let be the function:

$$F(t) = \int_0^{\pi} \log(1 - 2t\cos x + t^2)\,dx\,, F(0) = 0$$

$$F'(t) = \int_0^{\pi} \frac{2(t - \cos x)}{1 - 2t\cos x + t^2}dx = 4\int_0^{\infty} \frac{t - 1 + (t+1)u^2}{(1+u^2)[(1-t)^2 + (1+t^2)u^2]}du =$$

$$= 2\int_0^{\infty}\left(\frac{1}{t(1+u^2)} + \frac{t^2 - 1}{t}\cdot\frac{1}{(t-1)^2 + (t+1)^2u^2}\right)du =$$

$$= -\frac{2}{t}\left[\frac{\pi}{2} - \lim_{v\to\infty}\tan^{-1}\left(\frac{1+t}{1-t}u\right)\Big|_0^v\right] = 0 \Rightarrow F(t) = 0.$$

Application 4.6 Let $\boldsymbol{f}: \left[\frac{1}{a}, a\right] \to \mathbb{R}, \boldsymbol{a} > 1$ continuous function. Find:

$$\boldsymbol{\Omega = \int_{\frac{1}{a}}^{a} f\left(\frac{x^{2\lambda} + 1}{x^{\lambda}}\right)\frac{\log x}{x}\,dx};\ \boldsymbol{\lambda} \in \mathbb{R}$$

Solution: We have:

$$\Omega = \int_{\frac{1}{a}}^{a} f\left(\frac{x^{2\lambda} + 1}{x^{\lambda}}\right)\frac{\log x}{x}\,dx = \int_{\frac{1}{a}}^{1} f\left(\frac{x^{2\lambda} + 1}{x^{\lambda}}\right)\frac{\log x}{x}\,dx + \int_1^{a} f\left(\frac{x^{2\lambda} + 1}{x^{\lambda}}\right)\frac{\log x}{x}\,dx =$$

$$= I_1 + I_2, \text{where } I_1 = \int_{\frac{1}{a}}^{1} f\left(\frac{x^{2\lambda}+1}{x^{\lambda}}\right)\frac{\log x}{x}\,dx \overset{\left(x\to\frac{1}{x}\right)}{=} -I_2, \text{thus } \Omega = 0.$$

Application 4.7 Find:

$$\boldsymbol{\Omega = \int_0^{\frac{\pi}{2}} \log(\sin x)\, dx}$$

Solution:

$$\Omega = \int_0^{\frac{\pi}{2}} \log(\sin x)\, dx = \int_0^{\frac{\pi}{2}} \log\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right) dx =$$

$$= \frac{\pi}{2}\log 2 + \int_0^{\frac{\pi}{2}} \log\left(\sin\frac{x}{2}\right) dx + \int_0^{\frac{\pi}{2}} \log\left(\cos\frac{x}{2}\right) dx = \frac{\pi}{2}\log 2 + 2\int_0^{\frac{\pi}{4}} \log(\sin y)\, dy + 2\int_0^{\frac{\pi}{4}} \log(\cos y)\, dy =$$

$$= \frac{\pi}{2}\log 2 + 2\int_0^{\frac{\pi}{4}} \log(\sin y)\, dy + 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \log(\sin y)\, dy = \frac{\pi}{2}\log 2 + 2\Omega, \text{therefore } \Omega = -\frac{\pi}{2}\log 2$$

Application 4.8 Find:

$$\boldsymbol{\Omega = \int_0^1 \frac{\log(1 - t^2x^2)}{\sqrt{1 - x^2}}\, dx}, \qquad |t| < 1$$

Solution: Let us denote:

$$F(t) = \int_0^1 \frac{\log(1 - t^2x^2)}{\sqrt{1 - x^2}}\, dx, |t| < 1; F(0) = 0$$

$$F'(t) = \int_0^1 \frac{-2tx^2}{(1 - t^2x^2)\sqrt{1 - x^2}}\, dx = -2t\int_0^{\frac{\pi}{2}} \frac{\sin^2 u}{1 - t^2\sin^2 u}\, du =$$

$$= -2t\int_0^{\infty} \frac{v^2}{(1 + (1 - t^2)v^2)(1 + v^2)}\, dv = \frac{2}{t}\int_0^{\infty} \left(\frac{1}{1 + v^2} - \frac{1}{1 + (1 - t^2)v^2}\right) dv =$$

$$= \pi\left(\frac{1}{t} - \frac{1}{t\sqrt{1 - t^2}}\right), \qquad F(t) = \pi\log\left(\frac{1 + \sqrt{1 - t^2}}{2}\right)$$

Application 4.9 Find:

$$\boldsymbol{\Omega = \int_0^{2\pi} \log(1 - t\cos x)\, dx}; |t| < 1$$

Solution: Let be the function:

$$F(t) = \int_0^{2\pi} \log(1 - t\cos x)\, dx = 2\int_0^{\pi} \log(1 - t\cos x)\, dx; |t| < 1, F(0) = 0$$

$$F'(t_= - 2\int_0^{\pi} \frac{\cos x}{1 - t\cos x}\, dx = -4\int_0^{\infty} \frac{1 - u^2}{(1 + u^2)(1 - t + (1 + t)u^2)}\, du =$$

$$=\frac{4}{t}\int_0^\infty\left(\frac{1}{1+u^2}-\frac{1}{1-t+(1+t)u^2}\right)du=2\pi\left(\frac{1}{t}-\frac{1}{t\sqrt{1-t^2}}\right)$$

$$F(t)=2\pi\log\left(\frac{1+\sqrt{1-t^2}}{2}\right)$$

Application 4.10 Find:

$$\boldsymbol{\Omega}=\int_{\mathbf{0}}^{\frac{\pi}{2}}\mathbf{log}\left(\boldsymbol{t}^{\mathbf{2}}-\mathbf{sin}^{\mathbf{2}}\,\boldsymbol{x}\right)\boldsymbol{dx},|\boldsymbol{t}|>1$$

Solution: Let be the function:

$$F(t)=\ \Omega=\int_0^{\frac{\pi}{2}}\log(t^2-\sin^2 x)\,dx=\frac{\pi}{2}\log t+\int_0^{\frac{\pi}{2}}\log\left(1-\frac{sin^2 x}{t^2}\right)dx=$$

$$=\frac{\pi}{2}\log t+\int_0^1\frac{\log\left(1-\frac{y^2}{t^2}\right)}{\sqrt{1-y^2}}dy=\pi\log\left(\frac{t+\sqrt{t^2-1}}{2t}\right)$$

REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT THE PROBLEM 4705 FROM CRUX MATHEMATICORUM

By Florică Anastase-Romania

4705. Find the following limit:

$$\boldsymbol{\Omega}=\lim_{n\to\infty}\frac{\mathbf{1}}{\sqrt[3]{\boldsymbol{n}^{\mathbf{2}}}}\cdot\sum_{k=1}^{n}\frac{\mathbf{1}}{\sqrt[3]{\boldsymbol{k}}}$$

Proposed by Nguyen Viet Hung-Hanoi, Vietnam

First solution: It is well-know the double inequality:

$$\frac{3}{2}\left(\sqrt[3]{(n+1)^2}-1\right)\le\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}\le\frac{3}{2}\sqrt[3]{(n+1)^2};\quad(*)$$

Let $(u_n)_{n\ge1},(v_n)_{n\ge1}$ be sequences of real numbers defined by

$$u_n=\sum_{k=1}^n\frac{1}{\sqrt[3]{k}}\ge\frac{3}{2}\left(\sqrt[3]{(n+1)^2}-1\right)\xrightarrow{(n\to\infty)}\infty,\ v_n=\sqrt[3]{n^2}\xrightarrow{(n\to\infty)}\infty$$

Using Lemma Stolz-Cesaro, we get:

$$\Omega=\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}=\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{u_{n+1}-u_n}{v_{n+1}-v_n}=$$

$$=\lim_{n\to\infty}\frac{\frac{1}{\sqrt[3]{n+1}}}{\sqrt[3]{(n+1)^2}-\sqrt[3]{n^2}}=\lim_{n\to\infty}\frac{\sqrt[3]{(n+1)^4}+\sqrt[3]{n^2(n+1)^2}+\sqrt[3]{n^4}}{((n+1)^2-n^2)\sqrt[3]{n+1}}=$$

$$=\lim_{n\to\infty}\frac{n\sqrt[3]{n}\left[\sqrt[3]{\left(1+\frac{1}{n}\right)^4}+\sqrt[3]{\left(1+\frac{1}{n}\right)^2}+1\right]}{(2n+1)\sqrt[3]{n+1}}=\frac{3}{2}$$

Second solution: Let be the function $f:(0,\infty)\to(0,\infty), f(x)=\frac{1}{\sqrt[3]{x}}$ continuous and decreasing function with the primitive $F:(0,\infty)\to(0,\infty), F(x)=\frac{3}{2}\sqrt[3]{x}$. Using Mean Value Theorem for the function F on the interval $[k,k+1], k\in\mathbb{N}, k\geq 1$ exist $c_k\in(k,k+1)$ such that:

$$\frac{F(k+1)-F(k)}{k+1-k}=F'(c_k)\Leftrightarrow F(k+1)-F(k)=f(c_k), c_k\in(k,k+1)$$

How f −continuous and decreasing function and $c_k\in(k,k+1), k\geq 1$, we have:

$$f(k+1)\leq F(k+1)-F(k)\leq f(k), \forall c_k\in(k,k+1)\Leftrightarrow$$

$$\frac{1}{\sqrt[3]{k+1}}\leq\frac{3}{2}\left[\sqrt[3]{(k+1)^2}-\sqrt[3]{k^2}\right]\leq\frac{1}{\sqrt[3]{k}}$$

By adding for $k\in\{1,2,\dots,n\}$, we get:

$$\frac{3}{2}\sum_{k=1}^{n}\left[\sqrt[3]{(k+1)^2}-\sqrt[3]{k^2}\right]\leq\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}},\quad \frac{3}{2}\left(\sqrt[3]{(n+1)^2}-1\right)\leq\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}};\quad (1)$$

By adding for $k\in\{1,2,\dots,n\}$, we get:

$$\frac{3}{2}\sum_{k=1}^{n}\left[\sqrt[3]{(k+1)^2}-\sqrt[3]{k^2}\right]\geq\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k+1}},\qquad \frac{3}{2}\sqrt[3]{(n+1)^2}\geq\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}};\quad (2)$$

From (1) and (2), it follows that:

$$\frac{3}{2}\left(\sqrt[3]{(n+1)^2}-1\right)\leq\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}\leq\frac{3}{2}\sqrt[3]{(n+1)^2};\quad (*)$$

$$\frac{3}{2}\left(\sqrt[3]{(n+1)^2}-1\right)\cdot\frac{1}{\sqrt[3]{n^2}}\leq\frac{1}{\sqrt[3]{n^2}}.\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}\leq\frac{3}{2}\sqrt[3]{(n+1)^2}\cdot\frac{1}{\sqrt[3]{n^2}}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{1}{\sqrt[3]{n^2}}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}}=\frac{3}{2}$$

Third solution: It is well-know the double inequality:

$$\frac{3}{2}\left(\sqrt[3]{(n+1)^2}-1\right) \le \sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}} \le \frac{3}{2}\sqrt[3]{(n+1)^2}; \quad (*)$$

Now, let be the sequence $(a_n)_{n\ge1}$ of real numbers defined by

$$a_n = \sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}} - \frac{3}{2}\sqrt[3]{n^2}$$

First, we want to show that the sequence $(a_n)_{n\ge1}$ converges. We have:

$$a_{n+1}-a_n = \sum_{k=1}^{n+1} \frac{1}{\sqrt[3]{k}} - \frac{3}{2}\sqrt[3]{(n+1)^2} - \sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}} + \frac{3}{2}\sqrt[3]{n^2}$$

$$= \frac{1}{\sqrt[3]{n+1}} - \frac{3}{2}\left(\sqrt[3]{(n+1)^2}-\sqrt[3]{n^2}\right) \overset{(*)}{\le} \frac{1}{\sqrt[3]{n+1}} - \frac{1}{\sqrt[3]{n+1}} = 0$$

How $a_{n+1}-a_n \le 0, \forall n \ge 1$, the sequence $(a_n)_{n\ge1}$ is decreasing. Now, we have:

$$a_n = \sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}} - \frac{3}{2}\sqrt[3]{n^2} \overset{(*)}{\ge} \frac{3}{2}\left(\sqrt[3]{(n+1)^2}-\sqrt[3]{n^2}-1\right) \ge \frac{1}{\sqrt[3]{n+1}} - \frac{3}{2} \ge -\frac{3}{2}$$

So, $(a_n)_{n\ge1}$ converges and we have:

$$\Omega = \lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2}} \cdot \sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}} = \lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2}} \cdot \left(\sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}} - \frac{3}{2}\sqrt[3]{n^2} + \frac{3}{2}\sqrt[3]{n^2}\right) = \lim_{n\to\infty} \frac{a_n}{\sqrt[3]{n^2}} + \frac{3}{2} = \frac{3}{2}$$

Fourth solution and generalization

Lemma. Let $f:[1,\infty) \to [0,\infty)$continuous and decreasing function and let $(x_n)_{n\ge1}, (y_n)_{n\ge1}$ be sequences of real numbers such that:

$$x_n = \sum_{k=1}^{n} f(k), \quad y_n = \int_1^n f(x)\,dx, \forall n \ge 1$$

Then $(z_n)_{n\ge1}$, $z_n = x_n - y_n$ converge and if $y_n \overset{n\to\infty}{\longrightarrow} +\infty$, thus $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = 1$.

Proof: $z_n = x_n - y_n, \forall n \ge 1,$ then $z_{n+1} - z_n = f(n+1) - \int_n^{n+1} f(x)\,dx$

Now, for all functions $g:[a,b] \to \mathbb{R}$ continuous, holds:

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

where, $m = \inf\limits_{x\in[a,b]} f(x)$ and $M = \sup\limits_{x\in[a,b]} f(x)$.

How the function f is decreasing, we have:

$$f(n+1) \le f(x) \le f(n), \forall x \in [n, n+1], \quad f(n+1) \le \int_n^{n+1} f(x)\,dx \le f(n), \forall n \ge 1$$

Thus, $z_{n+1} - z_n \le 0, \forall n \ge 1$ and the sequence $(z_n)_{n\ge1}$ is decreasing. Remains to prove that the sequence $(z_n)_{n\ge1}$ is inferior bounded. Using: $f(n+1) \le \int_n^{n+1} f(x)\,dx \le f(n), \forall n \ge 1$

$$z_n = f(1) + f(2) + \cdots + f(n) - \int_1^2 f(x)\,dx - \int_2^3 f(x)\,dx - \cdots - \int_{n-1}^n f(x)\,dx =$$

$$= \left(f(1) - \int_1^2 f(x)\,dx\right) + \left(f(2) - \int_2^3 f(x)\,dx\right) + \cdots + \left(f(n-1) - \int_{n-1}^n f(x)\,dx\right) + f(n) \ge 0, \forall n \ge 2$$

Therefore, $(z_n)_{n\ge1}$ converges and if $y_n \to \infty$ when $n \to \infty$, we get $\frac{z_n}{y_n} \to 0$ or $\frac{x_n}{y_n} \to 1$.

In the above problem, taking $f: [1,\infty) \to [0,\infty), f(x) = \frac{1}{\sqrt[3]{x}}$ continuous and decreasing, and

$y_n = \int_1^n f(x)\,dx, y_n \to \infty$ we have: $\Omega = \lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2}} \cdot \sum_{k=1}^n \frac{1}{\sqrt[3]{k}} = \frac{3}{2} \cdot \lim_{n\to\infty} \frac{x_n}{y_n} = \frac{3}{2}$

REFERENCES: [1]. ROMANIAN MATHEMATICAL MAGAZINE JOURNAL-ww.ssmrmh.ro

[2]. CRUX MATHEMATICORUM JOURNAL-CANADA

GAKOPOULOS' LEMMAS AND THEOREMS

By Thanasis Gakopoulos-Farsala-Greece

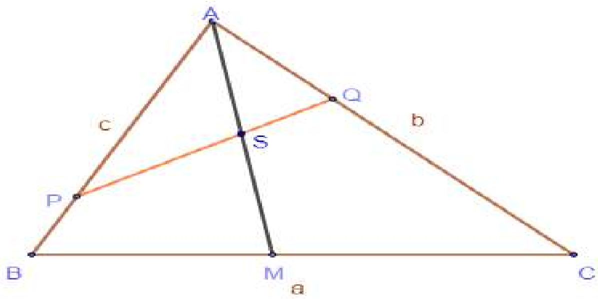
Introduction. *I have compiled into an article some basic exercises then I have created, proved and published in the past.* These exercises have been used by me and other team members as auxiliary exercises-lemmas-theorems to solve geometric problems. I think it is useful.

1. GAKOPOULOS' LEMMA or THEOREM:

$$\frac{PS}{SQ} = \frac{BM}{MC} \cdot \frac{AP}{AB} \cdot \frac{AC}{AQ}$$

Or

$$\frac{PS}{SQ} \cdot \frac{QA}{AP} = \frac{BM}{MC} \cdot \frac{CA}{AB}$$

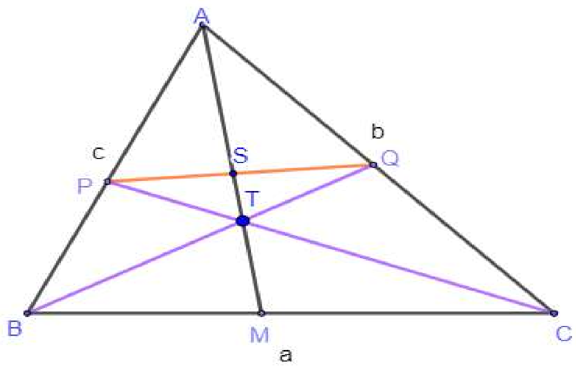


Proof: $*\ \Delta PQR, \overline{AST}$ – (Menelaus th.): $\frac{SP}{SQ} \cdot \frac{TR}{TP} \cdot \frac{AQ}{AR} = 1; \quad (1)$

$$*\ PR \parallel BC \Rightarrow \frac{AP}{AR} = \frac{AB}{AC} \Rightarrow AR = AP \cdot AB \cdot AC; \quad (2)$$

From (1)and (2) $\Rightarrow \frac{SP}{SQ}\cdot\frac{MC}{MB}\cdot\frac{AQ}{\frac{AP}{AB}\cdot AC}=1\Rightarrow\frac{PS}{PQ}=\frac{BM}{MC}\cdot\frac{AP}{AB}\cdot\frac{AC}{AQ}$ or $\frac{PS}{SQ}\cdot\frac{QA}{AC}=\frac{BM}{MC}\cdot\frac{CA}{AB}$

Application 1: $\frac{PS}{SQ}=\frac{PB}{QC}\cdot\frac{AC}{AB}$

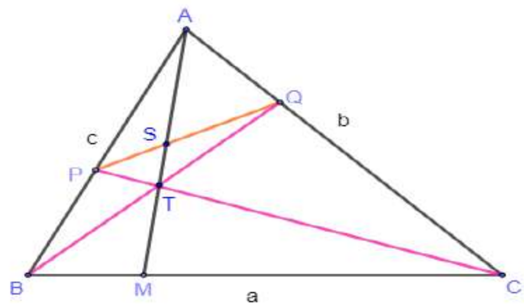


Proof:

$$\begin{cases}\frac{PS}{SQ}=\frac{BM}{MC}\cdot\frac{AP}{AB}\cdot\frac{AC}{AQ}\ (Gakopoulos\ th.)\\ \frac{BM}{MC}\cdot\frac{CQ}{QA}\cdot\frac{AP}{PB}=1\ (Ceva\ th.)\end{cases}$$

$$\frac{PS}{SQ}=\frac{AQ}{CQ}\cdot\frac{PB}{AP}\cdot\frac{AP}{AB}\cdot\frac{AC}{AQ}\text{ or }\frac{PS}{SQ}=\frac{PB}{QC}\cdot\frac{AC}{AB}$$

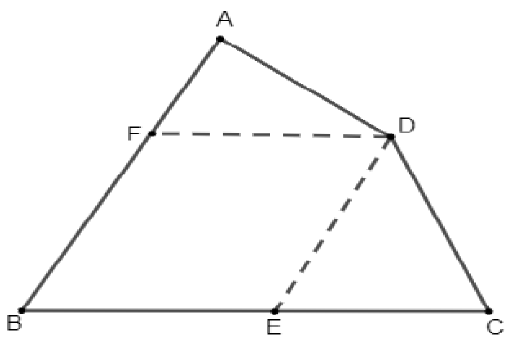
Application 2: $AB=AC;\frac{PS}{SQ}=\frac{PB}{QC}$ **Proof:** $\frac{PS}{SQ}=\frac{PB}{QC}\cdot\frac{AC}{AB}\overset{AB=AC}{\Longrightarrow}\frac{PS}{SQ}=\frac{PB}{QC}$

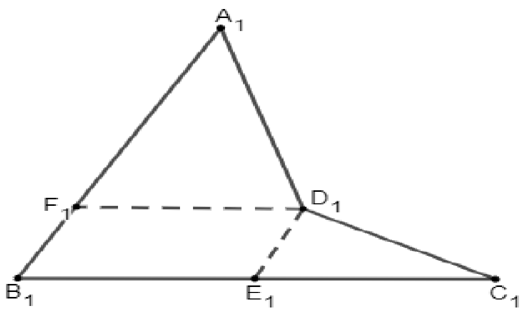


2. GAKOPOULOS-BLATSIS formulae:

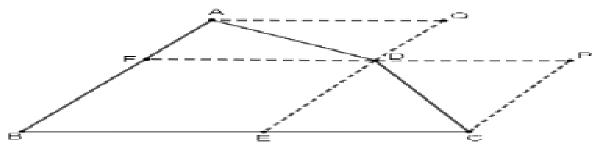
$DE\parallel AB, DF\parallel BC$

$D_1E_1\parallel A_1B_1, D_1F_1\parallel B_1CC_1$





$[ABCD]=\frac{\sin B}{2}(BC\cdot BF+BA+BE)$; $[A_1B_1C_1D_1]=\frac{\sin B}{2}(B_1C_1\cdot B_1F_1+B_1A_1+B_1E_1)$



Proof.

$$2[ABCD]=2[BFDE]+2[ECD]+2[ADF]=[BFDE]+[DPCE]+[BFDE]+[AQDF]$$

$$=[BCPF]+[ABEQ]=\frac{\sin B}{2}(BC\cdot BF)+\frac{\sin B}{2}(BA\cdot BE)$$

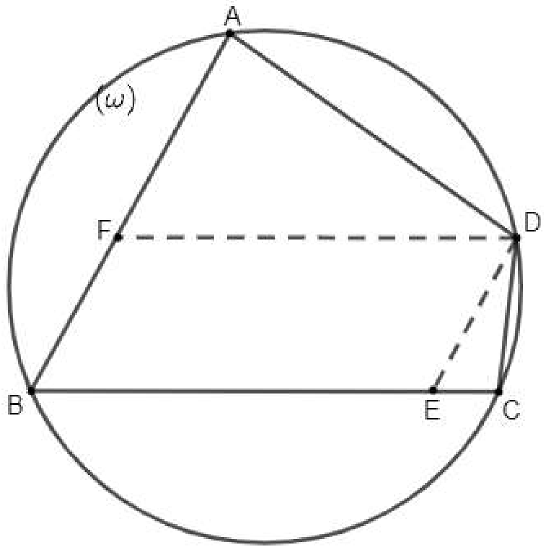
Hence: $[ABCD] = \frac{\sin B}{2}(BC \cdot BF + BA + BE)$;

3. $NCCQ$1-NEW CRITERION FOR CYCLIC QUADRILATERAL-(1)

$DE \parallel AB$; $FD \parallel BC$

$ABCD$ **–is cyclic⇔**

$$BD^2 = BC \cdot BE + BA \cdot BF$$



Proof.

Let $BC = a, BA = c, BE = d_1$,

$BF = d_2$

(ω) –circumcircle of ΔABC.

Plagiognal system: $BC \equiv BX, BA \equiv By$

$B(0,0), C(a,0), A(0,c), D(d_1, d_2)$

$$(\omega): x^2 + y^2 + 2xy \cdot \cos B - ax - cy = 0; \quad (1)$$

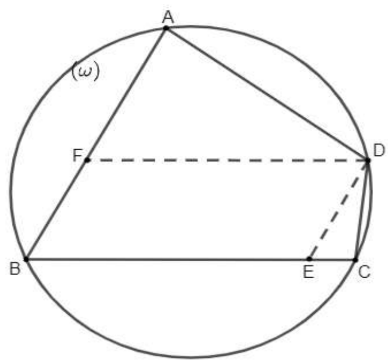
$$BD^2 = d_1^2 + d_2^2 + 2d_1d_2 \cdot \cos B; \quad (2)$$

$ABCD$ –is cyclic⇔ $D \in (\omega) \Leftrightarrow d_1^2 + d_2^2 + 2d_1d_2 \cdot \cos B - ad_1 - cd_2 = 0 \Leftrightarrow$

$$BD^2 = BC \cdot BE + BA \cdot BF$$

4. $NCCQ$2-NEW CRITERION FOR CYCLIC QUADRILATERAL-(2)

$$DE \parallel AB; \ FD \parallel BC; \ ABCD - \text{is cyclic} \Leftrightarrow \cos B = \frac{1}{2}\left(\frac{EC}{BF} + \frac{FA}{BE}\right)$$



Proof. $BD^2 = BC \cdot BE + BA \cdot BF \ (NCCQ1); \quad (1)$

$BD^2 = BE^2 + BF^2 + 2BE \cdot BF \cdot \cos B; (2)$

$ABCD$ –cyclic$\overset{(1)}{\Rightarrow} BD^2 = BC \cdot BE + BA \cdot BF \overset{(2)}{\Rightarrow}$

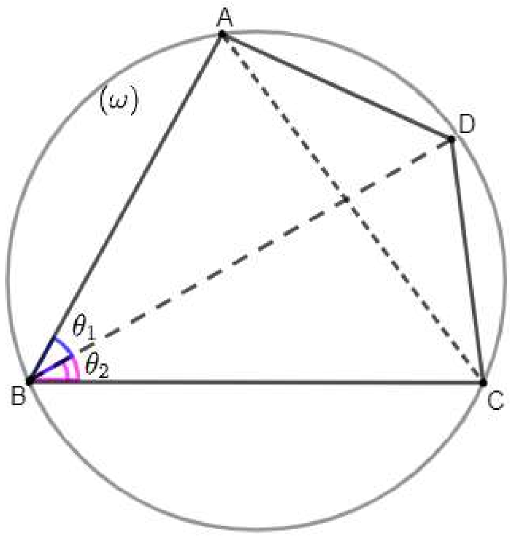
$BE^2 + BF^2 + 2BE \cdot BF \cos B = BC \cdot BE + BA \cdot BF$

$2BE \cdot BF \cos B = BC \cdot BE - BE^2 + BA \cdot BF - BF^2$

$2BE \cdot BF \cos B = BE(BC - BE) + BF(BA - BF)$

$$2BE \cdot BF \cos B = BE \cdot EC + BF \cdot FA$$

5. $NCCQ3$-NEW CRITERION FOR CYCLIC QUADRILATERAL-(3)



$$\boldsymbol{ABCD}\ -\textbf{cyclic}\Leftrightarrow$$

$$\boldsymbol{BD=\frac{BA\cdot\sin\theta_1+BC\sin\theta_2}{\sin(\theta_1+\theta_2)}}$$

Proof.(by Mansur Mansurov)

$$AD=2R\sin\theta_2\,;(1)\ \ CD=2R\sin\theta_1\,;(2)$$

$$AC=2R\sin(\theta_1+\theta_2);\ \ (3)$$

$$ABCD\ -\text{cyclic}\Leftrightarrow$$

$$BC\cdot 2R\sin\theta_2+BA\cdot 2R\sin\theta_1=BD\cdot 2R\sin(\theta_1+\theta_2)$$

$$\Leftrightarrow BD=\frac{BA\cdot\sin\theta_1+BC\sin\theta_2}{\sin(\theta_1+\theta_2)}$$

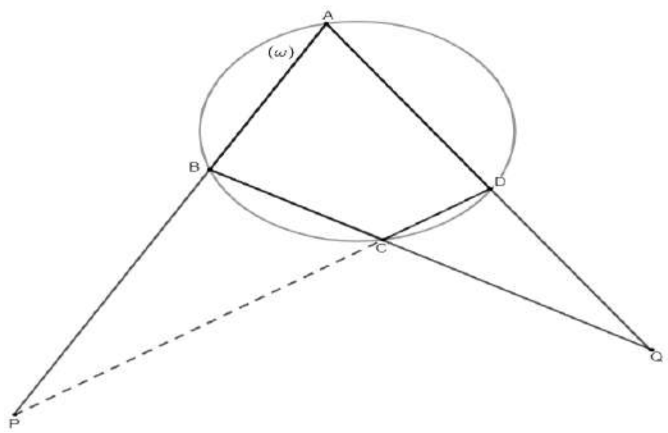
6. $NCCQ4$-NEW CRITERION FOR CYCLIC QUADRILATERAL-(4)

$$\boldsymbol{ABCD-\textbf{cyclic}\Leftrightarrow\cos A=\frac{1}{2}\cdot\frac{AB\cdot AP+AD\cdot AQ}{AP\cdot AQ}}$$

Proof. Let $AB=b, AP=p, AD=d, AQ=q$. Plagiogonal system: $AB\equiv Ax, AD\equiv Ay$

$$A(0,0),B(b,0),P(p,0),D(0,d),Q(0,q),C(c_1,c_2)$$

$$\begin{cases}BQ:\ \dfrac{x}{b}+\dfrac{y}{q}=1\\ PD:\ \dfrac{x}{p}+\dfrac{y}{d}=1\end{cases}\Rightarrow\begin{cases}x=c_1=\dfrac{bp(q-d)}{pq-bd}\\ y=c_2=\dfrac{qd(p-d)}{pq-bd}\end{cases};(1)$$



From $NCCQ1$, we have $ABCD$ is cyclic$\Leftrightarrow AC^2=AB\cdot c_1+AD\cdot c_2\Leftrightarrow$

$$c_1^2+c_2^2+2c_1c_2\cos A=bc_1+dc_2\Leftrightarrow\cos A=\frac{1}{2}\cdot\frac{bp+dq}{pq}$$

$$\Leftrightarrow \cos A = \frac{1}{2} \cdot \frac{AB \cdot AP + AD \cdot AQ}{AP \cdot AQ}$$

7. AREA OF CYCLIC QUADRILATERAL

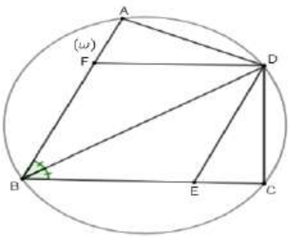
$$[ABCD] = BD^2 \cdot \frac{\sin B}{2}$$

Proof. $BD^2 = BE \cdot BC + BF \cdot BA(\text{ by } NCCQ1)$

$$[ABCD] = \frac{\sin B}{2}(BC \cdot BF + BA \cdot BE); (Gakopoulos - Blatsis\ formulae)$$

$$BE = BF \Rightarrow \frac{[ABCD]}{BD^2} = \frac{\sin B}{2}$$

$$[ABCD] = BD^2 \cdot \frac{\sin B}{2}$$



8. LENGTH OF ANGLE BISECTOR OF TRIANGLE

$$BE = \frac{BA \cdot BC}{BD}$$

Proof. From $NCCQ3$, we have:

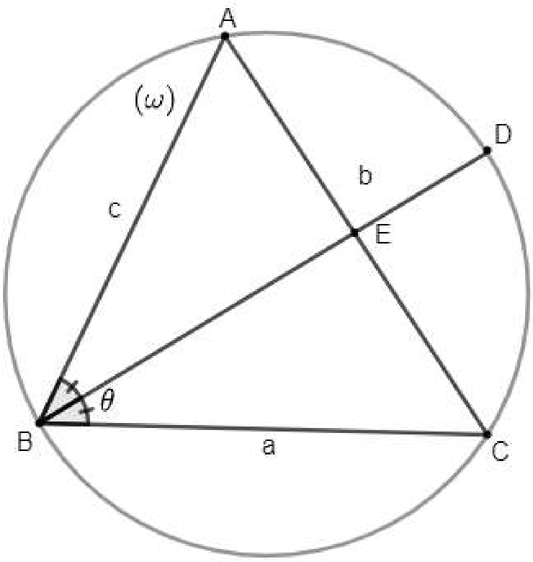
$$BD = \frac{BA \cdot \sin\theta + BC \sin\theta}{\sin(2\theta)} =$$

$$= \frac{BA + BC}{2\cos\theta}; (1)$$

$$BE = \frac{2BA \cdot BC}{BA + BC}\cos\theta$$

$$\frac{BA + BC}{2\cos\theta} = \frac{BA \cdot BC}{BE}; (2)$$

From (1) and (2): $BE = \frac{BA \cdot BC}{BD}$



9. CIRCUMRADIUS OF TRIANGLE

$$R^2 = \frac{OP^2 + OQ^2 - PQ^2}{2}$$

Proof. From $NCCQ4$, we have:

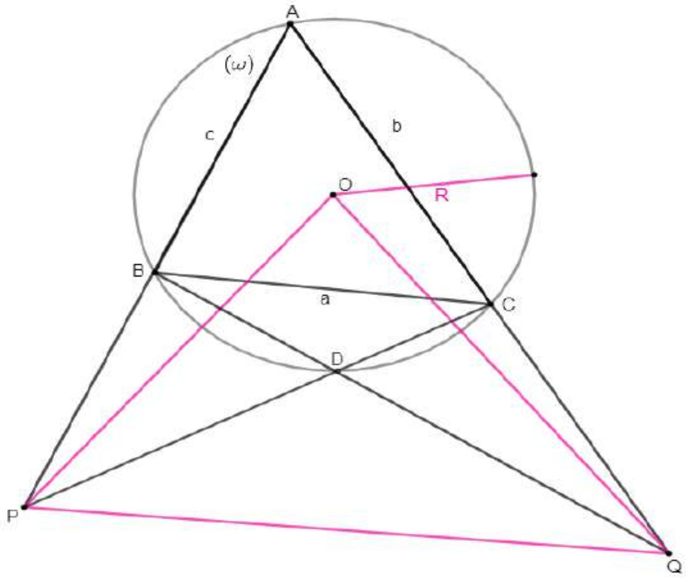
$$\cos A = \frac{1}{2} \cdot \frac{AB \cdot AP + AD \cdot AQ}{AP \cdot AQ}; (1)$$

$$\cos A = \frac{AP^2 + AQ^2 - PQ^2}{2AP \cdot PQ}; (2)$$

From (1) and (2), we get: $AP^2 + AQ^2 - PQ^2 = AB \cdot AP + AC \cdot AQ \Leftrightarrow$

$$AP(AP - AB) + AQ(AQ - AC) - PQ^2 = 0 \Leftrightarrow AP \cdot PB + AQ \cdot QC - PQ^2 = 0 \Leftrightarrow$$

$$OP^2 - R^2 + OQ^2 - R^2 - PQ^2 = 0 \Leftrightarrow R^2 = \frac{OP^2 + OQ^2 - PQ^2}{2}$$

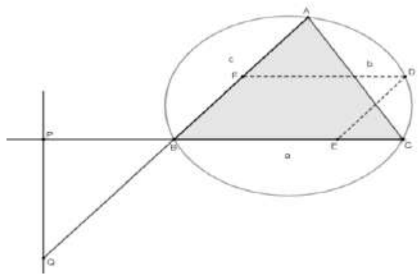


10. PERPENDICULAR CRITERION: $D \in (\widehat{AC}), DE \parallel AB, \ DF \parallel BC$

$$PQ \perp BC \Leftrightarrow \frac{EF}{BF} + \frac{FA}{BE} = 2 \cdot \frac{BP}{PQ}$$

Proof. From $NCCQ2$, we have: $\frac{1}{2} \cdot \frac{EC}{BF} + \frac{FA}{BE} = \cos B\,; (1)$

$$PQ \perp BC \Leftrightarrow \cos B = \frac{BP}{BQ} \Rightarrow \frac{EC}{BF} + \frac{FA}{BE} = 2 \cdot \frac{BP}{BQ}$$



11. LINE PASSES THROUGH THE INCENTER OF TRIANGLE

$$D, I, F - \text{collinear} \Leftrightarrow \left(\frac{1}{BD} - \frac{1}{BC}\right) + \left(\frac{1}{BF} - \frac{1}{BA}\right) = \frac{AC}{BC \cdot BA}$$

Proof:

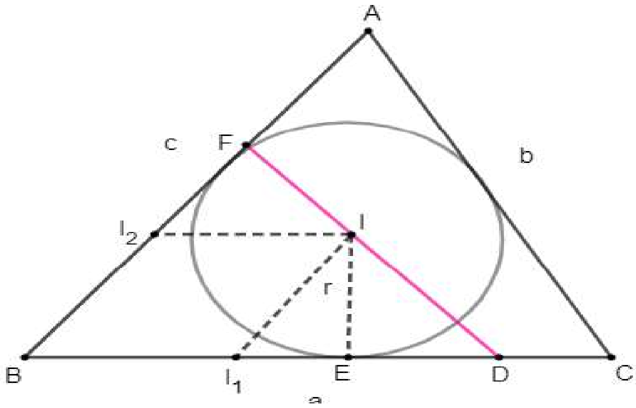
$$BI_1 = BI_2 = II_1 = II_2 = i$$

$$\sphericalangle II_1C = B,$$

$$\Delta IEI_1: \sin(II_1E) = \frac{r}{i}$$

$$i = \frac{F}{s} \cdot \frac{1}{\sin B} = \frac{ac \sin B \cdot 1/2}{\frac{a+b+c}{2} \sin B}$$

$$i = \frac{ac}{a+b+c}; (1)$$



$$\begin{cases} \Delta DII_1 \sim \Delta DBF \\ \Delta FII_2 \sim \Delta FDB \end{cases} \Rightarrow \begin{cases} \frac{DI}{DF} = \frac{i}{BF} \\ \frac{FI}{FD} = \frac{i}{BD} \end{cases} \Rightarrow \frac{i}{BF} + \frac{i}{BD} = 1; (2), D, I, F - \text{collinear} \Leftrightarrow \frac{i}{BF} + \frac{i}{BD} = 1$$

$$\Leftrightarrow \frac{1}{BF} + \frac{1}{BD} = \frac{1}{i}, \frac{1}{BF} + \frac{1}{BD} = \frac{AB + BC + CA}{BC \cdot BA} \Leftrightarrow \frac{1}{BD} + \frac{1}{BF} = \frac{1}{BC} + \frac{1}{BA} + \frac{AC}{BC \cdot BA}$$

$\Leftrightarrow$

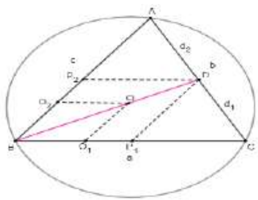
$$\left(\frac{1}{BD} - \frac{1}{BC}\right) + \left(\frac{1}{BF} - \frac{1}{BA}\right) = \frac{AC}{BC \cdot BA}$$

12. LINE PASSES THROW CIRCUMCENTER OF TRIANGLE

B, O, D **–collinear**$\Leftrightarrow$

$$\frac{AD}{DC} = \frac{c(a - c\cos B)}{a(c - a\cos B)} \Leftrightarrow$$

$$\cos B = \frac{ac(AD - DC)}{a^2 AD - c^2 DC}$$



Proof: Let $AD = d_2, DC = d_1$. From Plagiogonal system theory, we have:

$$BD_1 = \frac{ad_2}{d_1 + d_2}, \quad BD_2 = \frac{cd_1}{d_1 + d_2}, \quad BO_1 = \frac{a - c\cos B}{2\sin^2 B}, \quad BO_2 = \frac{c - a\cos B}{2\sin^2 B}$$

$$B, O, D - \text{collinear} \Leftrightarrow \frac{BD_1}{BD_2} = \frac{BO_1}{BO_2} \Leftrightarrow \frac{ad_2}{cd_1} = \frac{a - c\cos B}{c - a\cos B} \Leftrightarrow$$

$$\frac{d_2}{d_1} = \frac{AD}{DC} = c(a - c\cos B)/a(c - a\cos B) \Leftrightarrow \cos B = \frac{ac(AD - DC)}{a^2 AD - c^2 DC}$$

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A SIMPLE PROOF FOR LEHMMER'S INEQUALITY

By Daniel Sitaru-Romania

Abstract: *In this paper it's presented a simple proof for Lehmer's inequality, corollaries and applications.*

LEHMMER'S INEQUALITY: If $a, b, x > 0$ then:

$$\mathbf{0 < \log\left(1+\frac{x}{a}\right)\log\left(1+\frac{b}{x}\right) < \frac{b}{a}}; \quad (\mathbf{1})$$

Proof: Lemma: If $x > 0$ then: $\log(1+x) < x$

Proof of lemma: Let be the function $f: (0, \infty) \to \mathbb{R}; f(x) = \log(1+x) - x$, then

$$f'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x} < 0 \Rightarrow f - \text{decreasing}.$$

For $x > 0$: $\sup f(x) = \lim_{x\to 0_+} f(x) = \lim_{x\to 0_+}[\log(1+x) - x] = 0 \Rightarrow f(x) < 0, \forall x > 0.$

By Lemma: $0 = \log 1 < \log\left(1+\frac{x}{a}\right) < \frac{x}{a}$; (2)

$$0 = \log 1 < \log\left(1+\frac{b}{x}\right) < \frac{b}{x}; \quad (3)$$

By multiplying (2) and (3):

$$0 < \log\left(1+\frac{x}{a}\right)\log\left(1+\frac{b}{x}\right) < \frac{x}{a}\cdot\frac{b}{x} = \frac{b}{a}$$

Corollary 1: If $a, b > 0$ then: $0 < \log\left(1+\frac{1}{a}\right)\log(1+b) < \frac{b}{a}$

Proof: We take $x = 1$ in (1).

Corollary 2: If $a, b > 0$ then: $0 < \log^2\left(1+\sqrt{\frac{b}{a}}\right) < \frac{b}{a}$

Proof: We take $x = \sqrt{ab}$ in (1): $0 < \log\left(1+\frac{\sqrt{ab}}{a}\right)\log\left(1+\frac{b}{\sqrt{ab}}\right) < \frac{b}{a}$

$$0<\log\left(1+\sqrt{\frac{b}{a}}\right)\log\left(1+\sqrt{\frac{a}{b}}\right)<\frac{b}{a},\qquad 0<\log^2\left(1+\sqrt{\frac{b}{a}}\right)<\frac{b}{a}$$

Application 1: If $a,b>0$ then: $e^{2\sqrt{\frac{b}{a}}}>1+2\sqrt{\frac{b}{a}}+\frac{b}{a}$

Proof: By Lemma: $\log(1+x)<x\Rightarrow e^x>1+x,\forall x>0$

$$e^{\frac{x}{a}}>1+\frac{x}{a};\qquad (4)$$

$$e^{\frac{b}{x}}>1+\frac{b}{x};\qquad (5)$$

By multiplying (4) and (5): $e^{\frac{x}{a}}\cdot e^{\frac{b}{x}}>\left(1+\frac{x}{a}\right)\left(1+\frac{b}{x}\right)$

$$e^{\frac{x}{a}+\frac{b}{x}}>1+\frac{x}{a}+\frac{b}{x}+\frac{b}{a};\qquad (6)$$

For $x=\sqrt{ab}$ in (6):

$$e^{\frac{\sqrt{ab}}{a}+\frac{b}{\sqrt{ab}}}>1+\frac{\sqrt{ab}}{a}+\frac{b}{\sqrt{ab}}+\frac{b}{a}$$

Application 2:

$$\boldsymbol{\Omega=\lim_{x\to\infty}\log\left(1+\frac{1}{x^4}\right)\log(1+x^2)=0}$$

Proof: By (1): $0<\log\left(1+\frac{1}{x^4}\right)\log\left(1+\frac{x^2}{1}\right)<\frac{1}{x^4}\cdot x^2=\frac{1}{x^2},\ 0<\log\left(1+\frac{1}{x^4}\right)\log\left(1+\frac{x^2}{1}\right)<\frac{1}{x^2}$

Therefore:

$$0\le\lim_{x\to\infty}\log\left(1+\frac{1}{x^4}\right)\log(1+x^2)\le\lim_{x\to 0}\frac{1}{x^2}=0,\qquad \Omega=\lim_{x\to\infty}\log\left(1+\frac{1}{x^4}\right)\log(1+x^2)=0$$

Application 3:

$$\boldsymbol{\Omega=\sum_{n=1}^{\infty}\log\left(1+\frac{1}{n^4}\right)\log(1+n^2)<\frac{\pi^2}{6}}$$

Proof: By (1):

$$0 < \log\left(1+\frac{1}{n^4}\right)\log(1+n^2) < \frac{1}{n^4}\cdot n^2 = \frac{1}{n^2}$$

$$\Omega = \sum_{n=1}^{\infty}\log\left(1+\frac{1}{n^4}\right)\log(1+n^2) < \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

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LIFTING THE EXPONENT LEMMA-(LTE)

By Albert Iulian Romaniuc –10th grade

National College "Roman-Vodă" Roman-Romania

Definition. We define $v_p(x)$ to be the greatest power in which a prime p divides x: if $v_p(x) = m$, then $p^m \mid x$ and $p^{m+1} \nmid x$. We also write $p^m || x$ if and only if $v_p(x) = m$.

Properties.

1. $\mathbf{v_p(n) = m \in \mathbb{N}^* \Leftrightarrow p^m \mid n \text{ and } p^{m+1} \nmid n}$.
2. $\mathbf{v_p(n) = 0 \Leftrightarrow gcd(p,n) = 1}$.
3. $\mathbf{v_p(p) = 1, \text{for all primes } p}$.
4. $\mathbf{v_p(m+n) \geq min\{v_p(m), v_p(n)\}}$.
5. $\mathbf{v_p(mn) = v_p(m) + v_p(n)}$.

Note. We have $v_p(0) = \infty$ for all primes p.

Lemma 1. Let x and y be 2 integers and let n be a positive integer. Given an arbitrary prime p(in particular, we can have p = 2) such that $gcd(n,p) = 1, p \mid x - y$ and neither x, nor y is divisible by p, we have:

$$\mathbf{v_p(x^n - y^n) = v_p(x - y)}.$$

Proof. $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-1})$. Let's show that $p \nmid x^{n-1} + x^{n-2}y + \cdots + y^{n-1}$. From $p \mid x - y \Rightarrow x \equiv y (\text{mod } p) \Rightarrow x^{n-1} + x^{n-2}y + \cdots + y^{n-1} \equiv x^{n-1} + x^{n-2} \cdot x + \cdots + x^{n-1} \equiv \equiv nx^{n-1} (\text{mod } p)$. Now, because we know that $gcd(n,p) = 1$ and $p \nmid x \Rightarrow p \nmid nx^{n-1}$.

Therefore, since $p \nmid nx^{n-1} \Rightarrow v_p(x^n - y^n) = v_p(x - y)$, q.e.d.

Lemma 2. Let x and y be 2 integers and let n be an odd positive integer. Given an arbitrary prime p(in particular, we can have p = 2) such that $gcd(n,p) = 1$, $p \mid x + y$ and neither x, nor y is divisible by p, we have:

$$\mathbf{v_p(x^n + y^n) = v_p(x + y)}.$$

Proof. Since n is an odd positive integer, we know that $y^n = -(-y)^n \xRightarrow{\text{Lemma 1}} v_p(x^n + y^n) = = v_p(x^n - (-y)^n) = v_p(x - (-y)) \Rightarrow v_p(x^n + y^n) = v_p(x + y)$, q.e.d.

Theorem 1 (First Form of LTE). Let x and y be (not necessary positive) integers, let n be a positive integer, and let p be an odd prime such that $p \mid x - y$, $p \nmid x$ and $p \nmid y$. We have:

$$v_p(x^n - y^n) = v_P(x - y) + v_p(n).$$

Theorem 2 (Second Form of LTE). Let x, y be two integers, n be an odd positive integer, and p be an odd prime such that $p \mid x + y$, $p \nmid x$ and $p \nmid y$. We have:

$$\mathbf{v_p(x^n + y^n) = v_p(x + y) + v_p(n)}.$$

Theorem 3 (LTE for p = 2). Let x and y be two odd integers such that 4 | $x-y$. We have:

$$v_2(x^n-y^n)=v_2(x-y)+v_2(n).$$

Theorem 4. Let x and y be two odd integers and let n be an even positive integer. We have:

$$v_2(x^n-y^n)=v_2(x-y)+v_2(x+y)+v_2(n)-1.$$

Problem 1. Find all possible values of n, where n is a positive integer, such that $\frac{3^n-1}{2^n}$ is also an integer.

Solution. If n is even $\xRightarrow{\text{Theorem 4}} v_2(3^n-1^n)=v_2(3-1)+v_2(3+1)+v_2(n)-1=v_2(n)+2$. Because $\frac{3^n-1}{2^n}$ is an integer $\Rightarrow v_2(3^n-1^n)\geq n \Rightarrow v_2(n)+2\ \geq n$, but we also know that $v_2(n)\leq \log_2 n \Rightarrow 2+\log_2 n\geq n \Leftrightarrow \log_2 4+\log_2 n\geq n\Leftrightarrow \log_2(4n)\geq n\Leftrightarrow 4n\geq 2^n$, which is true only for n ≤ 4 (for n ≥ 5, it's easy to show that $2^n>4n$ with the Principle of Mathematical Induction). Therefore, in this case we have the solutions n = 2 and n = 4.

If n = 1 $\Rightarrow \frac{3^1-1}{2^1}=1$, which is an integer and so n = 1 is a solution. If n is odd and n ≥ 3 $\Rightarrow\Rightarrow$ n $=2k+1$, where k is a positive integer. For n ≥ 3, it's clear that $v_2(2^n)\geq 3\Rightarrow 4 \mid 2^n$, but $3^n-1=(4-1)^n-1\ \equiv -1-1\equiv -2\equiv 2 \pmod 4 \Rightarrow 4\nmid 3^n-1$ for n ≥ 3.

In conclusion, n ∈ {1, 2, 4}, q.e.d.

Problem 2. Find all positive integers a such that $\frac{5^a+1}{3^a}$ is an integer.

Solution. From $\frac{5^a+1}{3^a}\Rightarrow 3^a \mid 5^a+1$. If a is even, then: $5^a+1\equiv(-1)^a+1\equiv 2 \pmod 3$, which is false. So, a must be an odd positive integer $\xRightarrow{\text{Theorem 2}} v_3(5^a+1^a)=v_3(5^a+1)=v_3(5+1)+v_3(a)\Rightarrow v_3(5^a+1)=v_3(a)+1$. Let $a=3^r s$, where r ≥ 0 and s ≥ 1 are 2 integers $\Rightarrow v_3(a)=r$, but $v_3(3^a)=a$ and because $\frac{5^a+1}{3^a}$ is an integer $\Rightarrow v_3(3^a)\leq v_3(5^a+1)\Leftrightarrow 3^r s\leq r+1$. For $r\geq 1$, it's obvious that $3^r>r+1$ (it's easy to show this with the Principle of Mathematical Induction). Therefore, r = 0 ⟹ s = 1 ⟹ a = 3.

Problem 3. Let p > 2013 be a prime. Also, let a and b be positive integers such that p | a + b, but $p^2\nmid a+b$. If $p^2 \mid a^{2013}+b^{2013}$, then find the number of positive integer $n\leq 2013$ such that $p^n \mid a^{2013}+b^{2013}$.

Solution. From p | a + b and $p^2\nmid a+b \Rightarrow v_p(a+b)=1$. We also must have $v_p(a^{2013}+b^{2013})\geq 2$. If $p\nmid a$ and $p\nmid b$ $\xRightarrow{\text{Theorem 2}} v_p(a^{2013}+b^{2013})=v_p(a+b)+v_p(2013)=1$, which is obvious false. Now, WLOG let's consider that $p \mid a$ and $p\nmid b \Rightarrow p\nmid a+b$, which is false. Therefore $p \mid a$ and $p \mid b$. If $p \mid a$ and $p \mid b \Rightarrow p^{2013} \mid a^{2013}$ and $p^{2013} \mid b^{2013}\Rightarrow p^{2013} \mid a^{2013}+b^{2013}\Rightarrow p^k \mid a^{2013}+b^{2013}$ for every k positive integer, k ≤ 2013. In conclusion, the answer is: n can take all positive integers less than or equal to 2013 and so the number of positive integer n ≤ 2013 is 2013.

Problem 4. Let a and b two integers and p ≠ 3 a prime number such that p | a + b and $p^2 \mid a^3+b^3$. Show that $p^2 \mid a+b$ or $p^3 \mid a^3+b^3$.

Solution. If p | a, from p | a + b $\Longrightarrow$ p | b $\Longrightarrow p^3 \mid a^3$ and $p^3 \mid b^3 \Longrightarrow p^3 \mid a^3+b^3$. Analogous if p | b. Now, let's consider that p ∤ ab. Because p | a + b $\xRightarrow{\text{Theorem 2}} v_p(a^3+b^3)=v_p(a+b)+v_p(3)==v_p(a+b)$. From $p^2 \mid a^3+b^3 \Longrightarrow v_p(a^3+b^3)\geq 2 \Longrightarrow v_p(a+b)\geq 2 \Longrightarrow p^2 \mid a+b$.

Problem 5. Find all positive integer solutions of the equation $\mathbf{x^{2009}+y^{2009}=7^z}$.

Solution. Because $x+y \mid x^{2009}+y^{2009}$ and $x+y>1 \Longrightarrow 7 \mid x+y$. Removing the highest possible power of 7 from x, y, we get from Theorem 2 that: $v_7(x^{2009}+y^{2009})=v_7(x+y)+v_7(2009)==v_7(x+y)+2 \Longrightarrow x^{2009}+y^{2009}=49k(x+y)$, where $7\nmid k$. From $x^{2009}+y^{2009}=7^z \Longrightarrow$ the only prime factor of $x^{2009}+y^{2009}$ is 7 $\Longrightarrow$ k = 1. Therefore, $x^{2009}+y^{2009}=49(x+y)$. If x = 1 or y = 1 $\Longrightarrow$ $y^{2009}=48+49y$ or $x^{2009}=48+49x$, which obvious doesn't have any solutions in $\mathbb{Z}_+$ because LHS is always greater than RHS. In conclusion, the equation $x^{2009}+y^{2009}=7^z$ doesn't have any solutions in $\mathbb{Z}_+$.

Problem 6. Let k > 1 be an integer. Show that there exists infinitely many positive integers n such that n | $\mathbf{1^n+2^n+\cdots k^n}$.

Solution. Case I. k is not a power of 2. Let p be any odd prime divisor of k. Let's show that $n=p^m$ works for any positive integer m.

Consider $i^n+(p-i)^n$, where $i=1,2,3,\ldots,p-1$. From Theorem 2, we have: $v_p(i^n+(p-i)^n)==v_p(p)+v_p(n)=1+m$. Therefore, $p^{m+1} \mid i^n+(p-i)^n$. Summing, we have: $p^{m+1} \mid 2(1^n++2^n+\cdots+(p-i)^n)$ and so $p^{m+1} \mid 1^n+2^n+\cdots+(p-1)^n+p^n+(p+1)^n+\cdots+k^n$.

In conclusion, $n=p^m$ works for every positive integer m.

Case II. k is a power of 2.

Let p be any odd prime divisor of k + 1. Using a similar proof above, it's easy to show that $n=p^m$ works again for any positive integer m.

Problem 7. Let k be a positive integer. Find all positive integers n such that $\mathbf{3^k \mid 2^n-1}$.

Solution. If n is an odd positive integer $\Longrightarrow$ n = 2a + 1, where a is a nonnegative integer. Then, $2^n-1=2^{2a+1}-1=(3-1)^{2a+1}-1\equiv -1-1\equiv -2\equiv 1(\text{mod } 3)$, but because $v_3(3^k)>0$, this case is impossible. So, n is an even number, n = 2m, where m is a positive integer. Now, we have: $3^k \mid 4^m-1$. From Theorem 1: $v_3(4^m-1)=v_3(4^m-1^m)=v_3(4-1)+v_3(m)=1+v_3(m)\Longrightarrow\Longrightarrow v_3(m)\geq k-1$.Therefore, the answer is $n=2\cdot 3^{k-1}\cdot t$, where t is a nonnegative integer.

Problem 8. Prove that for all positive integers n, there is a positive integer m that

$$\mathbf{7^n \mid 3^m+5^m-1}.$$

Solution. We will show that m = 7^{n-1} works. From Theorem 1 $\Longrightarrow v_7\left(3^{7^{n-1}}+4^{7^{n-1}}\right)=v_7(3+4)+$ $+v_7(7^{n-1})=1+n-1=n \Longrightarrow 3^{7^{n-1}}\equiv -4^{7^{n-1}}(\text{mod } 7^n)$.

In a similar way, we get $5^m\equiv -2^m(\text{mod } 7^n) \Leftrightarrow 5^{7^{n-1}}\equiv -2^{7^{n-1}}(\text{mod } 7^n)$. So, we get: $3^{7^{n-1}}+5^{7^{n-1}}\equiv -4^{7^{n-1}}-2^{7^{n-1}}(\text{mod } 7^n) \Leftrightarrow 3^{7^{n-1}}+5^{7^{n-1}}-1\equiv -\left(4^{7^{n-1}}+2^{7^{n-1}}+1\right)(\text{mod } 7^n)$. Since we want to show that $3^{7^{n-1}}+5^{7^{n-1}}-1\equiv 0(\text{mod } 7^n)$, it's enough to show that $4^{7^{n-1}}+2^{7^{n-1}}+1\equiv\ \equiv 0(\text{mod } 7^n)$. Since $7\nmid 2^{7^{n-1}}-1$ (since $2^i\equiv 2,4,1\ (\text{mod } 7)$ and $2^i\equiv 1(\text{mod } 7)\Leftrightarrow i\equiv$

$0(\text{mod } 3))$, it is enough to show that: $(4^{7^{n-1}} + 2^{7^{n-1}} + 1)(2^{7^{n-1}} - 1) \equiv 0(\text{mod } 7) \Leftrightarrow 8^{7^{n-1}} - 1 \equiv 0(\text{mod } 7)$, which is actually true from Theorem 1: $v_7(8^{7^{n-1}} - 1) = v_7(8-1) + v_7(7^{n-1}) = n.$

In conclusion, there is a positive integer m such that $7^n \mid 3^m + 5^m - 1$, m = 7^{n-1}.

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A SIMPLE PROOF FOR WANG'S INEQUALITY AND APPLICATIONS

By Daniel Sitaru-Romania

Abstract: *In this paper it's gived a simple proof for the famous Wang's inequality and a few applications.*

WANG'S INEQUALTIY: If $x > 0, x \neq 1$ then:

$$\log x < n\left(\sqrt[n]{x} - 1\right) < \sqrt[n]{x} \log x \,;\, n \in \mathbb{N}, n \geq 1; \qquad (1)$$

Proof: Let be the function $f: (0, \infty) \to \mathbb{R}, f(x) = \log x - n\sqrt[n]{x} - n$, then

$$f'(x) = \frac{1}{x} - n \cdot \frac{1}{n} \cdot x^{\frac{1}{n}-1} = \frac{1 - \sqrt[n]{x}}{x}, \qquad f'(x) = 0 \Rightarrow 1 - \sqrt[n]{x} = 0 \Rightarrow 1 = \sqrt[n]{x} \Rightarrow x = 1$$

$$\lim_{x \to 0_+} f(x) = \lim_{x \to 0_+} \left(\log x - n\sqrt[n]{x} - n\right) = -\infty, \qquad \lim_{x \to \infty} \frac{\log x - n}{\sqrt[n]{x}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{n} x^{\frac{1}{n}-1}} = \lim_{x \to \infty} \frac{n}{x^{2-\frac{1}{n}}} = 0$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left(\log x - n\sqrt[n]{x} - n\right) = \lim_{x \to \infty} \sqrt[n]{x}\left(\frac{\log x - n}{\sqrt[n]{x}} - n\right) = \infty(0 - n) = -\infty$$

$$M = \sup_{x > 0} f(x) = f(1) = -n - 1 < 0$$

$$f(x) < 0, \forall x > 0, x \neq 1$$

Let be $g:(0,\infty)\to\mathbb{R}, g(x)=\log x-n+x^{-\frac{1}{n}}$, then $g'(x)=\frac{1}{x}-\frac{1}{n}\cdot\frac{x^{\frac{1}{n}}}{x}=\frac{n-\sqrt[n]{x}}{nx}$

$$g'(x)=0\Rightarrow n-\sqrt[n]{x}=0\Rightarrow x=n^n,\qquad g(n^n)=\log(n^n)-n+(n^n)^{-\frac{1}{n}}=\frac{1}{n}>0$$

$$\lim_{x\to\infty}g(x)=\lim_{x\to\infty}\left(\log x-n+\frac{1}{\sqrt[n]{x}}\right)=\infty,\qquad \lim_{x\to 0_+}\frac{\log x}{x^{-\frac{1}{n}}}=\lim_{x\to 0_+}\frac{\frac{1}{x}}{-\frac{1}{n}x^{-\frac{1}{n}-1}}=-n\sqrt[n]{0}=0$$

$$\lim_{x\to 0_+}g(x)=\lim_{x\to 0_+}\left(\log x-n+\frac{1}{\sqrt[n]{x}}\right)=-n+\lim_{x\to 0_+}\frac{\sqrt[n]{x}\log x+1}{\sqrt[n]{x}}=-n+\frac{1}{0_+}=\infty$$

$$m=\inf_{x>0}g(x)=g(n^n)=\frac{1}{n}>0\Rightarrow g(x)>0$$

$$\begin{cases}f(x)<0\\g(x)>0\end{cases}\Rightarrow\begin{cases}\log x-n\sqrt[n]{x}-n<0\\ \log x-n+\frac{1}{\sqrt[n]{x}}>0\end{cases}\Rightarrow\begin{cases}n\left(\sqrt[n]{x}-n\right)>\log x\\ \sqrt[n]{x}\log x>n\left(\sqrt[n]{x}-1\right)\end{cases}$$

Application 1: If $x,y,z>0$ then: $xyz\exp(3n)\le\exp\left(\sqrt[n]{x}+\sqrt[n]{y}+\sqrt[n]{z}\right); n\in\mathbb{N}^*$

Proof: By (1):

$$\log x\le n\left(\sqrt[n]{x}-n\right);\qquad (2)$$

$$\log y\le n\left(\sqrt[n]{y}-n\right);\qquad (3)$$

$$\log z\le n\left(\sqrt[n]{z}-n\right);\qquad (4)$$

By adding (2), (3) and (4): $\log x+\log y+\log z\le n\left(\sqrt[n]{x}+\sqrt[n]{y}+\sqrt[n]{z}\right)-3n$

$$\log(xyz)\le n\left(\sqrt[n]{x}+\sqrt[n]{y}+\sqrt[n]{z}\right)-3n,\ \ xyz\le\exp\left(n\left(\sqrt[n]{x}+\sqrt[n]{y}+\sqrt[n]{z}\right)-3n\right)$$

$$xyz\exp(3n)\le\exp\left(\sqrt[n]{x}+\sqrt[n]{y}+\sqrt[n]{z}\right); n\in\mathbb{N}^*$$

Equality holds for $x=y=z=1$.

Application 2: If $x,y,z>0; m,n,p\in\mathbb{N}^*$ then: $x^3\exp(m+n+p)\le\exp\left(n\sqrt[n]{x}+m\sqrt[m]{x}+p\sqrt[p]{x}\right)$

Proof: By (1):

$$\log x\le n\sqrt[n]{x}-n;\qquad (5)$$

$$\log x \le m\sqrt[m]{x} - m; \quad (6)$$

$$\log x \le p\sqrt[p]{x} - p; \quad (7)$$

By adding (5),)6) and (7): $3\log x \le n\sqrt[n]{x} + m\sqrt[m]{x} + p\sqrt[p]{x} - (m+n+p)$

$$\log(x^3) \le n\sqrt[n]{x} + m\sqrt[m]{x} + p\sqrt[p]{x} - (m+n+p)$$

$x^3 \le \exp\left(n\sqrt[n]{x} + m\sqrt[m]{x} + p\sqrt[p]{x} - (m+n+p)\right), x^3\exp(m+n+p) \le \exp\left(n\sqrt[n]{x} + m\sqrt[m]{x} + p\sqrt[p]{x}\right)$

Equality holds for $x = 1$.

Application 3: If $\boldsymbol{x,y,z} > 0, n \in \mathbb{N}^*$ then: $\boldsymbol{xyz\exp\left(n\left(\frac{1}{\sqrt[n]{x}} + \frac{1}{\sqrt[n]{y}} + \frac{1}{\sqrt[n]{z}}\right)\right) \ge \exp(3n)}$

Proof: By (1): $n\left(\sqrt[n]{x} - 1\right) \le \sqrt[n]{x}\log x$

$$\log x \ge n\left(1 - \frac{1}{\sqrt[n]{x}}\right); \quad (8)$$

$$\log y \ge n\left(1 - \frac{1}{\sqrt[n]{y}}\right); \quad (9)$$

$$\log z \ge n\left(1 - \frac{1}{\sqrt[n]{z}}\right); \quad (10)$$

By adding (8), (9) and (1): $\log(xyz) \ge n\left(3 - \frac{1}{\sqrt[n]{x}} + \frac{1}{\sqrt[n]{y}} + \frac{1}{\sqrt[n]{z}}\right)$,

$$xyz \ge \exp\left(3n - n\left(\frac{1}{\sqrt[n]{x}} + \frac{1}{\sqrt[n]{y}} + \frac{1}{\sqrt[n]{z}}\right)\right)$$

$$xyz\exp\left(n\left(\frac{1}{\sqrt[n]{x}} + \frac{1}{\sqrt[n]{y}} + \frac{1}{\sqrt[n]{z}}\right)\right) \ge \exp(3n)$$

Equality holds for $x = y = z = 1$.

Application 4: If $\boldsymbol{x} > 0; m,n,p \in \mathbb{N}^*$ then:

$$\boldsymbol{m\sqrt[m]{x} + n\sqrt[n]{x} + p\sqrt[p]{x} \le m + n + p + \left(m\sqrt[m]{x} + n\sqrt[n]{x} + p\sqrt[p]{x}\right)\log x}$$

Proof: By (1):

$$n\left(\sqrt[n]{x}-1\right)<\sqrt[n]{x}\log x\,; \qquad (11)$$

$$m\left(\sqrt[m]{x}-1\right)<\sqrt[m]{x}\log x\,; \qquad (12)$$

$$p\left(\sqrt[p]{x}-1\right)<\sqrt[p]{x}\log x\,; \qquad (13)$$

By adding (11), (12) and (13):

$$m\sqrt[m]{x}+n\sqrt[n]{x}+p\sqrt[p]{x}-(m+n+p)\le\left(m\sqrt[m]{x}+n\sqrt[n]{x}+p\sqrt[p]{x}\right)\log x$$

$$m\sqrt[m]{x}+n\sqrt[n]{x}+p\sqrt[p]{x}\le m+n+p+\left(m\sqrt[m]{x}+n\sqrt[n]{x}+p\sqrt[p]{x}\right)\log x$$

Equality holds for $x=1$.

Application 5: Prove without any software: $\sqrt[3]{\mathbf{3}}(\mathbf{3}-\mathbf{log\,3})<3$

Proof: Replace $x=n=3$ in (1): $3\left(\sqrt[3]{3}-1\right)<\sqrt[3]{3}\log 3$, $3\sqrt[3]{3}-\sqrt[3]{3}\log 3<3$, $\sqrt[3]{3}(3-\log 3)<3$

Application 6: Prove without any software: $\mathbf{2\sqrt{2}-log\,2}>2$

Proof: Replace $x=n=2$ in (1): $\log 2<2\left(\sqrt{2}-1\right)$, $\log 2<2\sqrt{2}-2$, $2\sqrt{2}-\log 2>2$

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NEW INEQUALITIES IN TRIANGLES

By D.M.Bătinețu-Giurgiu, Claudia Nănuți, Florică Anastase-Romania

Abstract: *In this paper are presented new inequalities in triangles and his applications.*

Let $n \in \mathbb{N}^*, n \geq 2$ and triangles $T_k = A_k B_k C_k$ with areas F_k, semiperimeter s_k, length sides a_k, b_k, c_k and circumradii $R_k, k = \overline{1,n}$.

Theorem 1: If $t, x, y, z > 0$, then:

$$\left(\frac{(x+y)^2}{z^2 \cdot \sqrt[n]{(a_1 a_2 \dots a_n)^2}} + t\right)\left(\frac{(y+z)^2}{x^2 \cdot \sqrt[n]{(b_1 b_2 \dots b_n)^2}} + t\right)\left(\frac{(z+x)^2}{y^2 \cdot \sqrt[n]{(c_1 c_2 \dots c_n)^2}} + t\right) \geq \frac{9t^2}{\sqrt[n]{R_1^2 R_2^2 \dots R_n^2}}; \quad (1)$$

Proof. Let be $u, v, w > 0$, then: $\left(u^2 + t\right)\left(v^2 + t\right) \geq \frac{3}{4} t\left((u+v)^2 + t\right)$; $\quad (2)$

$$\left(v^2 + t\right)\left(w^2 + t\right) \geq t(v+w)^2; \quad (3)$$

We have: $(u^2 + t)(v^2 + t) \geq \frac{3}{4} t((u+v)^2 + t) \Leftrightarrow$

$$4u^2v^2 + 4t(u^2 + v^2) + 4t^2 \geq 3t(u^2 + v^2) + 6tuv + 3t^2 \Leftrightarrow$$

$$4u^2v^2 - 4tuv + t^2 + t(u^2 + v^2 - 2uv) \geq 0 \Leftrightarrow (2uv - t)^2 + t(u-v)^2 \geq 0$$

Equality holds for $2uv = t$ and $u = v$ and hence: $(v^2 + t)(w^2 + t) \geq t(v+w)^2 \Leftrightarrow$

$$v^2w^2+t(v^2+w^2)+t^2\ge t(v^2+w^2)+2tuv \Leftrightarrow$$

$$v^2w^2-2tvw+t^2\ge 0 \Leftrightarrow (vw-t)^2\ge 0.$$ Equality holds for $vw=t$.

$$(\boldsymbol{u^2+t})(\boldsymbol{v^2+t})(\boldsymbol{w^2+t}) \overset{(2)}{\ge} \frac{\boldsymbol{3}}{\boldsymbol{4}}\boldsymbol{t}((\boldsymbol{u+v})^2+\boldsymbol{t})(\boldsymbol{w^2+t}) \overset{(3)}{\ge} \frac{\boldsymbol{3}}{\boldsymbol{4}}\boldsymbol{t^2}(\boldsymbol{u+v+w})^2; \qquad (\boldsymbol{4})$$

If in (4) we take:

$$u=\frac{x+y}{z\cdot\sqrt[n]{a_1a_2\dots a_n}};v=\frac{y+z}{x\cdot\sqrt[n]{b_1b_2\dots b_n}};w=\frac{z+x}{y\cdot\sqrt[n]{c_1c_2\dots c_n}}$$

$$\left(\frac{(x+y)^2}{z^2\cdot\sqrt[n]{(a_1a_2\dots a_n)^2}}+t\right)\left(\frac{(y+z)^2}{x^2\cdot\sqrt[n]{(b_1b_2\dots b_n)^2}}+t\right)\left(\frac{(z+x)^2}{y^2\cdot\sqrt[n]{(c_1c_2\dots c_n)^2}}+t\right)\ge$$

$$\ge\frac{3}{4}t^2\left(\frac{x+y}{z\cdot\sqrt[n]{a_1a_2\dots a_n}}+\frac{y+z}{x\cdot\sqrt[n]{b_1b_2\dots b_n}}+\frac{z+x}{y\cdot\sqrt[n]{c_1c_2\dots c_n}}\right)^2; \qquad (5)$$

In [1]. has proved:

$$\sum_{cyc}\frac{\boldsymbol{x+y}}{\boldsymbol{z}\cdot\sqrt[n]{\boldsymbol{a_1a_2\dots a_n}}}\ge\frac{\boldsymbol{2\sqrt{3}}}{\sqrt[n]{\boldsymbol{R_1R_2\dots R_n}}}; \qquad (\boldsymbol{6})$$

From (5) and (6), it follows:

$$\prod_{cyc}\left(\frac{(x+y)^2}{z^2\cdot\sqrt[n]{(a_1a_2\dots a_n)^2}}+t\right)\ge\frac{3}{4}t^2\left(\frac{2\sqrt{3}}{\sqrt[n]{R_1R_2\dots R_n}}\right)^2=\frac{9t^2}{\sqrt[n]{R_1^2R_2^2\dots R_n^2}}$$

Theorem 2: If $m,x,y,z\in[1,\infty)$ and $x+y+z=3m$, then:

$$\left(\frac{(\boldsymbol{x^x+y^x+z^x})^2}{\sqrt[n]{(\boldsymbol{a_1a_2\dots a_n})^2}}+\boldsymbol{t}\right)\left(\frac{(\boldsymbol{x^y+y^y+z^y})^2}{\sqrt[n]{(\boldsymbol{b_1b_2\dots b_n})^2}}+\boldsymbol{t}\right)\left(\frac{(\boldsymbol{x^z+y^z+z^z})^2}{\sqrt[n]{(\boldsymbol{c_1c_2\dots c_n})^2}}+\boldsymbol{t}\right)$$
$$\ge\frac{\boldsymbol{81}}{\boldsymbol{4}}\cdot\frac{\boldsymbol{m^{2m}}}{\sqrt[n]{\boldsymbol{R_1^2R_2^2\dots R_n^2}}}; \qquad (\boldsymbol{7})$$

Proof. In in (4) we take:

$$u=\frac{x^x+y^x+z^x}{\sqrt[n]{a_1a_2\dots a_n}};v=\frac{x^y+y^y+z^y}{\sqrt[n]{b_1b_2\dots b_n}};w=\frac{x^z+y^z+z^z}{\sqrt[n]{c_1c_2\dots c_n}}$$

$$\prod_{cyc}\left(\frac{(\boldsymbol{x^x+y^x+z^x})^2}{\sqrt[n]{(\boldsymbol{a_1a_2\dots a_n})^2}}+\boldsymbol{t}\right)\ge\frac{\boldsymbol{3}}{\boldsymbol{4}}\boldsymbol{t^2}\left(\sum_{cyc}\frac{\boldsymbol{x^x+y^x+z^x}}{\sqrt[n]{\boldsymbol{a_1a_2\dots a_n}}}\right)^2; \qquad (\boldsymbol{8})$$

In [1]. has proved:

$$\sum_{cyc}\frac{\boldsymbol{x^x+y^x+z^x}}{\sqrt[n]{\boldsymbol{a_1a_2\dots a_n}}}\ge\frac{\boldsymbol{3\sqrt{3}\cdot m^m}}{\sqrt[n]{\boldsymbol{R_1R_2\dots R_n}}};\qquad(\boldsymbol{9})$$

$$\prod_{cyc}\left(\frac{(x^x+y^x+z^x)^2}{\sqrt[n]{(a_1a_2\dots a_n)^2}}+t\right)\ge\frac{3}{4}t^2\cdot\left(\frac{3\sqrt{3}\cdot m^m}{\sqrt[n]{R_1R_2\dots R_n}}\right)^2=\frac{81}{4}\cdot\frac{81m^{2m}}{\sqrt[n]{R_1^2R_2^2\dots R_n^2}}$$

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ELEGANT, CLASSIC AND NEW IN INTEGRAL CALCULUS

By D.M. Bătineţu-Giurgiu and Neculai Stanciu-Romania

Proposition 1. If $a\in R_+^*$ $f,g,h:R\to R$ are continue functions with f and g odd and h even, then

$$\int_{-a}^{a} f(x)\cdot\ln\left(1+e^{g(x)}\right)\cdot arctg\left(h(x)\right)dx=\int_0^a f(x)g(x)arctg\left(h(x)\right)dx.$$

Proof. $I=\int_{-a}^{a} f(x)\ln\left(1+e^{g(x)}\right)arctg\left(h(x)\right)dx$, and putting $x=u(t)=-t$, $u'(t)=-1$, $u(a)=-a$, $u(-a)=a$ we obtain

$$I=\int_{-a}^{a} f(-x)\ln\left(1+e^{g(-x)}\right)arctg\left(h(-x)\right)dx=-\int_{-a}^{a} f(x)\ln\left(1+e^{-g(x)}\right)arctg\left(h(x)\right)dx=$$

$$=-\int_{-a}^{a} f(x)\ln\left(\frac{1+e^{g(x)}}{e^{g(x)}}\right)arctg\left(h(x)\right)dx=-\int_{-a}^{a} f(x)\ln\left(1+e^{g(x)}\right)arctg\left(h(x)\right)dx+$$

$$+\int_{-a}^{a} f(x)g(x)arctg\left(h(x)\right)dx=-I+2\cdot\int_0^a f(x)g(x)arctg(h(x))dx\text{, so}$$

$$I=\int_0^a f(x)g(x)arctg\left(h(x)\right)dx\text{, q.e.d.}$$

Proposition 2. If $a,b\in R$, $a<b$, $c\in R_+^*$ iar $f:R\to R_+^*$ is continue, then Să se calculeze:

$$\int_a^b\frac{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}}{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}+e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}}dx=\frac{b-a}{2}.$$

Proof. $I=\int_a^b \frac{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}}{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}+e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}}dx$, where we putting

$x=u(t)=a+b-t,\ u'(t)=-1, u(a)=b, u(b)=a$ and we obtain

$$I=\int_a^b \frac{e^{f(a+b-x-a)}\left(f(a+b-x-a)\right)^{\frac{1}{c}}}{e^{f(a+b-x-a)}\left(f(a+b-x-a)\right)^{\frac{1}{c}}+e^{f(b-a-b+x)}\left(f(b-a-b+x)\right)^{\frac{1}{c}}}dx=$$

$$=\int_a^b \frac{e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}}{e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}+e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}}dx \text{, so}$$

$$2I=I+I=\int_a^b \frac{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}+e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}}{e^{f(x-a)}\left(f(x-a)\right)^{\frac{1}{c}}+e^{f(b-x)}\left(f(b-x)\right)^{\frac{1}{c}}}dx=\int_a^b dx=x\Big|_a^b=b-a \text{, hence}$$

$$I=\frac{b-a}{2}.$$

Proposition 3. If $a\in R_+$, $b,c\in(1,\infty)$ and $f,g:R\to R$ are continue and odd, then

$$\int_{-a}^{a} f(x)\ln\left(b^{g(x)}+c^{g(x)}\right)dx=\ln(bc)\int_0^a f(x)g(x)dx.$$

Proof. $I=\int_{-a}^{a} f(x)\ln\left(b^{g(x)}+c^{g(x)}\right)dx$, where we putting $x=u(t)=-t$,

$u'(t)=-1, u(a)=-a, u(-a)=a$, so

$$I=\int_a^{-a} -f(t)\ln\left(b^{-g(t)}+c^{-g(t)}\right)(-1)dt=-\int_{-a}^{a} f(x)\ln\frac{b^{g(x)}+c^{g(x)}}{(bc)^{g(x)}}dx=$$

$$=-I+\int_{-a}^{a} f(x)\ln(bc)^{g(x)}dx=-I+\ln(bc)\int_{-a}^{a} f(x)g(x)dx \qquad (1)$$

Also we have $(fg)(-x)=f(-x)g(-x)=-f(x)(-g(x))=(fg)(x)$, i.e. $fg:R\to R$, is even – so by (1) we obtain $2I=\ln(bc)\int_{-a}^{a}(fg)(x)dx=2\ln(bc)\int_0^a f(x)g(x)dx$, hence

$$I=\ln(bc)\int_0^a f(x)g(x)dx.$$

Proposition 4. If $a,b\in R,\ a<b$ and $f:R\to R$ is derivanble with the derivative continue and $g:R\to R_+$ is such that $f(a+b-x)=f(x), g(a+b-x)g(x)=1, \forall x\in R$, then

$$\int_a^b\left(\frac{f(x)}{1+g(x)}+f'(x)\ln(1+g(x))\right)dx=\frac{1}{2}\int_a^b\left(f(x)+f'(x)\ln g(x)\right)dx.$$

Proof. $f(a+b-x)=f(x), \forall x\in R$, so: $f'(a+b-x)=-f'(x), \forall x\in R$.

$$I=\int_a^b\left(\frac{f(x)}{1+g(x)}+f'(x)\ln(1+g(x))\right)dx$$, where we putting

$x=u(t)=a+b-t,\ u'(t)=-1, u(a)=b, u(b)=a$, therefore

$$I=-\int_b^a\left(\frac{f(a+b-t)}{1+g(a+b-t)}+f'(a+b-t)\ln(1+g(a+b-t))\right)dt=$$

$$=\int_a^b\left(\frac{f(x)}{1+\frac{1}{g(x)}}-f'(x)\ln\left(1+\frac{1}{g(x)}\right)\right)dt=$$

$$=\int_a^b\left(\frac{f(x)g(x)}{1+g(x)}-f'(x)\ln\left(1+g(x)\right)+f'(x)\ln g(x)\right)dt.$$

$$2I=\int_a^b\left(\frac{(1+g(x))f(x)}{1+g(x)}+f'(x)\ln g(x)\right)dx=\int_a^b\left(f(x)+f'(x)\ln g(x)\right)dx$$, q.e.d.

Proposition 5. If $a,b\in R,\ a<b$ and $f,g,h:R\to R$ are continue, such that

$f(a+b-x)=-f(x), g(a+b-x)=g(x), h(a+b-x)=-h(x), \forall x\in R$, then

$$\int_a^b f(x)\left(arctgg(x)\right)\ln\left(1+e^{h(x)}\right)dx=\frac{1}{2}\int_a^b f(x)h(x)arctgg(x)dx.$$

Proof. $I=\int_a^b f(x)\left(arctgg(x)\right)\ln\left(1+e^{h(x)}\right)dx$, where we make the changes

$x=u(t)=a+b-t,\ u'(t)=-1, u(a)=b, u(b)=a$, then

$$I=-\int_b^a f(a+b-t)\left(arctgg(a+b-t)\right)\ln\left(1+e^{h(a+b-t)}\right)dt=$$

$$=-\int_a^b f(x)(arctgg(x))\ln\left(1+e^{-h(x)}\right)dx=-\int_a^b f(x)(arctgg(x))\ln\frac{1+e^{h(x)}}{e^{h(x)}}dx=$$

$$=-\int_a^b f(x)(arctgg(x))\ln\left(1+e^{h(x)}\right)dx+\int_a^b f(x)(arctgg(x))h(x)dx.$$

Hence, $2I=\int_a^b f(x)h(x)arctgg(x)dx$, q.e.d

Proposition 6. If $a,b\in R_+^*$ and $f:R\to R$ is continue and even, then

$$\int_{-a}^a \frac{f(x)}{b^2+arctgx+\sqrt{b^4+arctg^2x}}dx=\frac{1}{b^2}\int_0^a f(x)dx.$$

Proof. Putting $x=u(t)=-t$, $u'(t)=-1, u(a)=-a, u(-a)=a$, then

$$I=\int_{-a}^a \frac{f(x)}{b^2+arctgx+\sqrt{b^4+arctg^2x}}dx=\int_a^{-a}\frac{f(-t)}{b^2-arctgt+\sqrt{b^4+arctg^2t}}(-1)dt=$$

$$=\int_{-a}^a \frac{f(t)}{b^2-arctgt+\sqrt{b^4+arctg^2t}}dt.$$

$$2I=I+I=\int_{-a}^a f(x)\left(\frac{1}{b^2+arctgx+\sqrt{b^4+arctg^2x}}+\frac{1}{b^2-arctgx+\sqrt{b^4+arctg^2x}}\right)dx=$$

$$=\int_{-a}^a f(x)\cdot\frac{2\left(b^2+\sqrt{b^4+arctg^2x}\right)}{\left(b^4+\sqrt{b^4+arctg^2x}\right)^2-arctg^2x}dx=$$

$$=\int_{-a}^a f(x)\cdot\frac{2\left(b^2+\sqrt{b^4+arctg^2x}\right)}{2b^2\left(b^2+\sqrt{b^4+arctg^2x}\right)}dx=\frac{1}{b^2}\int_{-a}^a f(x)dx=\frac{2}{b^2}\int_0^a f(x)dx,$$

$$I=\frac{1}{b^2}\int_0^a f(x)dx.$$

Proposition 7. If $a\in R_+^*$ and $f,g,h:R\to R$ are continue and odd and $k:R\to(1,\infty)$ is continue and even, then

$$\int_{-a}^a f(x)\ln\left((k(x)^{g(x)}+(k(x))^{h(x)}\right)dx=\int_0^a f(x)(g(x)+h(x))\ln k(x)dx.$$

Proof. Putting $x=u(t)=-t$, $u'(t)=-1, u(a)=-a, u(-a)=a$, then

$$I=\int_{-a}^{a} f(x)\ln\left((k(x)^{g(x)}+(k(x))^{h(x)}\right)dx=\int_{a}^{-a} f(-t)\ln\left((k(-t)^{g(-t)}+(k(-t))^{h(-t)}\right)(-1)dt=$$

$$=-\int_{-a}^{a} f(x)\ln\left((k(x)^{-g(x)}+(k(x))^{-h(x)}\right)dx=-\int_{-a}^{a} f(x)\ln\frac{(h(x))^{g(x)}+(k(x))^{h(x)}}{(k(x))^{g(x)+h(x)}}dx=$$

$$=-I+\int_{-a}^{a} f(x)\ln\left(k(x)\right)^{g(x)+h(x)}dx=-I+\int_{-a}^{a} f(x)\left(g(x)+h(x)\right)\ln(k(x))dx,$$

so , $2I=\int_{-a}^{a} f(x)\left(g(x)+h(x)\right)\ln(k(x))dx$, and

$$f(-x)\left(g(-x)+h(-x)\right)\ln(k(-x))=-f(x)\left(-g(x)-h(x)\right)\ln(k(x))=f(x)\left(g(x)+h(x)\right)\ln(k(x)),$$

i.e. $f(x)\left(g(x)+h(x)\right)\ln(k(x))$ is even. Therefore,

$$2I=2\int_{0}^{a} f(x)\left(g(x)+h(x)\right)\ln(k(x))dx,\text{ so } I=\int_{0}^{a} f(x)\left(g(x)+h(x)\right)\ln(k(x))dx,$$

Proposition 8. If $f:R\to R$ is continue such that $f(x)=f(1-x)$, $\forall x\in R$, then

$$\int_{0}^{1}\frac{\sqrt{1-x}+\sqrt{x}}{1+\sqrt{2x}}f(x)dx=\sqrt{2}\cdot\int_{0}^{1}f(x)dx.$$

Proof. Let $x=u(t)=1-t$, $u'(t)=-1, u(0)=1, u(1)=0$. Therefore,

$$I=\int_{0}^{1}\frac{\sqrt{1-x}+\sqrt{x}}{1+\sqrt{2x}}f(x)dx=-\int_{1}^{0}\frac{\sqrt{t}+\sqrt{1-t}}{1+\sqrt{2(1-t)}}f(1-t)dt=\int_{0}^{1}\frac{\sqrt{t}+\sqrt{1-t}}{1+\sqrt{2(1-t)}}f(t)dt.$$

Hence,

$$2I=\int_{0}^{1}\left(\frac{\sqrt{x}+\sqrt{1-x}}{1+\sqrt{2x}}+\frac{\sqrt{x}+\sqrt{1-x}}{1+\sqrt{2(1-x)}}\right)f(x)dx=$$

$$=\int_{0}^{1}(\sqrt{x}+\sqrt{1-x})f(x)\left(\frac{1}{1+\sqrt{2x}}+\frac{1}{1+\sqrt{2(1-x)}}\right)dx=$$

$$=\int_{0}^{1}(\sqrt{x}+\sqrt{1-x})f(x)\cdot\frac{2+\sqrt{2x}+\sqrt{2(1-x)}}{1+\sqrt{2x}+\sqrt{2(1-x)}+2\sqrt{x(1-x)}}dx=$$

$$=\int_{0}^{1}\frac{(\sqrt{x}+\sqrt{1-x})f(x)\sqrt{2}(\sqrt{2}+\sqrt{x}+\sqrt{1-x})}{\sqrt{2}(\sqrt{x}+\sqrt{1-x})+1+2\sqrt{x(1-x)}}dx=$$

$$=\int_0^1 \frac{(\sqrt{x}+\sqrt{1-x})f(x)\sqrt{2}(\sqrt{2}+\sqrt{x}+\sqrt{1-x})}{\sqrt{2}(\sqrt{x}+\sqrt{1-x})+(\sqrt{x}+\sqrt{1-x})^2}dx=$$

$$=\int_0^1 \frac{\sqrt{2}\left(\sqrt{2}+\sqrt{x}+\sqrt{1-x}\right)f(x)}{\sqrt{2}+\sqrt{x}+\sqrt{1-x}}dx=\sqrt{2}\cdot\int_0^1 f(x)dx .$$

Proposition 9. If $a,b\in R,\ c\in R-\{1\}$ and $f,g:R\to R$ are continue such that $f(a+b-x)=cf(x)$, $g(a+b-x)=-g(x)$, $\forall x\in R$, then

$$\int_a^b f(x)\ln\left(1+e^{g(x)}\right)dx=\frac{c}{c-1}\int_a^b f(x)g(x)dx .$$

Proof: $I=\int_a^b f(x)\ln\left(1+e^{g(x)}\right)dx$, $x=u(t)=a+b-t$, $u(a)=b, u(b)=a, u'(t)=-1$, then

$$I=\int_b^a f(a+b-t)\ln\left(1+e^{g(a+b-t)}\right)(-1)dt=\int_a^b cf(t)\ln\left(1+e^{-g(t)}\right)dt=$$

$$=c\cdot\int_a^b f(t)\ln\frac{1+e^{g(t)}}{e^{g(t)}}dt=c\cdot\int_a^b f(t)\ln\left(1+e^{g(t)}\right)dt-c\cdot\int_a^b f(t)\ln e^{g(t)}dt=$$

$$=cI-c\cdot\int_a^b f(x)g(x)dx\Leftrightarrow(1-c)I=-c\cdot\int_a^b f(x)g(x)dx\Leftrightarrow$$

$$\Leftrightarrow(c-1)I=c\cdot\int_a^b f(x)g(x)dx\Leftrightarrow I=\frac{c}{c-1}\int_a^b f(x)g(x)dx .$$

Proposition 10. If $a,\ m\in R_+^*$ şi $f:R\to R$ is continue and odd, then $\int_{\frac{1}{a}}^{a}\frac{f\left(\ln^{2n+1}x\right)}{\left(1+x^{2m}\right)^{\frac{1}{m}}}dx=0$.

Proof. Putting $x=u(t)=\frac{1}{t},\ u'(t)=-\frac{1}{t^2},\ u(a)=\frac{1}{a},\ u\left(\frac{1}{a}\right)=a$ we obtain

$$I=\int_{\frac{1}{a}}^{a}\frac{f\left(\ln^{2n+1}x\right)}{\left(1+x^{2m}\right)^{\frac{1}{m}}}dx=\int_a^{\frac{1}{a}}\frac{f\left(-\ln^{2n+1}t\right)}{\left(1+\frac{1}{t^{2m}}\right)^{\frac{1}{m}}}\left(-\frac{1}{t^2}\right)dt=$$

$$=\int_{\frac{1}{a}}^{a}\frac{f\left(-\ln^{2n+1}t\right)}{\frac{\left(1+t^{2m}\right)^{\frac{1}{m}}}{t^2}}\cdot\frac{1}{t^2}dt=-\int_{\frac{1}{a}}^{a}\frac{f\left(\ln^{2n+1}t\right)}{\left(1+t^{2m}\right)^{\frac{1}{m}}}dt=-I ,$$

Hence, $I=0$.

References: **[*] Romanian Mathematical Magazine-www.ssmrmh.ro**

ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-(IX)

By Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$\sum \boldsymbol{m_a r_a} \le \frac{\boldsymbol{3R}}{\boldsymbol{2r}}(\boldsymbol{2R^2}+\boldsymbol{r^2})$$

Proposed by Marian Ursărescu-Romania

Solution by Marin Chirciu-Romania

Lemma. 2) In ΔABC the following relationship holds:

$$\sum \boldsymbol{m_a r_a} \le \boldsymbol{8R^2-3Rr+r^2}$$

Proof. Using well-known identities:

$$\sum m_a^2 = \frac{3}{4}\sum a^2;\ \sum a^2 = 2(s^2-r^2-4Rr),$$

$$\sum m_a^2 = \frac{3}{2}(s^2-r^2-4Rr), \sum r_a^2 = (4R+r)^2-2s^2$$

From AM-GM, we have: $m_a r_a \le \frac{m_a^2+r_a^2}{2} \Rightarrow$

$$\sum m_a r_a \le \sum \frac{m_a^2+r_a^2}{2} = \frac{1}{2}\left(\sum m_a^2+\sum r_a^2\right) =$$

$$= \frac{1}{2}\left[\frac{3}{2}(s^2-r^2-4Rr)+(4R+r)^2-2s^2\right] = \frac{1}{4}(32R^2+4Rr-r^2-s^2) \le$$

$$\overset{Gerretsen}{\le} \frac{1}{4}32R^2+4Rr-r^2-16Rr+5r^2) = \frac{1}{4}(32R^2-12Rr+4r^2) =$$

$= 8R^2-3Rr+r^2$.Let's get back to the main problem.

Using Lemma, it is suffices to prove that:

$$8R^2-3Rr+r^2 \le \frac{3R}{2r}(2R^2+r^2) \Leftrightarrow 2r(8R^2-3Rr+r^2) \le 3R(2R^2+r^2)$$

$\Leftrightarrow 6R^3-16R^2r+9Rr^2-2r^3 \ge 0 \Leftrightarrow (R-2r)(6R^2-4Rr+r^2) \ge 0$ which is true from $R \ge 2r(Euler)$. Remark. The problem it can be developed.

3) In ΔABC the following relationship holds:

$$\sum \boldsymbol{m_a r_a} \le \frac{\boldsymbol{1}}{\boldsymbol{3}}(\boldsymbol{4R}+\boldsymbol{r})^{\boldsymbol{2}}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Triplets $(m_a, m_b, m_c), (r_a, r_b, r_c)$ are reverse ordered, then applying Chebyshev's inequality, we get:

$$\sum m_a r_a \le \frac{1}{3}\sum m_a \sum r_a \le \frac{1}{3}(4R+r)(4R+r) = \frac{1}{3}(4R+r)^2$$

which follows from $\sum m_a \le 4R + r$ and $\sum r_a \le 4R + r$.

Equality holds if and only if triangle is equilateral.

4) In ΔABC the following relationship holds:

$$\sum \boldsymbol{m_a r_a} \le \frac{\mathbf{1}}{\mathbf{3}}(\mathbf{4R}+\boldsymbol{r})^{\mathbf{2}} \le \mathbf{8R^2 - 3Rr + r^2} \le \frac{\mathbf{3R}}{\mathbf{2r}}(\mathbf{2R^2 + r^2})$$

Proposed by Marin Chirciu-Romania

Solution by proposer See inequality (3), $\frac{1}{3}(4R+r)^2 \overset{(1)}{\le} 8R^2 - 3Rr + r^2$ and $8R^2 - 3Rr + r^2 \overset{(2)}{\le} \frac{3R}{2r}(2R^2 + r^2)$, where

$$(1) \Leftrightarrow \frac{1}{3}(4R+r)^2 \le 8R^2 - 3Rr + r^2 \Leftrightarrow (4R+r)^2 \le 3(8R^2 - 3Rr + r^2) \Leftrightarrow$$

$8R^2 - 17Rr + 2r^2 \ge 0 \Leftrightarrow (R-2r)(8R-r) \ge 0$, which is true from $R \ge 2r (Euler)$.

$(2) \Leftrightarrow 2r(8R^2 - 3Rr + r^2) \le 3R(2R^2 + r^2) \Leftrightarrow (R-2r)(6R^2 - 4Rr + r^2) \ge 0$, which is true from $R \ge 2r (Euler)$.

Equality holds if and only if triangle is equilateral.

5) In ΔABC the following relationship holds:

$$\sum \boldsymbol{m_a r_a} \ge \mathbf{27r^2}$$

Proposed by Marin Chirciu-Romania

Solution by proposer Using AM-GM inequality: $\prod m_a \ge \prod r_a \ge 27r^3$, we get:

$$\sum m_a r_a \ge 3\sqrt[3]{\prod m_a \prod r_a} \ge 3\sqrt[3]{27r^3 \cdot 27R^3} = 3 \cdot 3r \cdot 3r = 27r^2$$

Equality holds if and only if triangle is equilateral.

6) In ΔABC the following relationship holds:

$$\mathbf{27r^2} \le \sum \boldsymbol{m_a r_a} \le \frac{\mathbf{1}}{\mathbf{3}}(\mathbf{4R}+\boldsymbol{r})^{\mathbf{2}}$$

Proposed by Marin Chirciu-Romania

Solution by proposer For LHS, we have:

Using AM-GM inequality: $\prod m_a \ge \prod r_a \ge 27r^3$, we get:

$$\sum m_a r_a \ge 3\sqrt[3]{\prod m_a \prod r_a} \ge 3\sqrt[3]{27r^3 \cdot 27R^3} = 3 \cdot 3r \cdot 3r = 27r^2$$

Equality holds if and only if triangle is equilateral. For RHS, we have:

Triplets $(m_a, m_b, m_c), (r_a, r_b, r_c)$ are reverse ordered, then applying Chebyshev's inequality, we get:

$$\sum m_a r_a \le \frac{1}{3}\sum m_a \sum r_a \le \frac{1}{3}(4R+r)(4R+r) = \frac{1}{3}(4R+r)^2$$

which follows from $\sum m_a \le 4R+r$ and $\sum r_a \le 4R+r$.

Equality holds if and only if triangle is equilateral.

Reference: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

PROBLEMS DEDICATED TO 140-TH ANIVERSARY OF NATIONAL COLLEGE "FRAȚII BUZEȘTI"-CRAIOVA-ROMANIA

1. **Make an increasing order:**
   a. $\frac{10^{138}+1}{10^{139}+1}, \frac{10^{139}+1}{10^{140}+1}$
   b. $\frac{10^{1880}+1}{10^{1881}+1}, \frac{10^{1881}+1}{10^{1882}+1}$
   c. $\frac{10^{2020}+1}{10^{2021}+1}, \frac{10^{2021}+1}{10^{2022}+1}$

Proposed by Lucian Tuțescu, Camelia Dană -Romania

2. **If $x, y, z > 0, x+y+z=3$ then:**

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x} \le \frac{6}{xy+yz+zx}$$

Proposed by Dorina Goiceanu, Simona Dascălu-Romania

3. **Find the reminder of $\Omega = 1+2+2^2+\cdots+2^{2022}$ at $3, 5, 15$.**

Proposed by Dîrnu Daniela, Camelia Dană -Romania

4. **Let H −be the orthocenter of ΔABC. If $AB = HC, AC = HB$ then find:**

$$m(\measuredangle A), m(\measuredangle B), m(\measuredangle C).$$

Proposed by Rareș Tudorașcu-Romania

5. **If $x, y \in \mathbb{R},\ x-y=1$ then find $\Omega = min(x^3-y^3-xy)$ and values of x, y where Ω is reached.**

Proposed by Carina Maria Viespescu-Romania

6. **In acute $\Delta ABC, h_a > h_b, h_a > h_c, h_a = m_b$. Prove that:**

$$m(\measuredangle ABC) < 60^\circ$$

Proposed by Iulia Sanda, Ramona Nălbaru– Romania

7. **Let be the sequence:**

$$a_1 = 5, a_{n+1} = \begin{cases} \frac{a_n}{2}, n - even \\ \frac{a_n+51}{2}, n - odd \end{cases}, n \in \mathbb{N}, n \geq 1$$

Find: $a_{140}, a_{1822}, a_{2022}$.

Proposed by Ileana Didu,Dorina Goiceanu– Romania

ABOUT UPPER AND LOWER BOUNDS OF RECIPROCAL FIBONACCI AND LUCAS SERIES

By Seyran Ibrahimov, Ahmad Issa-Azerbaijan

Abstract: In this paper, introduces upper and lower bounds for $\sum_{n=1}^{\infty} \frac{1}{(F_1 \cdot F_2 \cdot \ldots \cdot F_n)^{\frac{1}{n}}}$ and $\sum_{n=1}^{\infty} \frac{1}{(L_1 \cdot L_2 \cdot \ldots \cdot L_n)^{\frac{1}{n}}}$ series.

Keywords: Fibonacci Numbers, Lucas Numbers, Reciprocal Fibonacci series, Reciprocal Lucas series.

1. Introduction

The Fibonacci numbers were described in work by Italian mathematician Leonardo Fibonacci, which has a lot of applications in cryptology along with mathematics. Many studies have been done by mathematicians about Fibonacci numbers.Fibonacci numbers are strongly related to Lucas numbers which $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}, n \geq 2$, $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}, n \geq 2$ are Fibonacci and Lucas numbers, respectively. These n^{th} numbers can be found by Binet's formula given as [1].

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}, L_n = \varphi^n + (-\varphi)^{-n}, \varphi = \frac{\sqrt{5}+1}{2}$$

2. Preliminaries

Lemma [1], [2] if F_n and L_n are Fibonacci and Lucas numbers, respectively. Then the following inequalities are satisfied:

$$\varphi^{n-1} \leq F_n \leq \varphi^n, \quad \varphi^{n-1} \leq L_n \leq 2\varphi^n$$

3. Main Results

Theorem 2.1 If F_n are Fibonacci numbers then the following inequality holds:

$$\frac{1}{\sqrt{\varphi}(\sqrt{\varphi}-1)} < \sum_{n=1}^{\infty} \frac{1}{(F_1 \cdot F_2 \cdot \ldots \cdot F_n)^{\frac{1}{n}}} < \frac{\sqrt{\varphi}}{\sqrt{\varphi}-1}; \quad (1)$$

Proof. We know that for all $n \geq 1, F_n \geq \varphi^{n-1}$ then we obtain

$$F_1 \cdot F_2 \cdot \ldots \cdot F_n \geq \varphi^0 \cdot \varphi^1 \cdot \ldots \cdot \varphi^{n-1} = \varphi^{\frac{(n-1)n}{2}}$$

that means

$$\sum_{n=1}^{\infty} \frac{1}{(F_1 \cdot F_2 \cdot \ldots \cdot F_n)^{\frac{1}{n}}} < \frac{\sqrt{\varphi}}{\sqrt{\varphi}-1}; \qquad (2)$$

and for all $n \geq 1, F_n \leq \varphi^n$ then we obtain $F_1 \cdot F_2 \cdot \ldots \cdot F_n \geq \varphi^0 \cdot \varphi^1 \cdot \ldots \cdot \varphi^n = \varphi^{\frac{(n+1)n}{2}}$ that means

$$\sum_{n=1}^{\infty} \frac{1}{(F_1 \cdot F_2 \cdot \ldots \cdot F_n)^{\frac{1}{n}}} < \frac{1}{\sqrt{\varphi}(\sqrt{\varphi}-1)}; \qquad (3)$$

From (2) and (3) we obtain:

$$\frac{1}{\sqrt{\varphi}(\sqrt{\varphi}-1)} < \sum_{n=1}^{\infty} \frac{1}{(F_1 \cdot F_2 \cdot \ldots \cdot F_n)^{\frac{1}{n}}} < \frac{\sqrt{\varphi}}{\sqrt{\varphi}-1};$$

Theorem 2.2 If L_n are Lucas numbers then the following inequality holds:

$$\frac{\mathbf{1}}{\sqrt{2\boldsymbol{\varphi}}(\sqrt{2\boldsymbol{\varphi}}-\mathbf{1})} < \sum_{n=1}^{\infty} \frac{\mathbf{1}}{(\boldsymbol{L_1} \cdot \boldsymbol{L_2} \cdot \ldots \cdot \boldsymbol{L_n})^{\frac{1}{n}}} < \frac{\sqrt{\boldsymbol{\varphi}}}{\sqrt{\boldsymbol{\varphi}}-\mathbf{1}}; \qquad (\mathbf{4})$$

Proof. We know that for all $n \geq 1, L_n \geq \varphi^{n-1}$, then we obtain

$$L_1 \cdot L_2 \cdot \ldots \cdot L_n \geq \varphi^0 \cdot \varphi^1 \cdot \ldots \cdot \varphi^{n-1} = \varphi^{\frac{(n-1)n}{2}} \text{ that means}$$

$$\sum_{n=1}^{\infty} \frac{1}{(L_1 \cdot L_2 \cdot \ldots \cdot L_n)^{\frac{1}{n}}} < \frac{\sqrt{\varphi}}{\sqrt{\varphi}-1}; \qquad (5)$$

and for all $n \geq 1, L_n < 2\varphi^n$ then we obtain

$$L_1 \cdot L_2 \cdot \ldots \cdot L_n \leq 2\varphi^2 \cdot 2\varphi^2 \cdot \ldots \cdot 2\varphi^n = (2\varphi)^{\frac{n(n+1)}{2}}$$

$$\sum_{n=1}^{\infty} \frac{1}{(L_1 \cdot L_2 \cdot \ldots \cdot L_n)^{\frac{1}{n}}} > \frac{1}{\sqrt{2\varphi}(\sqrt{2\varphi}-1)}; \qquad (6)$$

From (5) and (6) we obtain:

$$\frac{1}{\sqrt{2\varphi}(\sqrt{2\varphi}-1)} < \sum_{n=1}^{\infty} \frac{1}{(L_1 \cdot L_2 \cdot \ldots \cdot L_n)^{\frac{1}{n}}} < \frac{\sqrt{\varphi}}{\sqrt{\varphi}-1};$$

References:

[1]. Thomas Koshy. ***Fibonacci and Lucas Numbers with Applications**, Volume 2*. John Wiley & Sons, 2019.

[2]. Runnan Liu and Andrew YZ Wang. ***Sums of products of two reciprocal fibonacci numbers. Advances in Difference Equations***, 2016(1):1–26, 2016.

ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-(III)

By Marin Chirciu – Romania

Let a, b, c be the lengths of the sides of ABC triangle with inradius r and circumradius R. Show that:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \le \frac{3\sqrt{6}R}{4r}\sqrt{R(R-r)}$$

Proposed by George Apostolopoulos –Messolonghi- Greece

Solution: *We prove the following lemma:*

Lemma:

1) In ΔABC the following relationship holds:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} = \frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}$$

Proof:

We have $\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} = \frac{\sum a^2(a+b)(a+c)}{\prod(b+c)} = \frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}$, *which follows from the known identities in triangle* $\sum a^2(a+b)(a+c) = 4s(s^2-3r^2-4Rr)$ *and*

$$\prod(b+c) = 2s(s^2+r^2+2Rr)$$

Let's get back to the main problem.

Using the Lemma, we write the inequality: $\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr} \le \frac{3\sqrt{6}R}{4r}\sqrt{R(R-r)}$

Using Mitrinovic's inequality $s \le \frac{3R\sqrt{3}}{2}$ *it suffices to prove that:*

$$\frac{3R\sqrt{3}(s^2-3r^2-4Rr)}{s^2+r^2+2Rr} \le \frac{3\sqrt{6}R}{4r}\sqrt{R(R-r)} \Leftrightarrow \frac{s^2-3r^2-4Rr}{s^2+r^2+2Rr} \le \frac{1}{4r}\sqrt{2R(R-r)} \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{s^2-3r^2-4Rr}{s^2+r^2+2Rr}\right)^2 \le \frac{2R(R-r)}{16r^2} \Leftrightarrow$$

$$\Leftrightarrow s^2[s^2(R^2-Rr-8r^2)+r(4R^3-2R^2r+62Rr^2+48r^3)]+$$

$$+r^2(4R^4-128R^3r-192R^2r^2-192Rr^3-72r^4) \ge 0$$

We distinguish the following cases:

Case 1). If $[s^2(R^2 - Rr - 8r^2) + r(4R^3 - 2R^2r + 62Rr^2 + 48r^3)] \geq 0$*, the inequality is obvious.*

Case 2). If $[s^2(R^2 - Rr - 8r^2) + r(4R^3 - 2R^2r + 62Rr^2 + 48r^3)] < 0$*, the inequality rewrites its self:*

$$r^2(4R^4 - 128R^3r - 192R^2r^2 - 192Rr^3 - 72r^4) \geq$$

$$\geq s^2[s^2(8r^2 + Rr - R^2) - r(4R^3 - 2R^2r + 62Rr^2 + 48r^3)]$$

which follows from Gerretsen's inequality $s^2 \leq 4R^2 + 4Rr + 3r^2$*. It remains to prove that:*

$$r^2(4R^4 - 128R^3r - 192R^2r^2 - 192Rr^3 - 72r^4) \geq$$

$$\geq (4R^2 + 4Rr + 3r^2)[(4R^2 + 4Rr + 3r^2)(8r^2 + Rr - r^2) - r(4R^3 - 2R^2r + 62Rr^2 + 48r^3)] \Leftrightarrow$$

$\Leftrightarrow (R - 2r)(4R^4 + 16R^3r + 5R^2r^2 + 5Rr^3 + 2r^5) \geq 0$*, obviously with Euler's inequality* $R \geq 2r$*. Equality holds if and only if the triangle is equilateral.*

Remark: *We can strengthen the inequality:*

2) In $\boldsymbol{\Delta ABC}$ ***the following relationship holds:***

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \leq \frac{3R^2\sqrt{3}}{4r}$$

Proposed by Marin Chirciu – Romania

Solution: *Using the Lemma the inequality rewrites:* $\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr} \leq \frac{3R^2\sqrt{3}}{4r}$.

Using Mitrinovic's inequality $s \leq \frac{3R\sqrt{3}}{2}$ *it suffices to prove that:*

$$\frac{3R\sqrt{3}(s^2 - 3r^2 - 4Rr)}{s^2 + r^2 + 2Rr} \leq \frac{3R^2\sqrt{3}}{4r} \Leftrightarrow \frac{s^2 - 3r^2 - 4Rr}{s^2 + r^2 + 2Rr} \leq \frac{R}{4r} \Leftrightarrow$$

$\Leftrightarrow s^2(R - 4r) + r(2R^2 + 17Rr + 12r^2) \geq 0$*.We distinguish the following cases:*

Case 1) If $R - 4r \geq 0$*, inequality is obvious.*

Case 2). If $R - 4r < 0$*, the inequality rewrites itself:*

$r(2R^2 + 17Rr + 12r^2) \geq s^2(4r - R)$ *which follows from Gerretsen's inequality:*

$s^2 \leq 4R^2 + 4Rr + 3r^2$*. It remains to prove that:*

$$r(2R^2 + 17Rr + 12r^2) \geq (4R^2 + 4Rr + 3r^2)(4r - R) \Leftrightarrow 2R^2 - 5Rr + 2r^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(2R-r)\geq 0$, obviously from Euler's inequality $r\geq 2r$. Equality holds if and only if the triangle is equilateral. ***Remark.****Inequality 2) is stronger than the inequality from Problem 4462 from Crux Mathematicorum, Vol 45, Nr. 7.*

3) In ΔABC the following relationship holds:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\leq\frac{3R^2\sqrt{3}}{4r}\leq\frac{3\sqrt{6}R}{4r}\sqrt{R(R-r)}$$

Solution: *See inequality 2) and $\frac{3R^2\sqrt{3}}{4r}\leq\frac{3\sqrt{6}R}{4r}\sqrt{R(R-r)}\Leftrightarrow R\leq\sqrt{2R(R-r)}\Leftrightarrow R^2\leq 2R(R-r)\Leftrightarrow R\geq 2r$ (Euler's inequality). Equality holds if and only if the triangle is equilateral.* **Remark:** *Let's find an inequality having an opposite sense:*

4) In ΔABC the following relationship holds: $\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\geq s$

Solution: *Using Lemma we write the inequality: $\frac{2s(s^2-3r^2-4Rr)}{s^2+r^2+2Rr}\geq s\Leftrightarrow s^2\geq 10Rr+7r^2$, which follows from Gerretsen's inequality $s^2\geq 16Rr-5r^2$. It remains to prove that:*

$16Rr-5r^2\geq 10Rr+7r^2\Leftrightarrow R\geq 2r$ (Euler's inequality). Equality holds if and only if the triangle is equilateral. ***Remark:*** *We write the double inequality:*

5) In ΔABC the following relationship holds: $s\leq\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\leq\frac{3R^2\sqrt{3}}{4r}$

Solution: *See inequalities 2) and 4). Equality holds if and only if the triangle is equilateral.*

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

ABOUT THE PROBLEM 3326 FROM CRUX MATHEMATICORUM

By D.M.Bătinețu-Giurgiu, Daniel Sitaru, Florică Anastase-Romania

In Crux Mathematicorum Magazine, Vol. 34, No. 3 Mihaly Bencze proposed the problem $3326(a)$:

Let $a,b,c>0$, then:

$$\prod_{cyc}(a^2+2)+4\prod_{cyc}(a^2+1)\geq 6(a+b+c)^2;\qquad (M.B.)$$

We will to developed this problem.

Proposition 1: If $u,v,x>0$ then: $(u^2+x)(v^2+x)\geq\frac{3}{4}x((u+v)^2+x);\qquad (1)$

Proof: We have: $(u^2+x)(v^2+x)\geq\frac{3}{4}x((u+v)^2+x)\Leftrightarrow$

$$4u^2v^2+4x(u^2+v^2)+4x^2\geq 3x(u^2+v^2)+6xuv+3x^2\Leftrightarrow$$

$$4u^2v^2-4xuv+x^2+x(u^2+v^2-2uv)\geq 0\Leftrightarrow(2uv-x)^2+x(u-v)^2\geq 0$$

Equality holds for $2uv = x$ and $u = v = \sqrt{\frac{x}{2}}$.

Proposition 2: If $v, w, x > 0$ then: $\boldsymbol{(v^2 + x)(w^2 + x) \geq x(v + w)^2}$; $(\mathbf{2})$

Proof: We have: $(v^2 + x)(w^2 + x) \geq x(v + w)^2 \Leftrightarrow$

$$v^2w^2 + x(v^2 + w^2) + x^2 \geq x(v^2 + w^2) + 2xvw \Leftrightarrow$$

$$v^2w^2 - 2xvw + x^2 \geq 0 \Leftrightarrow (vw - x)^2 \geq 0.$$ Equality holds for $vw = x$.

Proposition 3. If $u, v, w, x > 0$ then: $\boldsymbol{(u^2 + x)(v^2 + x)(w^2 + x) \geq \frac{3}{4}x^2(u + v + w)^2}$; $(\mathbf{3})$

Proof: We have: $(u^2 + x)(v^2 + x)(w^2 + x) \overset{(1)}{\geq} \frac{3}{4}x((u + v)^2 + x)(w^2 + x) \overset{(2)}{\geq}$

$$\overset{(2)}{\geq} \frac{3}{4}x^2\big((u + v) + w\big)^2 = \frac{3}{4}x^2(u + v + w)^2 \geq \frac{9}{4}x^2(uv + vw + wu)$$

Theorem. If $a, b, c, m, n, t, s > 0$ and $p, q > 0, p + q > 0$ then:

$$\boldsymbol{p(m^2a^2 + t)(m^2b^2 + t)(m^2c^2 + t) + q(n^2a^2 + s)(n^2b^2 + s)(n^2c^2 + s) \geq}$$

$$\geq \frac{3}{4}(m^2t^2p + b^2s^2q)(a + b + c)^2 \geq \frac{9}{4}(m^2t^2p + n^2s^2q)(ab + bc + ca); \qquad (4)$$

Proof: In the relationship (3), taking $u = ma, v = mb, w = mc, x = t$, it follows:

$$(m^2a^2 + t)(m^2b^2 + t)(m^2c^2 + t) \geq \frac{3}{4}t^2(ma + mb + mc)^2 =$$

$$= \frac{3}{4}m^2t^2(a + b + c)^2 \geq \frac{9}{4}m^2t^2(ab + bc + ca); \qquad (5)$$

Analogous, in relation (3), taking $u = na, v = nb, w = nc$ and $x = s$, we get:

$$(n^2a^2 + s)(n^2b^2 + s)(n^2c^2 + s) \geq \frac{3}{4}s^2(na + nb + nc)^2 =$$

$= \frac{3}{4}n^2s^2(a + b + c)^2 \geq \frac{9}{4}n^2s^2(ab + bc + ca);$ (6). From (5) and (6) we get:

$$p(m^2a^2 + t)(m^2b^2 + t)(m^2c^2 + t) + q(n^2a^2 + s)(n^2b^2 + s)(n^2c^2 + s) \geq$$

$$\geq \frac{3}{4}m^2pt^2(a + b + c)^2 + \frac{3}{4}n^2qs^2(a + b + c)^2 = \frac{3}{4}(m^2t^2p + n^2s^2q)(a + b + c)^2 \geq$$

$$\geq \frac{9}{4}(m^2t^2p + n^2s^2q)(ab + bc + ca)$$

If in (4), we take $t = 2, s = 1, m = n = 1, p = 1, q = 4$ we get:

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) + 4(a^2 + 1)(b^2 + 1)(c^2 + 1) \geq$$

$$\geq \frac{3}{4} \cdot 4(1 \cdot 4 \cdot 1 + 1 \cdot 1 \cdot 4)(a+b+c)^2 = 6(a+b+c)^2 \geq 18(ab+bc+ca) \text{ i.e. } (M.B.)$$

If in (4) we take $q = 0, m = 1, p = 1$ we get:

$$(a^2+t)(b^2+t)(c^2+t) \geq \frac{3}{4}t^2(a+b+c)^2 \geq \frac{9}{4}t^2(ab+bc+ca); \qquad (7)$$

from which taking $t \to t^2$, results:

$$(a^2+t^2)(b^2+t^2)(c^2+t^2) \geq \frac{3}{4}t^4(a+b+c)^2 \geq \frac{9}{4}t^4(ab+bc+ca); \qquad (A.A.)$$

i.e. Arkady M. Alt inequality published in [1].

If in (7) we take $t = 2$, we obtain: $(a^2+2)(b^2+2)(c^2+2) \geq 3(a+b+c)^2 \geq 9(ab+bc+ca)$; $(H.L.)$ i.e. Hojoo's inequality proposed to APMO,2004.

REFERENCES: [1]. Alt M. Arkady-***ABOUT AN INEQUALITY FROM APMO, 2004-NEW SOLUTION AND GENERALIZATIONS***, *Octogon Mathematical Magazine, Vol.2,No.1, April 2019, pages 228-232*

PROOF WITHOUT WORDS:

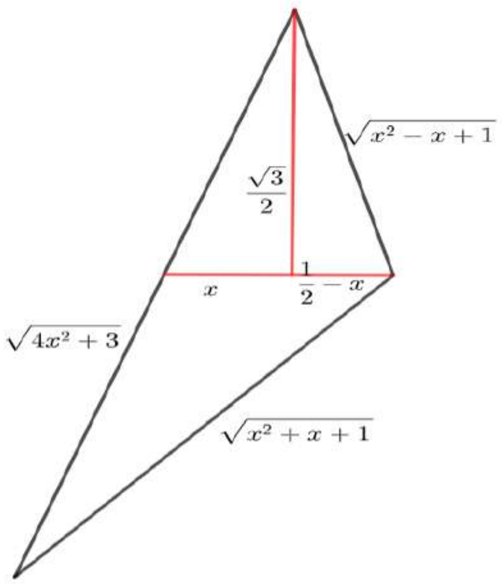
THE CHIRIȚĂ-SITARU-NĂNUȚI THEOREM AND ITS CONVERSE

By Martin Celli-Mexico

Theorem. The Chiriță triangles (triangles of sides $\sqrt{4x^2+3}, \sqrt{x^2-x+1}, \sqrt{x^2+x+1}$) are exactly the triangles of median $\frac{1}{2}$ and area $\frac{\sqrt{3}}{4}$.

— — —**Chiriță triangle**

— — —**Triangle of median $\frac{1}{2}$ and area $\frac{\sqrt{3}}{4}$**



The left-right implication is equivalent to propositions 2 and 3 of D. Sitaru, C, Nănuți. Metric relationships in Chiriță's triangle-Romanian Mathematical Magazine, November 2019.

ABOUT A RMM INEQUALITY-(XI)

By Marin Chirciu -Romania

1) If $a,b,c>0$ such that $(a+b)^3+(b+c)^3+(c+a)^3=24$ then:

$$\sum(b+c)(b^2+c^2)\geq 12$$

Proposed by Daniel Sitaru-Romania

Solution: We prove: **Lemma:**

2) If $x,y>0$ then:

$$x^2+y^2\geq\frac{(x+y)^2}{2}$$

Proof: We have $x^2+y^2\geq\frac{(x+y)^2}{2}\Leftrightarrow(x-y)^2\geq 0$, obviously with equality for $x=y$.

Let's get back to the main problem. Using the Lemma we obtain:

$$Ms=\sum(b+c)(b^2+c^2)\overset{Lemma}{\geq}\sum(b+c)\frac{(b+c)^2}{2}=\frac{1}{2}\sum(b+c)^3=\frac{1}{2}\cdot 24=12=Md$$

Equality holds if and only if $a=b=c=1$. **Remark:** The problem can be developed.

3) If $a,b,c>0$ such that $(a+b)^{n+1}+(b+c)^{n+1}+(c+a)^{n+1}=6\cdot 2^n, n\in\mathbb{N}, n\geq 2$ then:

$$\sum(b+c)(b^n+c^n)\geq 3\cdot 2^n$$

Marin Chirciu

Solution: We prove: **Lemma:**

4) If $x,y>0$ and $n\in\mathbb{N}, n\geq 2$ then:

$$x^n+y^n\geq\frac{(x+y)^n}{2^{n-1}}$$

Proof: Using Holder's inequality we obtain: $x^n+y^n=\frac{x^n}{1}+\frac{y^n}{1}\geq\frac{(x+y)^n}{2^{n-2}(1+1)}=\frac{(x+y)^n}{2^{n-1}}\Leftrightarrow(x-y)^2\geq 0$, obvious with equality for $x=y$. Let's get back to the main problem. Using the Lemma we obtain:

$$Ms=\sum(b+c)(b^n+c^n)\overset{Lemma}{\geq}\sum(b+c)\frac{(b+c)^n}{2^{n-1}}=\frac{1}{2^{n-1}}\sum(b+c)^{n+1}=$$

$$= \frac{1}{2^{n-1}} \cdot 6 \cdot 2^n = 12 = Md$$

Equality holds if and only if $a = b = c = 1$. **Note: For $n = 2$ we obtain the proposed problem by Daniel Sitaru in RMM 12/2020.**

Reference: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

WHY THE "RAVI SUBSTITUTIONS" SHOULD BE CALLED "VOICULESCU SUBSTITUTIONS"

By Dorin Mărghidanu, Marian Dincă-Romania

In 1964, Dan Voiculescu - then a young student, used a substitution which would later be called a 'Ravi substitution' - just six years before Ravi was born!.. Murray Klamkin also used this substitution at least ten years before Ravi. In order to restore the historical truth, a new name is required for this transformation.

- ***Keywords and phrases***:*Ravi substitution , Voiculescu substitution , principle of duality, inequalities in triangles*
- **2020 *Mathematics Subject Classification* : 01-02 , 01A60** , **26D15**

The notion of *Ravi substitution* is relatively often encountered and used in the mathe- matical practice of the last 30 years. This substitution is contained in the following very simple equivalence , but which is very useful in applications:

Theorem : *The numbers a, b, c represents the lengths of the sides of a triangle if and only if there are positive real numbers x, y, z such that $a = y + z, b = z + x, c = x + y$.*

Proof: If a, b, c represent the lengths of the sides of a triangle, then: $a + b > c, b + c > a, c + a > b$. The system of equations $x + y = c, y + z = a, z + x = b$ has the (unique) solution:

$$x = \frac{-a + b + c}{2}, y = \frac{a - b + c}{2}, z = \frac{a + b - c}{2}$$

so with the condition of triangularity it follows: $x > 0, y > 0, z > 0$.

Reciprocally, if $x > 0, y > 0, z > 0$, then we notice that:

$$a + b = x + y + 2z \overset{(z>0)}{>} x + y = c \Rightarrow a + b > c$$

$$b + c = y + z + 2x \overset{(x>0)}{>} y + z = a \Rightarrow b + c > a$$

$$c + a = z + x + 2y \overset{(y>0)}{>} z + x = b \Rightarrow c + a > b$$

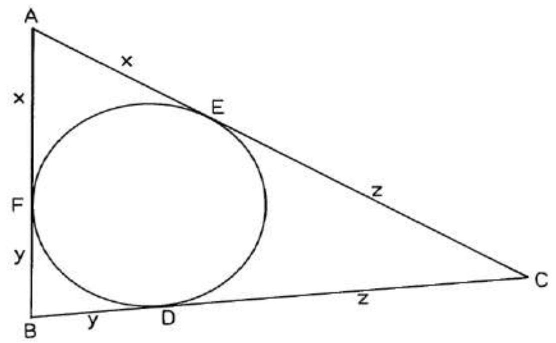
hence a, b, c can be the lengths of the sides of a triangle.

- ***Remark 1.***

With the substitution of the theorem above we can transfer properties (inequalities, identities or other relations) regarding triangles to equivalent properties in positive real numbers - and vice versa. The first implication is obviously interesting , because the properties that take place in the constrained conditions of the triangle are transferred to the properties that take place for positive numbers - for which there are many solving techniques provided by mathematics for a very long time.

- ***Remark 2.***

A very suggestive geometric interpretation of the substitution in the theorem is presented in the figure below. Geometrically x, y, z are the lengths of the segments on the sides of a triangle determined by the tangent points of the circle inscribed in the triangle (using the fact that the tangents from an outer point are equal) (see for example, [5]).

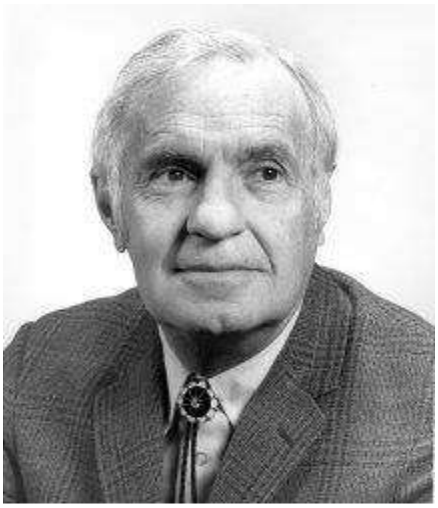


The name of this substitution comes from the name of the canadian mathematician of indian origin *Vakil D. Ravi ,* currently a professor at Stanford University . In adolescence, *Ravi* participated in various school competitions , including three international Olympics, where he won a silver medal (1986) and two gold medals (1987- perfect score and 1988) .

Vakil D. Ravi

He also had four participations at the Putnam competition and then was the training coordinator in preparation for the Putnam competition at Stanford University. He was one of the founders of the canadian journal *Mathematical Mayhem* (later merged with

Crux Mathematicorum).During his youth, *Ravi* often used the substitution that bears his name and successfully converted inequalities in triangle geometry into general inequalities. However, we have no knowledge of any work or reference by *Ravi* dedicated to this substitution. The name *Ravi substitution* was originally used locally - in canadian (olympic) math circles, then proliferated around the world. But this type of substitution was used at least 10 years before (even several times) by the well-known mathematician *Murray Klamkin* (1921-2004). For example, in each of the papers [1]-[4] is briefly presented the equivalence - as the one presented here, in the Theorem at the beginning of this paper .The equivalence relation is called *duality* (or *principle of duality*) by *Klamkin* and each of the equivalent inequalities is considered dual to the other. In addition to being one of the greatest problem-solvers in the world (or perhaps because of that) , *Klamkin* was also the coach of the United States IOM team from 1978-1985, and was the editor and coordinator at the *Olympiad corner* of the *Crux Mathematicorum* and then at other magazines. He is also the author of some interesting mathematical articles and notes, as well as some problem-solving books for competitions .



Murray Klamkin

Regarding the *substitution of Ravi*, in [4] *Klamkin* says verbatim that, "this transformation was known and used before he was born". And indeed (obviously, without *Klamkin*'s knowledge) , about six years before the birth of *Vakil Ravi* (born in february 22, 1970) , -more precisely in february 1964, in Romania, *Dan V. Voiculescu* -then a student for only 15 years. publishes a short mathematical note [6] , which uses exactly the substitution that is now called the *Ravi substitution* . Being a short work and perhaps harder to find now, we reproduce it below.

Asupra unei inegalități într-un triunghi

"Condiția necesară și suficientă pentru ca a, b, c să fie laturi ale unui triunghi este să existe x, y, z pozitivi, astfel ca:

$$\begin{cases} a = x + y \\ b = x + z\,; \\ c = y + z \end{cases} \qquad (1)$$

Într-adevăr, scriind că x, y, z sunt pozitivi avem:

$$x = \frac{a+b-c}{2} \geq 0; y = \frac{a-b+c}{2} \geq 0; z = \frac{-a+b+c}{2} \geq 0$$

Condiții necesare și suficiente pentru ca a, b, c să fie laturi ale unui triunghi. Rezultă deci, posibilitatea de a obține cu ajutorul formulelor (1) din orice inegalitate în care figurează a, b, c (laturi ale unui triunghi). Rezultă de asemenea, că pentru ca o inegalitate în a, b, c să fie adevărată este necesar și suficient ca inegalitatea în x, y, z obținută prin formulele (1) să fie adevărată. Spre a ilustra această metodă vom da în cele ce urmează, câteva exemple.

a) Cunoscuta inegalitate între raza cercului circumscris și raza cercului înscris unui triunghi. Din formulele uzuale în care am înlocuit conform lui (1), avem:

$$\frac{R}{r} = \frac{(x+y)(y+z)(z+x)}{4xyz} \geq \frac{8xyz}{4xyz} = 2$$

unde am aplicat fiecărei paranteze în parte, inegalitatea $u + v \geq 2\sqrt{uv}$.

b) Într-un triunghi avem: $a^2 + b^2 + c^2 = 2[x(x+y+z) + (y^2 + yz + z^2)] \geq$

$$\geq 4\sqrt{x(x+y+z)(y^2+yz+z^2)} \geq 4\sqrt{3xyz(x+y+z)} = 4F\sqrt{3}$$

unde am aplicat mai întâi, inegalitatea $u + v \geq 2\sqrt{uv}$, iar apoi, inegalitatea :$u + v + w \geq 3\sqrt[3]{uvw}$

c) Vom transforma cunoscuta inegalitate: $2(\sin A + \sin B + \sin C) \leq 3\sqrt{3}$

Avem, conform (1):

$$4\sqrt{(x+y+z)xyz}\left[\frac{1}{(x+y)(x+z)} + \frac{1}{(x+y)(y+z)} + \frac{1}{(x+z)(y+z)}\right] \leq 3\sqrt{3}$$

$$64(x+y+z)^3xyz \leq 27(x+y)^2 + (y+z)^2(x+z)^2$$

d) Din inegalitatea dată de Toma Albu în problema nr. 5696:

$$\sqrt{\sin A} + \sqrt{\sin B} + \sqrt{\sin C} \leq 3\sqrt[4]{\frac{3}{4}}$$

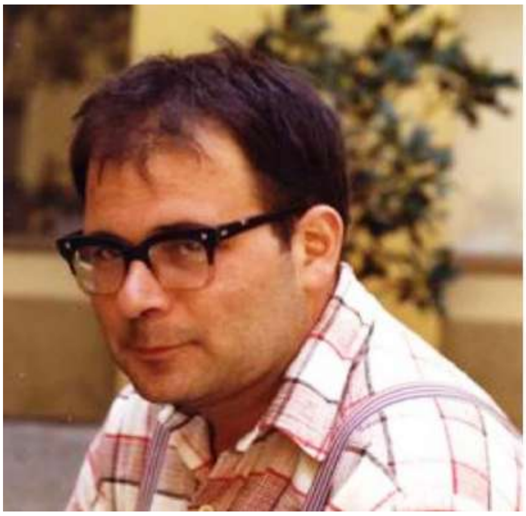
obținem: $16(x+y+z)\left(\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x}\right)^4 \leq 243(x+y)^2(y+z)^2(z+x)^2$

Urmând această cale se pot stabili lesne și alte inegalități între elemente ale unui triunghi

sau între numere pozitive.''

Voiculescu Dan, cl. VIII-a, șc.nr. 21 București

We must add that then, the young *Dan Voiculescu* evolved and performed very well in math competitions. He won (like *Ravi Vakil*) three IMO olympic medals : one silver in 1965 and two gold in 1966 and 1967. *Dan Voiculescu* then has an exceptional mathematical career. After being a brilliant researcher at the Institute of Mathematics of the Romanian Academy, he has been a professor at Berkeley University, California since 1986. He was nominated for the Fields Medal, but he did not obtain it, only because in the previous edition it had been obtained by a mathematician from the same field of research.. He is considered the greatest Romanian mathematician alive.



(1981) **Dan V. Voiculescu** **(1995)**

Returning to the substitution from the beginning of this paper and considering the ones presented , we believe that this type of substitution should be called *Voiculescu substitution*. Perhaps it would be just as correct to call this substitution *Voiculescu - Klamkin substitution* : the first for the chronological precedence of using this type of transformation , the second for the use and intense popularization of this type of relationship (that of *duality* and for the *principle of duality*) . In the last resort, the name of *Voiculescu - Klamkin - Ravi substitution* could be included!

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A NEW PROOF FOR V.O.GORDON'S INEQUALITY

By D.M. Bătinețu-Giurgiu, Daniel Sitaru, Florică Anastase-Romania

Abstract: *In this paper are presented a new proof for V.O. Gordon's inequality.*

Let $x, y, z > 0$, then holds:

$$(N) \qquad xy + yz + zx \geq \sqrt{3xyz(x+y+z)};$$

i.e. Newton's inequality.

Proof. We have: $xy + yz + zx \geq \sqrt{3xyz(x+y+z)} \Leftrightarrow$

$$(1) \qquad (xy + yz + zx)^2 \geq 3xyz(x+y+z);$$

Using:

$$(2) \qquad (u+v+w)^2 \geq 3(uv+vw+wu); (\forall)u, v, w > 0$$

and taking in (2): $u = xy; v = yz; w = zx$, we get:

$$(xy+yz+zx)^2 \geq 3(xy \cdot yz + yz \cdot zx + zx \cdot xy) = 3xyz(x+y+z)$$

i.e. inequality (N) was proved.

If in (N) we take $x = a; y = b; z = c$, where a, b, c are length sides of triangle ABC with F area and s semiperimeter, we get:

$$(*) \qquad ab + bc + ca \geq \sqrt{3abc(a+b+c)} = \sqrt{3} \cdot \sqrt{4RF \cdot 2s} \overset{Euler}{\geq} \sqrt{3} \cdot \sqrt{8rF \cdot 2s} =$$

$$= 4\sqrt{3} \cdot \sqrt{rs \cdot F} = 4\sqrt{3} \cdot \sqrt{F^2} = 4\sqrt{3} \cdot F$$

i.e. Gordon's inequality. Equality holds for $x = y = z \Leftrightarrow a = b = c$.

In ΔABC with a, b, c are length sides, F area and s semiperimeter holds:

$$(I - W) \qquad a^2 + b^2 + c^2 \geq 4\sqrt{3} \cdot F$$

i.e. Ionescu-Weitzenbock's inequality.

Proof. We have: $a^2 + b^2 + c^2 \geq ab + bc + ca \overset{(N)}{\geq} \sqrt{3abc(a+b+c)} \overset{(*)}{\geq} 4\sqrt{3} \cdot F$

Equality holds if and only if triangle is equilateral.

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ABOUT THE RMM PROBLEM UP495

By Marin Chirciu-Romania

Find a closed form:

$$\boldsymbol{\Omega = \sum_{n=1}^{\infty} \frac{7^n}{7^{2n} + 40 \cdot 7^n + 175}}$$

Proposed by Daniel Sitaru-Romania

Solution.Lemma: If $n \in \mathbb{N}^*$ then:

$$\frac{7^n}{7^{2n} + 40 \cdot 7^n + 175} = \frac{1}{6}\left(\frac{7}{7^{n-1} + 5} - \frac{1}{7^n + 5}\right)$$

Proof. Let us denote: $7^n = t$ and from $t^2 + 40t + 175 = (t + 5)(t + 35)$ we have:

$$\frac{t}{(t+5)(t+35)} = \frac{a}{t+5} + \frac{b}{t+35}, \qquad a(t+35) + b(t+5) = t$$

$$t(a+b) + 35a + 5b = t, \qquad \begin{cases} a + b = 1 \\ a + b = 0 \end{cases} \Rightarrow \begin{cases} a = -\frac{1}{6} \\ b = \frac{7}{6} \end{cases}$$

Hence,

$$\frac{t}{t^2 + 40t + 135} = -\frac{1}{6} \cdot \frac{1}{t+5} + \frac{7}{6} \cdot \frac{1}{t+35} = \frac{1}{6}\left(\frac{7}{t+35} - \frac{1}{t+5}\right)$$

$$t = 7^n \Rightarrow \frac{7^n}{7^{2n} + 40 \cdot 7^n + 135} = \frac{1}{6}\left(\frac{7}{7^n + 35} - \frac{1}{7^n + 5}\right) =$$

$$= \frac{1}{6}\left(\frac{7}{7(7^{n-1} + 5)} - \frac{1}{7^n + 5}\right) = \frac{1}{6}\left(\frac{1}{7^{n-1} + 5} - \frac{1}{7^n + 5}\right)$$

$$\Omega = \sum_{n=1}^{\infty} \frac{7^n}{7^{2n} + 40 \cdot 7^n + 175} = \frac{1}{6}\sum_{n=1}^{\infty}\left(\frac{1}{7^{n-1} + 5} - \frac{1}{7^n + 5}\right) =$$

$$= \frac{1}{6} \cdot \lim_{n \to \infty}\left(\sum_{k=1}^{n}\left(\frac{1}{7^{k-1} + 5} - \frac{1}{7^k + 5}\right)\right) = \frac{1}{6} \cdot \lim_{n \to \infty}\left(\frac{1}{6} - \frac{1}{7^n + 5}\right) = \frac{1}{36}$$

Remark. The problem can be developed: **Let $\boldsymbol{a > 1, b > 0}$. Find a closed form:**

$$\boldsymbol{\Omega = \sum_{n=1}^{\infty} \frac{a^n}{a^{2n} + (a+1)b \cdot a^n + ab^2}}$$

Proposed by Marin Chirciu-Romania

Solution.Lemma: If $a > 1, b > 0, n \in \mathbb{N}^*$, then:

$$\boldsymbol{\frac{a^n}{a^{2n} + (a+1)b \cdot a^n + ab^2} = \frac{1}{a-1}\left(\frac{1}{a^{n-1}+b} - \frac{1}{a^n+b}\right)}$$

Proof. Let us denote: $a^n = t$. We have: $t^2 + (a+1)b \cdot t + ab^2 = (t+b)(t+ab)$ and hence

$$\frac{t}{t^2 + (a+1)b \cdot t + ab^2} = \frac{A}{t+b} + \frac{B}{t+ab}$$

$$A(t+ab) + B(t+b) = t \Leftrightarrow t(A+B) + ab \cdot A + b \cdot B = t$$

$$\begin{cases} A + B = 1 \\ a \cdot A + B = 0 \end{cases} \Rightarrow \begin{cases} A = -\dfrac{1}{a-1} \\ B = \dfrac{a}{a-1} \end{cases}$$

$$\frac{t}{t^2 + (a+1)b \cdot t + ab^2} = \frac{1}{a-1}\left(\frac{a}{t+ab} - \frac{1}{t+b}\right) =$$

$$= \frac{1}{a-1}\left(\frac{a}{a(a^{n-1}+b)} - \frac{1}{a^n+b}\right) = \frac{1}{a-1}\left(\frac{1}{a^{n-1}+b} - \frac{1}{a^n+b}\right)$$

$$\Omega = \sum_{n=1}^{\infty} \frac{a^n}{a^{2n} + (a+1)b \cdot a^n + ab^2} = \frac{1}{a-1}\sum_{n=1}^{\infty}\left(\frac{1}{a^{n-1}+b} - \frac{1}{a^n+b}\right) =$$

$$= \frac{1}{a-1} \cdot \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(\frac{1}{a^{k-1}+b} - \frac{1}{a^k+b}\right)\right) = \frac{1}{a-1} \cdot \lim_{n\to\infty}\left(\frac{1}{b+1} - \frac{1}{a^n+b}\right) = \frac{1}{(a-1)(b+1)}$$

Note: For $a = 7, b = 5$ w obtain the Proposed Problem UP.495-RMM-33, by Daniel Sitaru-Romania.

REFERENCE: **ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro**

HOJOO LEE'S INEQUALITY REVISITED

By D.M. Bătinețu-Giurgiu, Daniel Sitaru, Florică Anastase-Romania

Abstract: *In this paper are presented an revisited form of Hojoo's Lee inequality and generalization.*

Proposition 1. If $u, v, x > 0$, then: $\boldsymbol{(u^2+x)(v^2+x) \ge \frac{3}{4}x\left((u+v)^2+x\right)}$; $\quad (\mathbf{1})$

Proof. We have:

$$(u^2+x)(v^2+x)\geq\frac{3}{4}x((u+v)^2+x)\Leftrightarrow$$

$$4u^2v^2+4x(u^2+v^2)+4x^2\geq 3x(u^2+v^2)+3uvx+3x^2\Leftrightarrow$$

$$4u^2v^2-4uvx+x^2+x(u^2+v^2-2uv)\geq 0\Leftrightarrow(2uv-x)^2+x(u-v)^2\geq 0$$

Equality holds for $2uv=x$ and $u=v\Leftrightarrow u=v=\sqrt{\frac{x}{2}}$.

Proposition 2. If $v,w,x>0$, then: $(v^2+x)(w^2+x)\geq x(v+w)^2$; $\quad(2)$

Proof. We have: $(v^2+x)(w^2+x)\geq x(v+w)^2\Leftrightarrow$

$$v^2w^2+x(v^2+w^2)+x^2\geq x(v^2+w^2)+2xvw\Leftrightarrow v^2w^2-2vwx+x^2\geq 0\Leftrightarrow(vw-x)^2\geq 0$$

Equality holds for $vw=x$.

Proposition 3. If $u,v,w,x>0$, then: $(u^2+x)(v^2+x)(w^2+x)\geq\frac{3}{4}(u+v+w)^2x^2$; $\quad(3)$

Proof. We have:

$$(u^2+x)(v^2+x)(w^2+x)\overset{(1)}{\geq}\frac{3}{4}x((u+v)^2+x)(w^2+x)\overset{(2)}{\geq}\frac{3}{4}x^2\big((u+v)+w\big)^2=$$

$$=\frac{3}{4}x^2(u+v+w)^2\geq\frac{9}{4}x^2(uv+vw+wu)^2$$

Theorem. If $a,b,c,m,n,t,s>0$ and $x,y,x+y\geq 0$, then:

$$\boldsymbol{x(ma^2+t)(mb^2+t)(mc^2+t)+y(na^2+s)(nb^2+s)(nc^2+s)\geq}$$

$$\boldsymbol{\geq\frac{3}{4}(mxt^2+nys^2)(a+b+c)^2\geq\frac{9}{4}(mxt^2+nys^2)(ab+bc+ca);}\quad(4)$$

Proof. In the relationship (3), taking $u=a\sqrt{m}; v=b\sqrt{m}; w=c\sqrt{m}$ and $x=t$ then:

$$(ma^2+t)(mb^2+t)(mc^2+t)\geq\frac{3}{4}t^2\big(a\sqrt{m}+b\sqrt{m}+c\sqrt{m}\big)^2=$$

$$=\frac{3}{4}mt^2(a+b+c)^2\geq\frac{9}{4}mt^2(ab+bc+ca);\quad(5)$$

Analogous, in (3) taking $u=a\sqrt{n}; v=b\sqrt{n}; w=c\sqrt{n}$ and $y=s$, we get:

$$(na^2+s)(nb^2+s)(nc^2+s) \geq \frac{3}{4}s^2\left(a\sqrt{n}+b\sqrt{n}+c\sqrt{n}\right)^2 =$$

$$=\frac{3}{4}ns^2(a+b+c)^2 \geq \frac{9}{4}ns^2(ab+bc+ca); \qquad (6)$$

From (5) and (6) we obtain:

$$x(ma^2+t)(mb^2+t)(mc^2+t)+y(na^2+s)(nb^2+s)(nc^2+s) \geq$$

$$\geq \frac{3}{4}mxt^2(a+b+c)^2+\frac{3}{4}nys^2(a+b+c)^2 =$$

$$=\frac{3}{4}(mxt^2+nys^2)(a+b+c)^2 \geq \frac{9}{4}(mxt^2+nys^2)(ab+bc+ca)$$

If in (4) we take $t=2, s=1, m=n=1$ and $x=1, y=4$ then we obtain

$$(a^2+2)(b^2+2)(c^2+2)+4(a^2+1)(b^2+1)(c^2+1) \geq$$

$$\geq \frac{3}{4}\cdot 4(a^2+2)(b^2+2)(c^2+2)+\frac{3}{4}\cdot 4(a^2+1)(b^2+1)(c^2+1) \geq$$

$$\geq \frac{3}{4}(1\cdot 1\cdot 4+1\cdot 4\cdot 1)(a+b+c)^2 = 6(a+b+c)^2 \geq 18(ab+bc+ca) \Leftrightarrow$$

$$(a^2+2)(b^2+2)(c^2+2)+4(a^2+1)(b^2+1)(c^2+1) \geq 6(a+b+c)^2 \geq 18(ab+bc+ca)$$

i.e. the proposed problem $\mathbf{3326(a)}$ **by Mihaly Bencze** from Crux Methematicorum Magazine.

If in (4) we take $y=0, m=1, x=1$ we get:

$$\boldsymbol{(a^2+t)(b^2+t)(c^2+t) \geq \frac{3}{4}t^2(a+b+c)^2 \geq \frac{9}{4}t^2(ab+bc+ca);} \qquad \mathbf{(7)}$$

from which taking $t \to t^2$, we have:

$$\boldsymbol{(a^2+t^2)(b^2+t^2)(c^2+t^2) \geq \frac{3}{4}t^4(a+b+c)^2 \geq \frac{9}{4}t^4(ab+bc+ca);} \qquad \mathbf{(8)}$$

i.e. Arkady M. Alt inequality published in [1]. If in (7) we take $t=2$ we get:

$$\boldsymbol{(a^2+2)(b^2+2)(c^2+2) \geq 3(a+b+c)^2 \geq 9(ab+bc+ca);} \qquad \mathbf{(9)}$$

i.e. Hojoo Lee's inequality proposed to APMO,2004.

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ABOUT THE PROBLEM 4720 FROM CRUX MATHEMATICORUM

By Marin Chirciu-Romania

Find all $x, y > 0$ such that:

$$\frac{1}{(x+1)^8} + \frac{1}{(y+1)^8} = \frac{1}{8(xy+1)^4}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution: Lemma. If $a, b > 0$ then:

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} \geq \frac{1}{1+ab}$$

Proof. We have:

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} \geq \frac{1}{1+ab}, \qquad 1 - 2ab + a^3b + ab^3 - a^2b^2 \geq 0$$

$$(1 - 2ab + a^2b^2) + (a^3b + ab^3 - 2a^2b^2) \geq 0, \qquad (1-ab)^2 + ab(a-b)^2 \geq 0$$

Equality holds for $a = b = 1$.

Now, using Holder's inequality and Lemma, we have:

$$\frac{1}{(x+1)^8} + \frac{1}{(y+1)^8} = \left(\frac{1}{(x+1)^2}\right)^4 + \left(\frac{1}{(y+1)^2}\right)^4 \overset{Holder}{\geq}$$

$$\geq \frac{1}{8}\left[\frac{1}{(x+1)^2} + \frac{1}{(y+1)^2}\right]^4 \overset{Lemma}{\geq} \frac{1}{8}\left(\frac{1}{xy+1}\right)^4 = \frac{1}{8(xy+1)^4}$$

Therefore, $S = \{(1,1)\}$.

CRUX MATHEMATICORUM CHALLENGES-(VII)

By Daniel Sitaru-Romania

4679. Let $(x_n)_{n\geq 1}$ be a sequence of real numbers such that $x_1 = \frac{1}{7}, x_2 = \frac{1}{5}$ and for $n \geq 2$,

$$2x_{n+1} \cdot x_{n-1} = (n+1)x_n \cdot x_{n-1} + (n-1)x_n \cdot x_{n+1}$$

Find:

$$\boldsymbol{\Omega = \lim_{n\to\infty}\left(\frac{2}{3}+x_n\right)^{nx_n}}$$

Proof. Let $y_n = \frac{n}{x_n}$, then $y_1 = 7, y_2 = 10$ and $2y_n = y_{n+1} + y_{n-1}$, so that $y_n = 3n + 4$, an arithmetic progression with common difference 3. Hence

$$x_n = \frac{n}{3n+4} = \frac{1}{3} - u_n \text{ and } nx_n = \frac{n^2}{3n+4} = \frac{1}{u_n}\cdot\frac{4n^2}{3(3n+4)^2},$$

where $u_n = \frac{4}{3(3n+4)}$ Therefore,

$$\Omega = \lim_{n\to\infty}\left(\frac{2}{3}+x_n\right)^{nx_n} = \lim_{n\to\infty}\left[(1-u_n)^{\frac{1}{u_n}}\right]^{\frac{4n^2}{3(3n+4)^2}} = (e^{-1})^{\frac{4}{27}} = e^{-\frac{4}{27}}$$

B104. Find all real roots of the equation: $\boldsymbol{5x^3 - 9x^2 - 15x + 3 = 0}$

Proof. We have:

$$5x^3 - 9x^2 - 15x + 3 = 0, \qquad 5x^3 - 9x^2 - 15x + 3 + 5(3x - x^3) = 5(3x - x^3)$$

$$3 - 9x^2 = 5(3x - x^3), \qquad 3(1 - 3x^2) = 5(3x - x^3)$$

Let's observe that $x = \pm\frac{\sqrt{3}}{3}$ are not solutions, so we can divide by $1 - 3x^2$ the equation:

$$\frac{3x - x^3}{1 - 3x^2} = \frac{5}{3}; \quad (1)$$

$$x \in \mathbb{R} - \left\{\pm\frac{\sqrt{3}}{3}\right\} \Rightarrow (\exists)\alpha \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right) - \left\{\pm\frac{\pi}{6}\right\}, x = \tan\alpha$$

$$(1) \Leftrightarrow \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha} = \frac{5}{3}$$

$$\tan 3\alpha = \frac{5}{3} \Rightarrow 3\alpha \in \left\{\tan^{-1}\frac{5}{3} + k\pi \middle| k \in \mathbb{Z}\right\}$$

$$\alpha \in \left\{\frac{1}{3}\tan^{-1}\frac{5}{3} + \frac{k\pi}{3}\middle| k \in \mathbb{Z}\right\} \cap \left(-\frac{\pi}{2},\frac{\pi}{2}\right) - \left\{\pm\frac{\pi}{6}\right\}$$

$$k = 0 \Rightarrow \alpha_1 = \frac{1}{3}\tan^{-1}\frac{5}{3} \Rightarrow x_1 = \tan\left(\frac{1}{3}tan^{-1}\frac{5}{3}\right)$$

$$k = 1 \Rightarrow \alpha_2 = \frac{1}{3}\tan^{-1}\frac{5}{3} + \frac{\pi}{3} \Rightarrow x_2 = \tan\left(\frac{1}{3}tan^{-1}\frac{5}{3} + \frac{\pi}{3}\right)$$

$$k = -1 \Rightarrow \alpha_3 = \frac{1}{3}\tan^{-1}\frac{5}{3} - \frac{\pi}{3} \Rightarrow x_3 = \tan\left(\frac{1}{3}tan^{-1}\frac{5}{3} - \frac{\pi}{3}\right)$$

B107. In $\boldsymbol{\Delta ABC, a, b, c}$ –sides, $\boldsymbol{s}$ –semiperimeter, $\boldsymbol{r}$ –inradii, the following relationship holds:

$$\frac{a}{\sqrt{s+b}}+\frac{b}{\sqrt{s+c}}+\frac{c}{\sqrt{s+a}}\geq\frac{108r^2}{a\sqrt{s+b}+b\sqrt{s+c}+c\sqrt{s+a}}$$

Solution 1 by proposer

$$\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{a}{\sqrt{s+b}}\right)\left(\sum_{cyc}a\sqrt{s+b}\right)=$$

$$=\left(\sum_{cyc}(\sqrt[3]{a})^3\right)\left(\sum_{cyc}\left(\sqrt[3]{\frac{a}{\sqrt{s+b}}}\right)^3\right)\left(\sum_{cyc}\left(\sqrt[3]{a\sqrt{s+b}}\right)^3\right)\overset{Holder}{\geq}$$

$$\geq\left(\sum_{cyc}\sqrt[3]{a}\cdot\sqrt[3]{\frac{a}{\sqrt{s+b}}}\cdot\sqrt[3]{a\sqrt{s+b}}\right)^3=\left(\sum_{cyc}(\sqrt[3]{a})^3\right)^3=(a+b+c)^3$$

$$\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{a}{\sqrt{s+b}}\right)\left(\sum_{cyc}a\sqrt{s+b}\right)\geq\left(\sum_{cyc}a\right)^3$$

$$\left(\sum_{cyc}\frac{a}{\sqrt{s+b}}\right)\left(\sum_{cyc}a\sqrt{s+b}\right)\geq\left(\sum_{cyc}a\right)^2=(2s)^2=4s^2\overset{MITRINOVIC}{\geq}4\left(3\sqrt{3}r\right)^2=108r^2$$

$$\sum_{cyc}\frac{a}{\sqrt{s+b}}\geq\frac{108r^2}{\sum_{cyc}a\sqrt{s+b}}$$

$$\frac{a}{\sqrt{s+b}}+\frac{b}{\sqrt{s+c}}+\frac{c}{\sqrt{s+a}}\geq\frac{108r^2}{a\sqrt{s+b}+b\sqrt{s+c}+c\sqrt{s+a}}$$

Solution 2 by Marin Chirciu-Romania

Inequality can be written:

$$\left(\frac{a}{\sqrt{s+b}}+\frac{b}{\sqrt{s+c}}+\frac{c}{\sqrt{s+a}}\right)\left(a\sqrt{s+b}+b\sqrt{s+c}+c\sqrt{s+a}\right)\geq 108r^2$$

which results by CBS: $\left(\frac{a}{\sqrt{s+b}}+\frac{b}{\sqrt{s+c}}+\frac{c}{\sqrt{s+a}}\right)\left(a\sqrt{s+b}+b\sqrt{s+c}+c\sqrt{s+a}\right)\overset{CBS}{\geq}$

$\geq(a+b+c)^2=4s^2\overset{(1)}{\geq}108r^2$, where $(1)\Leftrightarrow 4s^2\geq 108r^2\Leftrightarrow s^2\geq 27r^2(Mitrinovic)$.

Equality holds for an equilateral triangle.

B.120 In ΔABC the following relationship holds:

$$\frac{(1+x^2)(1+y^2)}{x^2y^2}+\frac{(1+y^2)(1+z^2)}{y^2z^2}+\frac{(1+z^2)(1+x^2)}{z^2x^2}\geq\frac{48}{x^2+y^2+z^2}$$

Solution by proposer: **Lemma 1:**

In ΔABC the following relationship holds:

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 1$$

Proof. Let be $f:(0,\pi)\to\mathbb{R}, f(x)=\tan^2\frac{x}{2}$

$$f'(x)=\frac{\sin\frac{x}{2}}{\cos^3\frac{x}{2}}; f''(x)=\frac{1}{2\cos^4\frac{x}{2}}>0; f-\text{convexe.}$$

By Jense's inequality:

$$\sum_{cyc}\tan^2\frac{A}{2}\geq 3\tan^2\left(\frac{\frac{A+B+C}{3}}{2}\right)=3\tan^2\left(\frac{\pi}{6}\right)=3\cdot\left(\frac{1}{\sqrt{3}}\right)^2=1$$

Lemma 2: In ΔABC the following relationship holds:

$$\frac{1}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}}+\frac{1}{\sin^2\frac{C}{2}\sin^2\frac{A}{2}}+\frac{1}{\sin^2\frac{A}{2}\sin^2\frac{B}{2}}\geq 48$$

Proof.

$$\sum_{cyc}\frac{1}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}}=\sum_{cyc}\frac{ac\cdot ab}{(s-a)(s-c)\cdot(s-a)(s-b)}=$$

$$=\frac{abc}{(s-a)(s-b)(s-c)}\cdot\sum_{cyc}\frac{a}{s-a}=\frac{abc}{(s-a)^2(s-b)^2(s-c)^2}\cdot\sum_{cyc}a(s-b)(s-c)=$$

$$=\frac{abcs^2}{s^2(s-a)^2(s-b)^2(s-c)^2}\cdot\sum_{cyc}a(s^2-s(b+c)+bc)=$$

$$=\frac{4RFs^2}{F^4}\cdot\sum_{cyc}(as^2-as(2s-a)+abc)=\frac{4Rs^2}{F^3}\cdot\sum_{cyc}(a^2s-as^2+abc)=$$

$$=\frac{4Rs^2}{F\cdot r^2s^2}\left(s\sum_{cyc}a^2-s^2\sum_{cyc}a+3abc\right)=\frac{4R}{Fr^2}(s\cdot 2(s^2-r^2-4Rr)-2s^3+12RF)=$$

$$=\frac{4R}{Fr^2}(2s^3-2sr^2-8Rrs-2s^3+12RF)=\frac{4R}{Fr^2}(-2rF-8RF+12RF)=$$

$$=\frac{4R}{r^2}(-2r-8R+12R)=\frac{4R(4R-2r)}{r^2}=\frac{8R(2R-r)}{r^2}\overset{EULER}{\geq}\frac{8\cdot 2r(2\cdot 2r-r)}{r^2}=$$

$$=\frac{16r\cdot 3r}{r^2}=\frac{48r^2}{r^2}=48$$

Back to the main problem: $x,y,z\in(0,1)\Rightarrow(\exists)\alpha,\beta,\gamma\in\left(0,\frac{\pi}{4}\right)$

$$x=\tan\alpha\,;y=\tan\beta\,;z=\tan\gamma$$

$$xy+yz+zx=1\Rightarrow x(y+z)=1-yz\Rightarrow x=\frac{1-yz}{y+z}$$

$$\tan\alpha=\frac{1-\tan\beta\tan\gamma}{\tan\beta+\tan\gamma}\Rightarrow\tan\alpha=\cot(\beta+\gamma)$$

$$\tan\alpha=\tan\left(\frac{\pi}{2}-(\beta+\gamma)\right)\Rightarrow\alpha=\frac{\pi}{2}-(\beta+\gamma)\Rightarrow 2\alpha+2\beta+2\gamma=\pi$$

Denote: $A=2\alpha;B=2\beta;C=2\gamma,\ \alpha=\frac{A}{2};\beta=\frac{B}{2};\gamma=\frac{C}{2};A+B+C=\pi$

$$x=\tan\frac{A}{2};y=\tan\frac{B}{2};z=\tan\frac{C}{2}$$

$$\sin^2\frac{A}{2}=\frac{x^2}{1+x^2};\sin^2\frac{B}{2}=\frac{y^2}{1+y^2};\sin^2\frac{C}{2}=\frac{z^2}{1+z^2}$$

$$\frac{(1+x^2)(1+y^2)}{x^2y^2}+\frac{(1+y^2)(1+z^2)}{y^2z^2}+\frac{(1+z^2)(1+x^2)}{z^2x^2}\geq\frac{48}{x^2+y^2+z^2}\Leftrightarrow$$

$$\left(\sum_{cyc}x\right)^2\left(\sum_{cyc}\frac{(1+x^2)(1+y^2)}{x^2y^2}\right)\geq 48,\qquad\left(\sum_{cyc}\tan^2\frac{A}{2}\right)\left(\sum_{cyc}\frac{1}{\frac{x^2}{1+x^2}\cdot\frac{y^2}{1+y^2}}\right)\geq 48$$

$$\left(\sum_{cyc}\tan^2\frac{A}{2}\right)\left(\sum_{cyc}\frac{1}{\sin^2\frac{A}{2}\sin^2\frac{B}{2}}\right)\geq 48$$

By Lemma 1 and Lemma 2:

$$\sum_{cyc}\tan^2\frac{A}{2}\geq 1;\qquad(1),\qquad\sum_{cyc}\frac{1}{\sin^2\frac{A}{2}\sin^2\frac{B}{2}}\geq 48;\ \ (2)$$

By multiplying (1); (2):

$$\left(\sum_{cyc}\tan^2\frac{A}{2}\right)\left(\sum_{cyc}\frac{1}{\sin^2\frac{A}{2}\sin^2\frac{B}{2}}\right)\geq 48$$

Equality holds if $A=B=C=\frac{\pi}{3}\Rightarrow x=y=z=\frac{\sqrt{3}}{3}$.

Solution 2 by Marin Chirciu-Romania

In the identity: $\sum \tan\frac{B}{2}\tan\frac{C}{2} = 1$ replace:

$$(x,y,z) = \left(\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}\right)$$

$$\sum_{cyc}\frac{(1+y^2)(1+z^2)}{y^2z^2} = \sum_{cyc}\frac{\left(1+\tan^2\frac{B}{2}\right)\left(1+\tan^2\frac{C}{2}\right)}{\tan^2\frac{B}{2}\tan^2\frac{C}{2}} = \frac{\sum\sin^2\frac{A}{2}}{\prod\sin^2\frac{A}{2}} = \frac{1-\frac{r}{2R}}{\frac{r^2}{4R^2}} = \frac{8R(2R-r)}{r^2}$$

$$x^2+y^2+z^2 = \sum_{cyc}\tan^2\frac{A}{2} = \frac{(4R+r)^2}{s^2}-2$$

Inequality can be written as:

$$\frac{8R(2R-r)}{r^2} \ge \frac{48}{\frac{(4R+r)^2}{s^2}-2} \Leftrightarrow \frac{8R(2R-r)}{r^2}\cdot\left[\frac{(4R+r)^2}{s^2}-2\right] \ge 48$$

which results by Blundon-Gerretsen:

$$s^2 \le \frac{R(4Rr+r)^2}{2(2R-r)}$$

Remains to prove:

$$\frac{8R(2R-r)}{r^2}\left[\frac{(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}-2\right] \ge 48 \Leftrightarrow \frac{R(2R-r)}{r^2}\left[\frac{2(2R-r)}{R}-2\right] \ge 6$$

$$(2R-r)(R-r) \ge r^2 \Leftrightarrow 2R^2-3Rr-2r^2 \ge 0 \Leftrightarrow (R-2r)(2R+r) \ge 0,$$

obviously true by Euler $R \ge 2r$.

Equality holds by an equilateral triangle.

In the initial inequality the equality holds for:

$$x = y = z = \frac{1}{\sqrt{3}}$$

B.124 In ΔABC the following relationship holds:

$$\frac{a^4}{w_a}+\frac{b^4}{w_b}+\frac{c^4}{w_c} \ge 144r^3$$

Solution by the proposer

$$w_a = \frac{2bc}{b+c}\cos\frac{A}{2} = \frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)} \le \sqrt{s(s-a)}$$

$$\text{because: } \frac{2\sqrt{bc}}{b+c} \le 1 \Leftrightarrow 0 \le \left(\sqrt{b}-\sqrt{c}\right)^2$$

$w_a \le \sqrt{s(s-a)}$ and analogous:

$w_b \le \sqrt{s(s-b)}$ and $w_c \le \sqrt{s(s-c)}$

$$(w_a+w_b+w_c)^2 \le \left(\sqrt{s(s-a)}+\sqrt{s(s-b)}+\sqrt{s(s-c)}\right)^2 \overset{CBS}{\le}$$

$$\le 3\left(3s^2 - s(a+b+c)\right) = 3(3s^2 - s\cdot 2s) = 3s^2$$

$$(w_a+w-b+w_c)^2 \le 3s^2 = \frac{3}{4}(a+b+c)^2 \le \frac{3}{4}(3a^2+3b^3+3c^2) = \frac{9}{4}(a^2+b^2+c^2)$$

$$w_a+w_b+w_c \le \frac{3}{2}\sqrt{a^2+b^2+c^2}; \quad (1)$$

$$\frac{a^4}{w_a}+\frac{b^4}{w_b}+\frac{c^4}{w_c} = \frac{(a^2)^2}{w_a}+\frac{(b^2)^2}{w_b}+\frac{(c^2)^2}{w_c} \overset{BERGSTROM}{\ge}$$

$$\ge \frac{(a^2+b^2+c^2)^2}{w_a+w_b+w_c} \overset{(1)}{\ge} \frac{a^2+b^2+c^2}{\frac{3}{2}\sqrt{a^2+b^2+c^2}} = \frac{3}{2}(a^2+b^2+c^2)^{\frac{3}{2}} \overset{WEITZENBOCK}{\ge}$$

$$\ge \frac{2}{3}\left(4F\sqrt{3}\right)^{\frac{3}{2}} = \frac{2}{3}\left(4\sqrt{3}rs\right)^{\frac{3}{2}} \overset{MITRINOVIC}{\ge} \frac{2}{3}\left(4\sqrt{3}r\cdot 3\sqrt{3}r\right)^{\frac{3}{2}} =$$

$$= \frac{2}{3}(36r^2)^{\frac{3}{2}} = \frac{2}{3}(6r)^3 = \frac{2}{3}\cdot 216r^3 = 144r^3$$

Equality holds for $a=b=c$.

Solution 2 by Marin Chirciu-Romania

By Bergstrom's inequality:

$$\sum_{cyc}\frac{a^4}{w_a} \ge \frac{(\sum a^2)^2}{\sum w_a} \overset{m_a\ge w_a}{\ge} \frac{(\sum a^2)^2}{\sum m_a} \overset{LEUNBERGER}{\ge} \frac{(\sum a^2)^2}{4R+r} =$$

$$= \frac{\left(2(s^2-r^2-4Rr)\right)^2}{4R+r} \overset{Gerretsen}{\ge} \frac{4(16Rr-5r^2-r^2-4Rr)^2}{4R+r} =$$

$$= \frac{4(12Rr-6r^2)^2}{4R+r} = \frac{4(12Rr-6r^2)^2}{4R+r} = \frac{4\cdot 36r^2(2R-r)^2}{4R+r} \overset{(1)}{\ge} 144r^3$$

$$(1) \Leftrightarrow \frac{4\cdot 36r^2(2R-r)^2}{4R+r} \ge 144r^3 \Leftrightarrow (2R-r)^2 \ge r(4R+r) \Leftrightarrow$$

$$4R^2-4Rr+r^2 \ge 4Rr+r^2 \Leftrightarrow R \ge 2r\ (Euler).$$

Equality holds for an equilateral triangle.

Remark. The problem can be generalized:

In ΔABC the following relationship holds:

$$\frac{a^{2n}}{w_a}+\frac{b^{2n}}{w_b}+\frac{c^{2n}}{w_c}\ge 12^n\cdot r^{2n-1}, n\in\mathbb{N}^*$$

Proposed by Marin Chirciu-Romania

Solution: By Holder's inequality:

$$\sum_{cyc}\frac{a^{2n}}{w_a}\ge\frac{(\sum a^2)^n}{3^{n-2}\sum w_a}\overset{m_a\ge w_a}{\ge}\frac{(\sum a^2)^n}{3^{n-2}\sum m_a}\overset{LEUENBERGER}{\ge}\frac{(\sum a^2)^n}{3^{n-2}(4R+r)}=$$

$$=\frac{\left(2(s^2-r^2-4Rr)\right)^n}{3^{n-2}(4R+r)}\overset{GERRETSEN}{\ge}\frac{2^n(16Rr-5r^2-r^2-4Rr)^n}{3^{n-2}(4R+r)}=$$

$$=\frac{2^n(12Rr-6r^2)^n}{3^{n-2}(4R+r)}=\frac{2^n\cdot(6r)^n(2R-r)^n}{3^{n-2}(4R+r)}\overset{(1)}{\ge}12^n r^{2n-1}$$

$$(1)\Leftrightarrow\frac{2^n\cdot(6r)^n(2R-r)^n}{3^{n-2}(4R+r)}\ge 12^n r^{2n-1}\Leftrightarrow\frac{(6r)^n(2R-r)^n}{3^{n-2}(4R+r)}\ge 6^n r^{2n-1}\Leftrightarrow$$

$$\frac{r^n(2R-r)^n}{3^{n-2}(4R+r)}\ge r^{2n-1}\Leftrightarrow\frac{(2R-r)^n}{3^{n-2}(4R+r)}\ge r^{n-1}\Leftrightarrow\left(\frac{2R-r}{3r}\right)^n\ge\frac{4R+r}{9r}$$

which will be proved by mathematical induction:

$$P(n):\ \left(\frac{2R-r}{3r}\right)^n\ge\frac{4R+r}{9r}; n\in\mathbb{N}^*,\qquad P(1):\frac{2R-r}{3r}\ge\frac{4R+r}{9r}\Leftrightarrow R\ge 2r(Euler)$$

$P(k)\Rightarrow P(k+1), k\ge 1$ it is equivalent with: $P(1):\frac{2R-r}{3r}\ge\frac{4R+r}{9r}\Leftrightarrow R\ge 2r(Euler)$

Equality holds for an equilateral triangle.

Note: For $n=2$ it's obtained Problem B124 from Crux Mathematicorum, No. 48 (3) proposed by Daniel Sitaru-Romania:

B.124 In ΔABC the following relationship holds:

$$\frac{a^4}{w_a}+\frac{b^4}{w_b}+\frac{c^4}{w_c}\ge 144r^3$$

REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

JENSEN AND NESBITT'S INEQUALITIES REVISITED

By Dorin Mărghidanu-Romania

This paper presents two possibilities for generating new inequalities obtained especially by the successive application of the Jensen and Nesbitt inequalities in certain conditions of monotony. By choosing specific convex / concave functions, it is get various applications.

Keywords*: *Jensen's inequality , Nesbitt's inequality , convex / concave function

2000 *Mathematics Subject Classification* : 26D15

The inequalities of *Nesbitt* and *Jensen*, - two already classical inequalities - are well known in mathematical literature and practice :

1. ***Proposition*** **(** ***Nesbitt's inequality*** , [5])

If $a, b, c > 0$, then ,

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}, \qquad (N)$$

having equality iff $a = b = c$.

2. ***Proposition*** **(** ***Jensen's inequality ,*** [1] **)**

Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ a *convex function* on the interval I . Then for any $x_k \in I$, we have

$$\frac{1}{n}\cdot\sum_{k=1}^{n} f(x_k) \geq f\left(\frac{1}{n}\cdot\sum_{k=1}^{n} x_k\right) \quad , \qquad (J)$$

If f is a *concave function* on I, the inequality sign in (J) is reversed.Equality in (J) occurs if and only if $x_1 = x_2 = ... = x_n$, or when the function f is a function *linear* (*affine*). In the following we will highlight some inequalities that result from the successive application of the *inequalities of Jensen* (for case $n = 3$) and **Nesbitt** , together with certain properties of monotony of the functions considered . Here is a first result of this kind .

3. ***Proposition :*** If the function $f : I \subset (0, \infty) \longrightarrow \mathbb{R}_+$ is a *convex* and *increasing* function on I , then :

$$f\left(\frac{a}{b+c}\right)+f\left(\frac{b}{c+a}\right)+f\left(\frac{c}{a+b}\right)\geq 3\cdot f\left(\frac{1}{2}\right) \quad , \qquad (1)$$

for any $a, b, c > 0$.

Proof: Indeed, using *Jensen's inequality* for *convex functions* in the first instance, then by *Nesbitt's inequality* and also taking into account the fact that the function is *increasing* , we obtain successively :

$$\sum_{cyc} f\left(\frac{a}{b+c}\right) \overset{(J)}{\geq} 3f\left(\frac{1}{3}\sum_{cyc}\frac{a}{b+c}\right) \overset{(N)}{\geq} 3f\left(\frac{3}{2}\cdot\frac{1}{3}\right) = 3f\left(\frac{1}{2}\right)$$

Equality occurs if ***a = b = c* .**

4. *Corollary* (*a generalization of Nesbitt's inequality*)

For any $p > 1$ and for any $a, b, c > 0$, the following inequality occurs ,

$$\left(\frac{a}{b+c}\right)^p + \left(\frac{b}{c+a}\right)^p + \left(\frac{c}{a+b}\right)^p \geq \frac{3}{2^p} \quad , \qquad \textbf{(2)}$$

Proof: Consider the function, $f : (0, \infty) \longrightarrow \mathbb{R}_+$, $f(x) = x^p$, $p > 1$, which is obviously *convex* and *ascending*, so with *Proposition* 3, we have : $\left(\frac{a}{b+c}\right)^p + \left(\frac{b}{c+a}\right)^p + \left(\frac{c}{a+b}\right)^p \geq 3\cdot\left(\frac{1}{2}\right)^p$,

with equally if ***a = b = c* .** For $p = 1$, inequality (N) is obtained .

5. *Application:* For any $q > 1$ and for any $a, b, c > 0$, the following inequality occurs ,

$$q^{\frac{a}{b+c}} + q^{\frac{b}{c+a}} + q^{\frac{c}{a+b}} \geq 3\cdot\sqrt{q} \quad , \qquad \textbf{(3)}$$

Proof **:**Let the function , $f : (0, \infty) \longrightarrow \mathbb{R}_+$, $f(x) = q^x$, $q > 1$, which is obviously *convex* and *ascending*, so with *Proposition* 3, we have : $q^{\frac{a}{b+c}} + q^{\frac{b}{c+a}} + q^{\frac{c}{a+b}} \geq 3\cdot q^{\frac{1}{2}}$, with equally if ***a = b = c* .**

6. *Application* , [2] For $a, b, c > 0$, there is the inequality ,

$$\frac{\sqrt{a^2+(b+c)^2}}{b+c} + \frac{\sqrt{b^2+(c+a)^2}}{c+a} + \frac{\sqrt{c^2+(a+b)^2}}{a+b} \geq \frac{3}{2}\cdot\sqrt{5} \quad . \qquad \textbf{(4)}$$

Proof: Let the function , $f : I \subset (0, \infty) \longrightarrow \mathbb{R}_+$, $f(x) = \sqrt{x^2+1}$, for which we have:

$$f'(x) = \frac{x}{\sqrt{x^2+1}} > 0 \ , \ f''(x) = \frac{1}{\left(x^2+1\right)\cdot\sqrt{x^2+1}} > 0$$, so the function f is convex and

increasing on **(0 , ∞) .** With Proposition 3, we have ,

$$\sqrt{\left(\frac{a}{b+c}\right)^2+1}+\sqrt{\left(\frac{b}{c+a}\right)^2+1}+\sqrt{\left(\frac{c}{a+b}\right)^2+1}\overset{(Prop.3)}{\geq}3\cdot f\left(\frac{1}{2}\right)=3\cdot\sqrt{\left(\frac{1}{2}\right)^2+1}=\frac{3}{2}\cdot\sqrt{5},$$

hence the inequality (2) , with equally if $a = b = c$.

7. *Remark:* If a, b, c are the lengths of the sides of a triangle , then ,

$$\frac{a}{b+c},\frac{b}{c+a},\frac{c}{a+b}\in(0,1), \quad (5)$$

Indeed, from $b + c > a$, result , $\frac{a}{b+c}<1$. Analogous : $\frac{b}{c+a}<1$, $\frac{c}{a+b}<1$ ×

8. *Application*

For a real number $p > 1$ and a, b, c ? sides of a triangle , we have the following inequality ,

$$\frac{a^p}{(-a+b+c)\cdot(b+c)^{p-1}}+\frac{b^p}{(a-b+c)\cdot(c+a)^{p-1}}+\frac{c^p}{(a+b-c)\cdot(a+b)^{p-1}}\geq\frac{3}{2^{p-1}}. \quad (6)$$

Proof : Consider the function , $f:(0,1)\longrightarrow\mathbb{R}_+$, $f(x)=\frac{x^p}{1-x}$, for which we have :

$$f'(x)=\frac{x^{p-1}[p-(p-1)x]}{(1-x)^2}>0 \text{ in intervalul } \left(0,1+\frac{1}{p-1}\right)\supset(0,1),$$

$$f''(x)=\frac{x^{p-2}[p(p-1)(1-x)^2+2x(p-(p-1)x)]}{(1-x)^3}>0,\ (\forall)\,x\in(0,1)\cdot$$ It turns out that

the function f is *convex* and *increasing* on (0, 1) , so with *Proposition* 3, and *Remark* 7, we have ,

$$\frac{\left(\frac{a}{b+c}\right)^p}{1-\left(\frac{a}{b+c}\right)}+\frac{\left(\frac{b}{c+a}\right)^p}{1-\left(\frac{b}{c+a}\right)}+\frac{\left(\frac{c}{a+b}\right)^p}{1-\left(\frac{c}{a+b}\right)}\geq3\cdot\frac{\left(\frac{1}{2}\right)^p}{1-\frac{1}{2}}\Leftrightarrow$$

$$\Leftrightarrow\frac{a^p}{(-a+b+c)\cdot(b+c)^{p-1}}+\frac{b^p}{(a-b+c)\cdot(c+a)^{p-1}}+\frac{c^p}{(a+b-c)\cdot(a+b)^{p-1}}\geq\frac{3}{2^{p-1}}.$$

Equality occurs if $a = b = c$.

9. *Application :* In the triangle **ABC** , on the sides a, b, c , we have the inequality ,

$$\arcsin\left(\frac{a}{b+c}\right)+\arcsin\left(\frac{b}{c+a}\right)+\arcsin\left(\frac{c}{a+b}\right)\geq\frac{\pi}{2}, \quad (7)$$

Proof: How :

$$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \in (0,1)$$

(*Remark* 7) , we consider the function , $f:(0,1) \longrightarrow (0, \pi/2)$, $f(x) = \arcsin x$, which is *convex* and *ascending* on $(0,1)$. Then with *Proposition* 3, we have :

$$\arcsin\left(\frac{a}{b+c}\right) + \arcsin\left(\frac{b}{c+a}\right) + \arcsin\left(\frac{c}{a+b}\right) \geq 3 \arcsin \frac{1}{2} \ ,$$

that is, the inequality in the statement . Equality occurs in the case of the equilateral triangle .

10. ***Proposition :*** If the function $f: I \subset (0, \infty) \longrightarrow \mathbb{R}_+$ is a *concave* and *descending* function on interval I ,

then :
$$f\left(\frac{a}{b+c}\right) + f\left(\frac{b}{c+a}\right) + f\left(\frac{c}{a+b}\right) \leq 3 \cdot f\left(\frac{1}{2}\right) \ , \qquad (8)$$

Proof: First , using *Jensen inequality* for *concave* functions , then *Nesbitt inequality* and taking into account and the fact that the function is decreasing , we obtain successively :

$$\sum_{cyc} f\left(\frac{a}{b+c}\right) \overset{(J)}{\leqq} 3f\left(\frac{1}{3}\sum_{cyc}\frac{a}{b+c}\right) \overset{(N)}{\leqq} 3f\left(\frac{3}{2}\cdot\frac{1}{3}\right) = 3f\left(\frac{1}{2}\right)$$

Equality occurs if $a = b = c$.

11. ***Application*** **, [3]** In triangle **ABC** , with sides a , b , c , we have inequality ,

$$\frac{\sqrt{a+b-c}}{a+b} + \frac{\sqrt{-a+b+c}}{b+c} + \frac{\sqrt{a-b+c}}{c+a} \leq \frac{3\cdot\sqrt{3}}{2\cdot\sqrt{a+b+c}} \ , \qquad (9)$$

Proof : The inequality in the statement can be written in the equivalent forms :

$$\frac{\sqrt{(a+b-c)(a+b+c)}}{a+b} + \frac{\sqrt{(-a+b+c)(a+b+c)}}{b+c} + \frac{\sqrt{(a-b+c)(a+b+c)}}{c+a} \leq \frac{3\cdot\sqrt{3}}{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\sqrt{(a+b)^2-c^2}}{a+b} + \frac{\sqrt{(b+c)^2-a^2}}{b+c} + \frac{\sqrt{(c+a)^2-b^2}}{c+a} \leq \frac{3\cdot\sqrt{3}}{2} \Leftrightarrow$$

$$\Leftrightarrow \sqrt{1-\left(\frac{a}{b+c}\right)^2} + \sqrt{1-\left(\frac{b}{c+a}\right)^2} + \sqrt{1-\left(\frac{c}{a+b}\right)^2} \leq \frac{3\cdot\sqrt{3}}{2} \ . \qquad (10)$$

Consider the function , $f:(0,1) \longrightarrow \mathbb{R}_+$, $f(x) = \sqrt{1-x^2}$ (*the circle function- in the first dial*),

which is obviously *concave* and *decreasing* on $(0,1)$. How ,

$$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \in (0, 1)$$

(*Remark* 7) , then with *Proposition* 10, we have :

$$\sqrt{1-\left(\frac{a}{b+c}\right)^2} + \sqrt{1-\left(\frac{b}{c+a}\right)^2} + \sqrt{1-\left(\frac{c}{a+b}\right)^2} \le 3 \cdot f\left(\frac{1}{2}\right) = 3 \cdot \sqrt{1-\left(\frac{1}{2}\right)^2} = \frac{3\cdot\sqrt{3}}{2} ,$$

so there is inequality (10) .

12. ***Application*** **, [4]** In triangle **ABC** , with sides ***a*** , ***b*** , ***c*** , we have inequality ,

$$\arccos\left(\frac{a}{b+c}\right) + \arccos\left(\frac{b}{c+a}\right) + \arccos\left(\frac{c}{a+b}\right) \le \pi , \qquad (11)$$

with equally if ***a*** **=** ***b*** **=** ***c*** **.**

Proof : With *Remark* 7 , we have ,

$$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \in (0, 1)$$

To solve we consider the function ,

$f : (0, 1) \longrightarrow (0, \pi/2)$, $f(x) = \arccos x$, which is *concave* and *decreasing* on **(0 , 1) .**

Then with *Proposition* 10, we have :

$$\arccos\left(\frac{a}{b+c}\right) + \arccos\left(\frac{b}{c+a}\right) + \arccos\left(\frac{c}{a+b}\right) \le 3 \arccos \frac{1}{2} = \pi$$

that is, the inequality in the statement. Equality occurs in the case of the equilateral triangle .

13. ***Remark:*** Inequality (11) can also be obtained from inequality (7) , using identity ,

$$\arccos x = \pi/2 - \arcsin x \cdot \qquad \textbf{(12)}$$

14. ***Remark:*** Note that only the possibilities of association : (***f*** *– convex* , ***f*** *– ascending*) - from *Proposition* 3, and (***f*** *– concave* , ***f*** *– descending*) - from *Proposition* 10 can be considered. The other two possibilities of association do not ensure the transitivity of the inequality relationship . For the above applications - demonstrated by the successive application of *Jensen* and *Nesbitt* inequalities - there are also other ways to demonstrate - as happened in the group posts : [2] , [3] , [4] . Obviously, many other applications of Sentences 3 and 10 can be obtained and demonstrated, respecting the above scenarios .

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GAMES WITH SUMS AND DIGITS

By Carmen-Victorița Chirfot-Romania

Abstract: *In this paper are presented two short calculus formulas and a few applications.*

Introduction.

We know that: $\boldsymbol{a^4-b^4=(a^2-b^2)(a^2+b^2)=(a-b)(a+b)(a^2+b^2), a,b}\in\mathbb{R};\ \ (\mathbf{1})$.

We want to find some a few sums in complex forms using two identities.

Replacing a and b with consecutive values, we have:

$$n^4-(n-1)^4=1\cdot(2n-1)((n-1)^2+n^2), n\in\mathbb{N}^*$$

$$(n-1)^4-(n-2)^4=1\cdot(2n-3)((n-2)^2+(n-1)^2)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$3^4-2^4=1\cdot5(2^2+3^2)$$

$$2^4-1^4=1\cdot3(1^2+2^2)$$

$$1^4-0^4=1\cdot1(0^2+1^2)$$

By adding these relationships, we get:

$$1(0^2+1^2)+3(1^2+2^2)+592^2+3^2)+\dots+(2n-3)((n-2)^2+(n-1)^2)$$
$$+(2n-1)((n-1)^2+n^2)=n^4, n\in\mathbb{N}^*$$

Another sum who we can find using the relationship (1) is:

$$S = 1\cdot 3(1^2+2^2)+2\cdot 6(2^2+4^2)+3\cdot 11(4^2+7^2)+4\cdot 18(7^2+11^2)+\dots+$$

$$+n\cdot(n^2+2)\left(\left(\frac{n^2-n+2}{2}\right)^2+\left(\frac{n^2+n+2}{2}\right)^2\right), n\in\mathbb{N}^*$$

For some terms, the relationship (1) has $a-b=k$ and $a+b=k^2, k=\overline{1,n}, n\in\mathbb{N}^*$. So,

$$1\cdot 3(1^2+2^2)=2^4-1^4$$

$$2\cdot 6(2^2+4^2)=4^4-2^4$$

$$3\cdot 11(4^2+7^2)=7^4-4^4$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$n\cdot(n^2+2)\left(\left(\frac{n^2-n+2}{2}\right)^2+\left(\frac{n^2+n+2}{2}\right)^2\right)=\left(\frac{n^2+n+2}{2}\right)^4-\left(\frac{n^2-n+2}{2}\right)^4$$

By adding, we get:

$$\boldsymbol{S=\left(\frac{n^2+n+2}{2}\right)^4-1;\ \ (2)}$$

Let's using the identity: $\boldsymbol{a^3-b^3=(a-b)(a^2+ab+b^2), a,b\in\mathbb{R};\ \ (3)}$. For example,

$$11^3-1^3=10(1^2+1\cdot 11+11^2)$$

$$111^3-11^3=100(11^2+11\cdot 111+111^2)$$

$$1111^3-111^3=1000(111^2+111\cdot 1111+1111^2)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\underbrace{\overline{11\dots 11}^3}_{2022-digits}-\underbrace{\overline{11\dots 11}^3}_{2021-digits}=\underbrace{\overline{100\dots 00}}_{2022-digits}\left(\underbrace{\overline{11\dots 11}^2}_{2021-digits}+\underbrace{\overline{11\dots 11}}_{2021-digits}\cdot\underbrace{\overline{11\dots 11}}_{2022-digits}+\underbrace{\overline{11\dots 11}^2}_{2022-digits}\right)$$

By adding, we get: $\boldsymbol{S=10(1^2+1\cdot 11+11^2)+10^2(11^2+11\cdot 111+111^2)+}$

$$\boldsymbol{+10^3(111^2+111\cdot 1111+1111^2)+\dots+}$$

$$\boldsymbol{\dots+10^{2021}\left(\underbrace{\overline{11\dots 11}^2}_{2021-digits}+\underbrace{\overline{11\dots 11}}_{2021-digits}\cdot\underbrace{\overline{11\dots 11}}_{2022-digits}+\underbrace{\overline{11\dots 11}^2}_{2022-digits}\right)=\underbrace{\overline{11\dots 11}^3}_{2022-digits}-1;\ \ (4)}$$

Applications.

1) Find: $\boldsymbol{S=1(0^2+1^2)+3(1^2+2^2)+5(2^2+3^2)+\dots+2021(1010^2+1011^2)}$

Solution: $1(0^2+1^2)+3(1^2+2^2)+5(2^2+3^2)+\dots+(2n-3)((n-2)^2+(n-1)^2)+(2n-1)((n-1)^2+n^2)=n^4$. The last term has $2n-1=2021\Rightarrow 2n=2022\Rightarrow n=1011$. So $S=1011^4$.

2) Prove that the sum

$\boldsymbol{S = 1\cdot 3(1^2+2^2)+2\cdot 6(2^2+4^2)+3\cdot 11(4^2+7^2)+4\cdot 18(7^2+11^2)+\cdots+10\cdot 102(46^2+56^2)}$ **is multiply of** $\boldsymbol{11}$.

Solution: From (2): $S=\left(\frac{n^2+n+2}{2}\right)^4-1=\left(\frac{100+10+2}{2}\right)^4-1=56^4-1=55\cdot 57(1+56^2)$.

3) Find:

$$\boldsymbol{S = 10\cdot 12(1^2+11^2)+10^2\cdot 122(11^2+111^2)+10^3\cdot 1222(111^2+1111^2)+\cdots+10^{2021}\cdot \underbrace{\overline{122\ldots 22}}_{2022-digits}\left(\underbrace{\overline{11\ldots 11^2}}_{2021-digits}+\underbrace{\overline{11\ldots 11^2}}_{2022-digits}\right)}$$

Solution: $11^4-1^4=10\cdot 12(1^2+11^2)$

$$111^4-11^4=100\cdot 122(11^2+111^2)$$

$$1111^4-111^4=1000\cdot 1222(111^2+1111^2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\underbrace{\overline{11\ldots 11}^4}_{2022-digits}-\underbrace{\overline{11\ldots 11}^4}_{2021-digits}=\underbrace{\overline{100\ldots 00}}_{2022-digits}\cdot\underbrace{\overline{122\ldots 22}}_{2022-digits}(\underbrace{\overline{11\ldots 11}^2}_{2021-digits}+\underbrace{\overline{11\ldots 11}^2}_{2022-diggits}$$

So, we have:

$$S=10\cdot 12(1^2+11^2)+10^2\cdot 122(11^2+111^2)+10^3\cdot 1222(111^2+1111^2)+\cdots+10^{2021}\cdot\underbrace{\overline{122\ldots 22}}_{2022-digits}\left(\underbrace{\overline{11\ldots 11}^2}_{2021-digits}+\underbrace{\overline{11\ldots 11}^2}_{2022-digits}\right)=\underbrace{\overline{11\ldots 11}^4}_{2022-digits}-1$$

4) Prove that the sum

$\boldsymbol{S = 11\cdot 13(1^2+12^2)+111\cdot 135(12^2+123^2)+1111\cdot 1357(123^2+1234^2)+11111\cdot 13579(1234^2+12345^2)}$ **cannot be a perfect square.**

Solution:

$$12^4-1^4=11\cdot 13(1^2+12^2)$$

$$123^4-12^4=111\cdot 135(12^2+123^2)$$

$$1234^4-123^4=1111\cdot 1357(123^2+1234^2)$$

$$12345^4-1234^4=11111\cdot 13579(1234^2+12345^2)$$

Therefore, $S=12345^4-1$.

5) Prove that:

$$\boldsymbol{1(1^2+3^2)+2(3^2+5^2)+3(5^2+7^2)+\cdots+1009(20172+2019^2)=\frac{2019^4-1}{8}}$$

Solution: We have:

$$2\cdot 4(1^2+3^2)+2\cdot 8(3^2+5^2)+2\cdot 12(5^2+7^2)+\cdots+2\cdot 4036(2017^2+2019^2)=2019^4-1$$

$$3^4-1=2\cdot 4(1^2+3^2),5^4-3^4=2\cdot 8(3^2+5^2)$$

$$7^4-5^4=2\cdot 12(5^2+7^2),\dots,2019^4-2017^4=2\cdot 4036(2017^2+2019^2)$$

Proposed problems.

P 1) Find: $\boldsymbol{S=11(1^2+1\cdot 12+12^2)+111(12^2+12\cdot 123+123^2)}$

$$\boldsymbol{+1111(123^2+123\cdot 1234+1234^2)+\cdots+}$$

$$\boldsymbol{+\underbrace{11\dots 11}_{9-digits}(\overline{12\dots 78}^2+\overline{12\dots 78}\cdot\overline{12\dots 89}+\overline{12\dots 89}^2)}$$

Indications. Using the relationship (3), we have:

$$12^3-1^3=11(1^2+1\cdot 12+12^2)$$

$$123^3-12^3=111(12^2+12\cdot 123+123^2)$$

$$1234^3-123^3=1111(123^2+123\cdot 1234+1234^2)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\overline{12\dots 89}^3-\overline{12\dots 78}^3=\underbrace{11\dots 11}_{9-digits}(\overline{12\dots 78}^2+\overline{12\dots 78}\cdot\overline{12\dots 89}+\overline{12\dots 89}^2)$$

Therefore, $S=\overline{12\dots 89}^3-1$.

P 2) Find: $\boldsymbol{S=2\cdot 10(1^2+1\cdot 21+21^2)+3\cdot 10^2(21^2+21\cdot 321+321^2)+}$

$$\boldsymbol{+4\cdot 10^3(321^2+321\cdot 4321+4321^2)+\cdots+}$$

$$\boldsymbol{+10^8(\overline{87\dots 21}^2+\overline{87\dots 21}\cdot\overline{98\dots 21}+\overline{98\dots 21}^2)}$$

Indications. By the relationship (3), we have: $21^3-1^3=20(1^2+1\cdot 21+21^2)$

$$321^3-21^3=300(21^2+21\cdot 321+321^2)$$

$$4321^3-321^3=4000(321^2+321\cdot 4321+4321^2)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\overline{98\dots 21}^3-\overline{87\dots 21}^3=\underbrace{10\dots 00}_{9-digits}(\overline{87\dots 21}^2+\overline{87\dots 21}\cdot\overline{98\dots 21}+\overline{98\dots 21}^2)$$

Therefore, $S=\overline{98\dots 21}^3-1$.

P 3) Determine the real number $\boldsymbol{a}$ such that

$$\boldsymbol{a\cdot 10(1^2+1\cdot 11+11^2)+a\cdot 10^2(11^2+11\cdot 111+111^2)+}$$

$$+\boldsymbol{a}\cdot\mathbf{10^3}\left(\mathbf{111^2+111\cdot 1111+1111^2}\right)+\cdots+$$

$$\ldots+\boldsymbol{a}\cdot\mathbf{10^{2021}}\left(\underbrace{\overline{\mathbf{11\ldots11}}^{\mathbf{2}}}_{\mathbf{2021-}\boldsymbol{digits}}+\underbrace{\overline{\mathbf{11\ldots11}}}_{\mathbf{2021-}\boldsymbol{digits}}\cdot\underbrace{\overline{\mathbf{11\ldots11}}}_{\mathbf{2022-}\boldsymbol{digits}}+\underbrace{\overline{\mathbf{11\ldots11}}^{\mathbf{2}}}_{\mathbf{2022-}\boldsymbol{digits}}\right)=\underbrace{\overline{\mathbf{11\ldots11}}^{\mathbf{3}}}_{\mathbf{2022-}\boldsymbol{digits}}-\mathbf{1}$$

Hints: $a[10(1^2+1\cdot 11+11^2)+10^2(11^2+11\cdot 111+111^2)+$

$$+10^3(111^2+111\cdot 1111+1111^2)+\cdots+$$

$$\ldots+10^{2021}\left(\underbrace{\overline{11\ldots11}^2}_{2021-digits}+\underbrace{\overline{11\ldots11}}_{2021-digits}\cdot\underbrace{\overline{11\ldots11}}_{2022-digits}+\underbrace{\overline{11\ldots11}^2}_{2022-digits}\right)]=\underbrace{\overline{11\ldots11}^3}_{2022-digits}-1$$

From (4): $S=10(1^2+1\cdot 11+11^2)+10^2(11^2+11\cdot 111+111^2)+$

$$+10^3(111^2+111\cdot 1111+1111^2)+\cdots+$$

$$\ldots+10^{2021}\left(\underbrace{\overline{11\ldots11}^2}_{2021-digits}+\underbrace{\overline{11\ldots11}}_{2021-digits}\cdot\underbrace{\overline{11\ldots11}}_{2022-digits}+\underbrace{\overline{11\ldots11}^2}_{2022-digits}\right)=\underbrace{\overline{11\ldots11}^3}_{2022-digits}-1$$

$$a\left(\underbrace{\overline{11\ldots11}^3}_{2022-digits}-1\right)\ldots+10^{2021}\left(\underbrace{\overline{11\ldots11}^2}_{2021-digits}+\underbrace{\overline{11\ldots11}}_{2021-digits}\cdot\underbrace{\overline{11\ldots11}}_{2022-digits}+\underbrace{\overline{11\ldots11}^2}_{2022-digits}\right)=$$

$$=\underbrace{\overline{11\ldots11}^3}_{2022-digits}-1$$

But from (3):

$$\underbrace{\overline{11\ldots11}^3}_{2022-digits}-\underbrace{\overline{11\ldots11}^3}_{2021-digits}=\underbrace{\overline{100\ldots00}}_{2022-digits}\left(\underbrace{\overline{11\ldots11}^2}_{2021-digits}+\underbrace{\overline{11\ldots11}}_{2021-digits}\cdot\underbrace{\overline{11\ldots11}}_{2022-digits}+\underbrace{\overline{11\ldots11}^2}_{2022-digits}\right)$$

$a\left(\underbrace{\overline{11\ldots11}^3}_{2022-digits}-1\right)=\underbrace{\overline{11\ldots11}^3}_{2022-digits}-1.$ So, $a=1$.

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[2]. Alpha Magazine Colections, Craiova, Romania

PROPOSED PROBLEMS

J.2201 Let $m \in (0,3]$ and h_A, h_B, h_c, h_D be the lengths altitudes of tetrahedron $ABCD$ with r radius of inscribed sphere, then

$$\prod_{cyc}\left(\left(\frac{h_A+mr}{h_A-mr}\right)^2+2\right) \geq 72\left(\frac{4+m}{4-m}\right)^2$$

Proposed by D.M. Bătinețu-Giurgiu, Mihaly Bencze-Romania

J.2202 In right triangle ABC with hypothenuse $BC = a$ and the cathetes $AC = b, AB = c$ holds:

$$4(b^4+1)(c^4+1) \geq 3(a^4+1)$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.2203 Let $m \geq 0$ and $a, b, c, t > 0$, then:

$$(a^2+2t^m)(b^2+2t^m)(c^2+2t^m) \geq 9(ab+bc+ca)\cdot t^{2m}$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

J.2204 If $m, t, a, b, c > 0$, then: $(m^2a^2+t)(m^2b^2+t)(m^2c^2+t) \geq \frac{9}{4}m^2t^2(ab+bc+ca)$

Proposed by D.M. Bătinețu-Giurgiu-Romania

J.2205 If $n \geq 2, x, y, a_k > 0, (\forall)k = \overline{1,n}$, then:

$$\left(x^2+\sum_{k=1}^{n}a_k^2\right)\left(y^2+\sum_{k=1}^{n}a_k^2\right) \geq \frac{3}{4n^2}\left(\sum_{k=1}^{n}a_k\right)^2\cdot\left(n(x+y)^2+\left(\sum_{k=1}^{n}a_k\right)^2\right)$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

J.2206 Let $m, a, b > 0$, then : $(m^2a^2+2)(m^2b^2+2) \geq \frac{3}{2}(m^2(a+b)^2+2)$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

J.2207 Let $a, b, x, y > 0$, then: $(a^2+x^2+y^2)(b^2+x^2+y^2) \geq \frac{3}{2}((a+b)^2+2xy)xy$

Proposed by D.M. Bătinețu-Giurgiu, Mihaly Bencze-Romania

J.2208 If $a, b, x, y > 0$, then: $(a^2+x^2+y^2)(b^2+x^2+y^2) \geq \frac{3}{16}(x+y)^2(2(a+b)^2+(x+y)^2)$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.2209 If $x, y > 0$, then in ΔABC holds:

$$((xr_a+yr_b)^2+R^2)(xr_b+yr_c)^2+R^2)((xr_c+yr_a)^2+R^2) \geq 2\sqrt{3}(x+y)^2\cdot F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2210 In ΔABC with area F holds: $(a^2+b^2+4)(b^2+c^2+4)(c^2+a^2+4)\geq 288\sqrt{3}\cdot F$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.2211 Let a,b,c be the length sides of an triangle ABC with area F, then holds:

$$(a^2+b^2+2)(b^2+c^2+2)(c^2+a^2+2)\geq 72\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

J.2212 If $M\in Int(\Delta ABC)$ and $x=MA, y=BM, z=CM$ then prove:

$$(t^2a^2+x^2)(t^2b^2+y^2)(t^2c^2+z^2)\geq 16\sqrt{3}t^4F^2, (\forall)t>0$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.2213 If $m,n\geq 0$ and $a,b,c,x,y,z\geq 0$ then:

$$(a^2+x^my^n)(b^2+y^mz^n)(c^2+z^mx^n)\geq \frac{3}{4}\left(a\sqrt{x^ny^mz^{n+m}}+b\sqrt{y^nz^mx^{m+n}}+c\sqrt{z^nx^my^{n+m}}\right)^2$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2214 If $a,b,c,x,y,z,t>0$ then:

$$(a^2+2txy)(b^2+2tyz)(c^2+2tzx)\geq 9t^2\left(ab\sqrt{xz}+bc\sqrt{yz}+ca\sqrt{xy}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Mihaly Bencze-Romania

J.2215 In $\Delta ABC, g_a$ –Gergonne's cevian from A, n_a –Nagel's cevian from B, F area, s_c –simmedian from C, then holds: $(g_a^4+2)(n_b^4+2)(s_c^4+2)\geq 48\cdot\frac{F^4}{R^4}$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2216 If n_a,n_b,n_c –Nagel's cevians in ΔABC with F area, then holds:

$$(m_an_a+2)(m_bn_b+2)(m_cn_c+2)\geq 36\cdot\frac{F^2}{R^2}$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

J.2217 In ΔABC with F area and $X,Y\in Int(\Delta ABC)$ holds:

$$(a^2\cdot AX^2\cdot AY^2+2)(b^2\cdot BX^2\cdot BY^2+2)(c^2\cdot CX^2\cdot CY^2+2)\geq\frac{64}{\sqrt{3}}\cdot F^3$$

Proposed by D.M. Bătinețu-Giurgiu, Mihaly Bencze-Romania

J.2218 In any ΔABC the following relationship holds: $\prod_{cyc}\left(\left(\frac{h_a-r}{h_a+r}\right)^2+2\right)\geq\frac{27}{4}$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.2219 Let $x, y, z > 0$, then: $\left(x^2+4(y^2+z^2)\right)\left(y^2+4(z^2+x^2)\right)\left(z^2+4(x^2+y^2)\right) \geq 27x^2y^2z^2$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

J.2220 In ΔABC the following relationship holds:

$$\frac{IA^2}{a^2}+\frac{IB^2}{b^2}+\frac{IC^2}{c^2} \geq \frac{a^4+b^4+c^4}{a^2b^2+b^2c^2+c^2a^2} \geq 1$$

Proposed by Nguyen Van Canh-Vietnam

J.2221 In ΔABC the following relationship holds:

$$a)\ \sum_{cyc} h_a h_b + \frac{r^2}{3R}(R-2r) \leq \sum_{cyc} h_a^2, \qquad b)\ \sum_{cyc} r_a r_b + r(R-2r) \leq \sum_{cyc} r_a^2$$

Proposed by Nguyen Van Canh-Vietnam

J.2222 In $\Delta ABC, p_a$ −Spieker's cevian, the following relationship holds:

$$\sum_{cyc} p_a^2 \geq \sum_{cyc} m_a^2 + \frac{1}{8}\sum_{cyc} \frac{a(b-c)^2}{s+a}$$

Proposed by Nguyen Van Canh-Vietnam

J.2223 Let $x, y, z > 0, x+y+z=3, \alpha \geq 12$. Prove that:

$$\alpha \geq (x-y)^2+(y-z)^2+(z-x)^2+\alpha\sqrt[3]{xyz}$$

Proposed by Nguyen Van Canh-Vietnam

J.2224 In $\Delta ABC, v_a$ −Bevan's cevian, the following relationship holds:

$$a)\ \sum_{cyc} m_a^2 \leq \sum_{cyc} v_a^2, \quad b)\ \sum_{cyc} v_a^2 \leq \sum_{cyc} m_a^2 + 8(R^2-4r^2)$$

Proposed by Nguyen Van Canh-Vietnam

J.2225 If $a, b, c > 0$ then:

$$a)\ \left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)^4 \geq \frac{3(a^3+b^3+c^3)(a^2+b^2+c^2)^2}{abc}$$

$$b)\ \frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b} \geq \frac{4(a+b+c)^2}{a^2+b^2+c^2+3(ab+bc+ca)}$$

$$c)\ \frac{(a^2+b^2)^2}{a^2-ab+b^2}+\frac{(b^2+c^2)^2}{b^2-bc+c^2}+\frac{(c^2+a^2)^2}{c^2-ca+a^2} > \frac{2}{3}(a+b+c)^2$$

Proposed by Nguyen Van Canh-Vietnam

J.2226 In ΔABC the following relationship holds:

$$\sum_{cyc}(m_a - w_a)(m_a + w_a) \geq 2(R^2 - 4r^2)$$

Proposed by Nguyen Van Canh-Vietnam

J.2227 Solve for real numbers:

$$\begin{cases} 4^x + 1 = 2^{y+1} \\ 4^y + 1 = 2^{z+1} \\ 4^z + 1 = 2^{x+1} \end{cases}$$

Proposed by Neculai Stanciu-Romania

J.2228 In $\Delta ABC, T$ –Toricelli' s point.Prove that: $TA^{2n} + TB^{2n} + TC^{2n} \geq 3(2r)^{2n}, n \in \mathbb{N}$

Proposed by Marin Chirciu-Romania

J.2229 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sqrt{b + \lambda c - a}}{b + \lambda c} \leq \frac{3\sqrt{\lambda^2 + 2\lambda}}{(\lambda + 1)\sqrt{a + b + \lambda c}}, \lambda \geq 1$$

Proposed by Marin Chirciu-Romania

J.2230 In $\Delta ABC, \varphi_1 = \measuredangle(g_a, n_a), \varphi_2 = \measuredangle(g_b, n_b), \varphi_3(g_c, n_c)$. Prove that: $0 \leq \varphi_1 + \varphi_2 + \varphi_3 < \pi$

Proposed by Nguyen Van Canh-Vietnam

J.2231 Let $a, b, c \geq 0, ab + bc + ca = 1, p = a + b + c$, prove that:

$$\sqrt{(2a + bc)(2b + ac)(2a + ab)} \leq p\sqrt{\frac{2p + 1}{p^2 + 1}}$$

Proposed by Phan Ngoc Chau-Vietnam

J.2232 In ΔABC the following relationship holds:

$$\left(\frac{r_a r_b}{r_a + r_b}\right)^n + \left(\frac{r_b r_c}{r_b + r_c}\right)^n + \left(\frac{r_c r_a}{r_c + r_a}\right)^n \geq 3 \cdot \left(\frac{3r}{2}\right)^n ; n \in \mathbb{N}$$

Proposed by Marin Chirciu-Romania

J.2233 If $M \in Int(\Delta ABC)$ then:

$$\frac{AM}{h_b} + \frac{BM}{h_c} + \frac{CM}{h_a} \geq \frac{1}{3}\left(\frac{b + c}{a} + \frac{c + a}{b} + \frac{a + b}{c}\right)$$

Proposed by Bogdan Fuștei-Romania

J.2234 If $a,b,c>0, abc=1$ then:

$$\sum_{cyc}\frac{a^3}{\sqrt{a+\lambda}}\geq\frac{3}{\sqrt{\lambda+1}}$$

Proposed by Marin Chirciu-Romania

J.2235 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a^3}{b+c}\leq\frac{1}{2}\sum_{cyc}a^2+\frac{R^3-8r^3}{r}$$

Proposed by Nguyen Van Canh-Vietnam

J.2236 Let $f:[0,1]\to\mathbb{R}$. Prove that $(\exists)x_0,x_1,x_2\in(0,1)$ such that

$$\frac{f(x_0)}{x_0^2}+\frac{f(x_1)}{2x_1^2}=3f(x_2)$$

Proposed by Nguyen Van Canh-Vietnam

J.2237 Let $x,y,z>0, x+y+z=9$. Prove that:

$$\frac{x^3+y^3+z^3}{x^2+y^2+z^2}+\frac{8xyz}{xy+yz+zx}\leq 11$$

Proposed by Nguyen Van Canh-Vietnam

J.2238 In ΔABC the following relationship holds:

$$9\leq\max\left\{\left(\sum_{cyc}\sqrt{\frac{m_a}{m_b^2+m_c^2}}\right)\left(\sum_{cyc}\sqrt{\frac{m_b^2+m_c^2}{m_a}}\right);\left(\sum_{cyc}\sqrt{\frac{a}{b^2+c^2}}\right)\left(\sum_{cyc}\sqrt{\frac{b^2+c^2}{a^2}}\right)\right\}\leq\frac{27R^2+8s^2}{36r^2}$$

Proposed by Nguyen Van Canh-Vietnam

J.2239 Let $\alpha,\beta\geq 0$. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$\beta f(\alpha x^3)=\alpha f(\beta x)+x^{\alpha+\beta+1};\forall x\in\mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

J.2240 Let $a,b,c>0, a^3+b^3+c^3+abc=4$ and $\alpha\geq 27$. Prove that:

$$(a+b+c)^3+\alpha(a^3+b^3+c^3)\geq 27+3\alpha$$

Proposed by Nguyen Van Canh-Vietnam

J.2241 If $m\in\mathbb{N}^*-\{1\}, a,c,s\in\mathbb{R}_+, b,d,x_k,y_k\in\mathbb{R}_+^*, k=\overline{1,m}, r\in[1,\infty)$ and $X_n=\sum_{k=1}^{m}x_k$, $Y_m=\sum_{k=1}^{m}y_k$, then prove:

$$\left(\sum_{k=1}^{m}(aX_m+bx_k)^r\right)\cdot\sum_{k=1}^{m}\frac{1}{(cY_m+dy_k)^2}\geq\frac{(am+b)^r}{(cm+d)^r}\cdot\frac{X_m^r}{Y_m^s}\cdot m^{s-r+2}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2242 In non isosceles $\Delta ABC, m\geq 0$ and $x,y>0$ the following relationship holds:

$$\sum_{cyc}\frac{a^{3(m+1)}}{(bx+cy)^m(a-b)^{m+1}(a-c)^{m+1}}>\frac{6\sqrt{3}r}{(x+y)^m}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2243 If $n\in\mathbb{N}^*-\{1\}, a,b\in\mathbb{R}_+^*$ such that $2^{n+1}a>b\max\limits_{1\leq k\leq n}\binom{n}{k}$, then prove:

$$\sum_{k=0}^{n}\frac{\binom{n}{k}}{2^{n+1}a-b\binom{n}{k}}\geq\frac{n+1}{2a(n+1)-b}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2244 If $x,y,z>0$, then:

$$\frac{x^2}{z^3(zx+y^2)}+\frac{y^2}{x^3(xy+z^2)}+\frac{z^2}{y^3(yz+x^2)}\geq\frac{3}{2xyz}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2245 In ΔABC the following relationship holds:

$$\sqrt{\frac{2R}{r}}\cdot\sum_{cyc}\left(\frac{b+c}{a}\right)\geq\sum_{cyc}\left(\frac{m_a}{w_a}+\sqrt{\frac{m_a}{r_a}}+\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)$$

Proposed by Bogdan Fuștei-Romania

J.2246 Let P point in plane of ΔABC and $x(y+z), y(z+x), z(x+y)>0, xyz(x+y+z)>0$. The following relationship holds:

$$\sum_{cyc}\sqrt{x(y+z)}\cdot AP\geq\sqrt{\frac{1}{2}\sum_{cyc}x(y+z)(b^2+c^2-a^2)+4F\sqrt{xyz(x+y+z)}}$$

Proposed by Bogdan Fuștei-Romania

J.2247 If x,y,z are real numbers, then in ΔABC holds:

$$x^2+y^2+z^2\geq\left(yz\sin\frac{A}{2}+zx\sin\frac{B}{2}+xy\sin\frac{C}{2}\right)\sec\frac{\pi}{3}$$

Proposed by Bogdan Fuștei-Romania

J.2248 In $\Delta ABC, x, y, z > 0$ the following relationship holds:

$$x^2 + y^2 + z^2 \geq \frac{1}{2\sqrt{2}} \cdot \sum_{cyc} \frac{yz \cdot \cos\frac{A}{2}}{\sqrt{1 + \sin\frac{A}{2}}}$$

Proposed by Bogdan Fuștei-Romania

J.2249 In acute ΔABC the following relationship holds:

$$\sum_{cyc} \frac{a}{w_a} \leq \frac{1}{2} \sum_{cyc} \frac{\left(\sqrt{w_a^2 + (s-b)^2} + \sqrt{s_a^2 + (s-c)^2}\right)\sqrt{m_b m_c}}{m_a h_a}$$

Proposed by Bogdan Fuștei-Romania

J.2250 In acute ΔABC the following relationship holds: $b + c \geq \sqrt{w_a^2 + (s-b)^2} + \sqrt{s_a^2 + (s-c)^2}$

Proposed by Bogdan Fuștei-Romania

J.2251 Let $M \in Int(\Delta ABC)$ and $r_i, i = 1,2,3$ —Malfatti's radies, then holds:

$$\sum_{cyc} \frac{g_a(r_b + r_c)}{h_a} \cdot AM \geq 2R\left(s + 2\sum_{cyc} \sqrt{r_1 r_2}\right)$$

Proposed by Bogdan Fuștei-Romania

J.2252 In $\Delta ABC, r_i i = 1,2,3$ —Malfatti's radies, the following relationship holds:

$$2\sum_{cyc} \frac{r_a}{w_a + a - n_a - 2\sqrt{r_1 r_2}} \leq \frac{s}{r} + \sum_{cyc} \frac{n_a}{h_a}$$

Proposed by Bogdan Fuștei-Romania

J.2253 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{s-a}{h_a - 2r} \geq 4 \cdot \sum_{cyc} \frac{m_a}{b+c}$$

Proposed by Bogdan Fuștei-Romania

J.2254 In ΔABC the following relationship holds: $\sum_{cyc} m_a \cdot \cos\frac{A}{2} \geq \frac{3s}{2}$

Proposed by Bogdan Fuștei-Romania

J.2255 If $xyz(x+y+z) > 0$ then: $\frac{1}{2}|ayz + bzx + cxy| \geq \sqrt{r(r_a + r_b + r_c)(x+y+z)xyz}$

Proposed by Bogdan Fuștei-Romania

J.2256 In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{a}{b}+\frac{b}{a}\right) \le \sqrt{\frac{2R}{r}} \cdot \sum_{cyc}\frac{s_a}{w_a}\sqrt{\frac{h_a}{r_a}}$$

Proposed by Bogdan Fuștei-Romania

J.2257 If $M \in Int(\Delta ABC)$ the following relationship holds:

$$\sum_{cyc} AM \cdot \cos\frac{A}{2} \ge \frac{1}{4} \cdot \sum_{cyc}\frac{ch_b + bh_c}{m_a}$$

Proposed by Bogdan Fuștei-Romania

J.2258 If $M \in \mathbb{R}^3, x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{MA^2}{yz} + \frac{MB^2}{zx} + \frac{MC^2}{xy} \ge \left(\frac{a+b+c}{x+y+z}\right)^2$$

Proposed by Bogdan Fuștei-Romania

J.2259 If $\alpha > 0$ is a root of the equation $x^2 - 2207x + 1 = 0$, then find the natural numbers a, b and c which satisfy the following identity $\sqrt[3]{\alpha} = \frac{a+\sqrt{b}}{c}$.

Proposed by Neculai Stanciu-Romania

J.2260 For $a > 0$ and $(a_n)_{n\ge1}$ –arithmetic progression with positive terms, solve for real numbers:

$$(a + a_{n-1})^x + (a + a_{n+2})^x = (a + a_n)^x + (a + a_{n+1})^x$$

Proposed by Neculai Stanciu-Romania

J.2261 If $x, y, z \in \mathbb{R}^*$ with $x \ge y \ge z$ then prove that:

$$\left(\frac{x}{z}\right)^y \cdot \left(\frac{y}{x}\right)^z \cdot \left(\frac{z}{y}\right)^x \le 1$$

Proposed by Neculai Stanciu-Romania

J.2262 Compute: $\Omega = \frac{\sqrt[5]{3+2\sqrt[4]{5}} - \sqrt[5]{4-4\sqrt[4]{5}}}{\left(\sqrt[5]{3+2\sqrt[4]{5}} + \sqrt[5]{4-4\sqrt[4]{5}}\right)\left(2+\sqrt[4]{5}+\sqrt[4]{25}+\sqrt[4]{125}\right)}$

Proposed by Neculai Stanciu-Romania

J.2263 Determine all triplets (x, y, z) of prime numbers, with $x < y$, which satisfy: $x + y \cdot y^y = z$

Proposed by Neculai Stanciu-Romania

J.2264 If $2^a = 3, 3^b = 4, 4^c = 5, 5^d = 6, 6^e = 7, 7^f = 8$, then compute $abcdef$.

Proposed by Neculai Stanciu-Romania

J.2265 If $A, B, C \in \left(0, \frac{\pi}{2}\right)$ and $\frac{1}{\cot A \cot B} + \frac{1}{\cot B \cot C} + \frac{1}{\cot C \cot A} = 1$, then compute:

$$E = \frac{\sin 2A + \sin 2B + \sin 2C}{\cos A \cos B \cos C}$$

Proposed by Neculai Stanciu-Romania

J.2266 If $xyz(x+y+z) > 0$ then in ΔABC the following relationship holds:

$$\sqrt{\frac{R}{2r}}|xy + yz + zx| \geq \sqrt{\left(2\sum_{cyc}\sqrt{\frac{m_a m_b}{h_a h_b}} - \sum_{cyc}\frac{m_a}{h_a}\right)xyz(x+y+z)}$$

Proposed by Bogdan Fuștei-Romania

J.2267 Let $a, b, c > 0$ and $ab + bc + ca = 3c^2$. Prove that:

$$\frac{1}{3}\left(\frac{a^4 + b^4 - 2c^4}{a^4 + b^4 + c^4}\right) \leq \frac{a^3 + b^3 - 2c^3}{a^3 + b^3 + c^3} \leq 3\left(\frac{a^2 + b^2 - 2c^2}{a^2 + b^2 + c^2}\right)$$

Proposed by Choy Fai Lam-Hong Kong

J.2268 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{bc}{m_b^2 + m_c^2 - a^2} = 2\sum_{cyc}\frac{s_a}{m_a}$$

Proposed by Ertan Yildirim-Turkiye

J.2269 In ΔABC the following relationship holds:

$$\frac{h_a r_a}{b+c} + \frac{h_b r_b}{c+a} + \frac{h_c r_c}{a+b} \leq \frac{3}{8}(a+b+c)$$

Proposed by Ertan Yildirim-Turkiye

J.2270 Let P be point in the plane of ΔABC the following relationship holds:

$$\sum_{cyc} AP\sqrt{m_a(m_b + m_c - m_a)} \geq \sqrt{\frac{1}{2}\sum_{cyc} m_a(m_b + m_c - m_a)(b^2 + c^2 - a^2) + 6F^2}$$

Proposed by Bogdan Fuștei-Romania

J.2271 If in $\Delta ABC, x, y, z, > 0$ then holds:

$$\max\left(\sum_{cyc} yz \cdot \cos\frac{A}{2}, \sum_{cyc} yz \cdot \sin A\right) \leq \frac{1}{2}(x+y+z)\sqrt{xy + yz + zx}$$

Proposed by Bogdan Fuștei-Romania

J.2272 Let $a, b, c > 0$. Prove that:

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a} \geq \sqrt{\frac{3}{2}(a^2+b^2+c^2+ab+bc+ca)}$$

Proposed by Choy Fai Lam-Hong Kong

J.2273 In $\Delta ABC, x, y, z > 0$ the following relationship holds:

$$x^2+y^2+z^2 \geq \sqrt{2}\sum_{cyc}\frac{yz\cdot\cos\frac{A}{2}}{\sqrt{1+\sin\frac{A}{2}}}$$

Proposed by Bogdan Fuștei-Romania

J.2274 In ΔABC the following relationship holds:

$$\sqrt{2}\sum_{cyc}\frac{\sqrt{1+\sin\frac{A}{2}}}{\cos\frac{A}{2}} \geq 3(2-\sqrt{3})+\frac{2R}{s}\left(\sum_{cyc}\cos\frac{A}{2}\right)^2$$

Proposed by Bogdan Fuștei-Romania

J.2275 In ΔABC the following relationship holds:

$$\frac{h_a+h_b+h_c-3r}{\sqrt{h_ah_b+h_bh_c+h_ch_a}} \geq 1+\frac{3(2-\sqrt{3})r}{s}$$

Proposed by Bogdan Fuștei-Romania

J.2276

$$\text{Let } X=\max\left\{\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right\}, Y=\min\left\{\frac{m_am_b}{h_ah_b},\frac{m_bm_c}{h_bh_c},\frac{m_cm_a}{h_ch_a}\right\}, \alpha=\prod_{cyc}\frac{m_aw_a}{n_ag_a}$$

In ΔABC the following relationship holds: $\frac{R}{r} \geq X^{\alpha}Y^{1-\alpha}+X^{1-\alpha}Y^{\alpha}$

Proposed by Bogdan Fuștei-Romania

J.2277 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sqrt{b^2+bc+c^2}}{h_a} \geq \frac{\sqrt{3}}{F}\left(R^2+7Rr-(R-2r)\sqrt{R(R-2r)}\right)$$

Proposed by Bogdan Fuștei-Romania

J.2278 In ΔABC the following relationship holds:

$$32Rr^2\prod_{cyc}\left(1+\sin\frac{A}{2}-\cos\frac{A}{2}\right)\le\prod_{cyc}(r_b+r_c+a-b-c)$$

Proposed by Bogdan Fuștei-Romania

J.2279 In ΔABC the following relationship holds:

$$\frac{s+\sqrt{(2R+5r)(h_a+h_b+h_c)}}{r}\ge 2\sum_{cyc}\frac{h_a}{g_a-s+a}$$

Proposed by Bogdan Fuștei-Romania

J.2280 In ΔABC the following relationship holds:

$$2R+5r\ge\sqrt{(2R+5r)(h_a+h_b+h_c)}\ge g_a+g_b+g_c\ge h_a+h_b+h_c$$

Proposed by Bogdan Fuștei-Romania

J.2281 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{h_a-2r}\ge\frac{s+3(2-\sqrt{3})r}{\sqrt{2R}}$$

Proposed by Bogdan Fuștei-Romania

J.2282 Let $x,y,z\in\mathbb{R}_+$ then in ΔABC the following relationship holds:

$$(x+y+z)^2\ge 2\sqrt{3}\sum_{cyc}yz\cdot\cos\frac{A}{2}$$

Proposed by Bogdan Fuștei-Romania

J.2283 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sqrt{m_a^2+m_am_b+m_b^2}}{m_am_b}>\frac{5\sqrt{3}}{s}$$

Proposed by Bogdan Fuștei-Romania

J.2284 In acute ΔABC the following relationship holds:

$$\frac{a}{w_a}+\frac{b}{w_b}+\frac{c}{w_c}\le\frac{1}{2}\sum_{cyc}\frac{(b+c)\sqrt{m_bm_c}}{m_ah_a}$$

Proposed by Bogdan Fuștei-Romania

J.2285 In $\Delta ABC, n_a$ –Nagel's cevian, holds:

$$\frac{n_a+n_b}{h_c^3}+\frac{n_b+n_c}{h_a^3}+\frac{n_c+n_a}{h_b^3}\geq\frac{2\sqrt{3}}{F}$$

Proposed by D.M. Bătinețu-Giurgiu,Mihaly Bencze-Romania

J.2286 Let $a,b,c,x,y,z>0$ then:

$$(a^2+x)(b^2+y)(c^2+z)\geq\frac{3}{4}\left(a\sqrt{yz}+b\sqrt{zx}+c\sqrt{xy}\right)^2$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

J.2287 Let $a,b,c,x,y,z>0$ then:

$$((ax+by)^2+z)((bx+cy)^2+z)((cx+ay)^2+z)\geq\frac{9}{4}(x+y)^2z^2(ab+bc+ca)$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.2288 Let $x,y,z>0$ then in ΔABC holds:

$$(a^2x^2+1)(b^2y^2+1)(c^2z^2+1)\geq 12\left(\frac{xy}{ab}+\frac{yz}{bc}+\frac{zx}{ca}\right)F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți-Romania

J.2289 In ΔABC holds: $(m_a^4+2)(m_b^4+2)(m_c^4+2)\geq 81F^2$

Proposed by D.M. Bătinețu-Giurgiu,Mihaly Bencze-Romania

J.2290 Let $x,y,z,t>0$ then in ΔABC holds:

$$(a^4x^2+t^2)(b^4y^2+t^2)(c^4z^2+t^2)\geq 3t^4F^2\left(\frac{xy}{\sin^2\frac{C}{2}}+\frac{yz}{\sin^2\frac{A}{2}}+\frac{zx}{\sin^2\frac{B}{2}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.2291 Let $m,x,y,z>0$ then in ΔABC holds:

$$\left(\frac{x^2a^4}{(y+z)^2}+m\right)\left(\frac{y^2b^4}{(z+x)^2}+m\right)\left(\frac{z^2c^4}{(x+y)^2}+m\right)\geq 9m^2F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.2292 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cot^5\frac{A}{2}}{\cot^3\frac{B}{2}}\geq 2\left(\frac{4R}{r}-3\right).$$

Proposed by Marin Chirciu-Romania

J.2293 Let $\lambda \in \mathbb{R}$ fixed. Solve in real numbers: $3^{2x^2-3\lambda x} + 3^{3\lambda x - x^2} + 3^{3\lambda x - x^2} = 3^{1+\lambda x}$.

Proposed by Marin Chirciu-Romania

J.2294 If $a_1, a_2, \dots, a_n > 0$ such that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \leq n$ then:

$$\frac{1}{\sqrt{a_1 + a_2}} + \frac{1}{\sqrt{a_2 + a_3}} + \cdots + \frac{1}{\sqrt{a_n + a_1}} \leq \frac{n}{\sqrt{2}}.$$

Proposed by Marin Chirciu-Romania

J.2295 In acute ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\sin^{2n+1} A}{\sin^{2n-1} B} \geq \left(1 + \frac{r}{R}\right)^2, n \in \mathbb{N}^*.$$

Proposed by Marin Chirciu-Romania

J.2296 In ΔABC the following relationship holds: $\sum_{cyc} \frac{a}{\sqrt{b+\lambda c}} \geq \sqrt{\frac{6s}{\lambda+1}}, \lambda \geq 0$.

Proposed by Marin Chirciu-Romania

J.2297 Let $\lambda > 0$ fixed. Solve for real numbers:

$$\big((\lambda+1)x + 1\big)\sqrt{1+\lambda x} = \sqrt{1-x} + \sqrt{1 + (\lambda - 1)x - \lambda x^2} + \lambda x + 2.$$

Proposed by Marin Chirciu-Romania

J.2298 If $x, y, z > 0, xy + yz + zx = 3$ then in acute ΔABC holds:

$$\sum_{cyc} \frac{\tan^{n+1} A \tan^{n+1} B}{x^n y^n} \geq 3^{n+2}, n \in \mathbb{N}.$$

Proposed by Marin Chirciu-Romania

J.2299 If $x, y, z > 0, x + y + z = 3$ and $\lambda \geq 0$ then holds:

$$\frac{x}{y^3 + \lambda y^2} + \frac{y}{z^3 + \lambda z^2} + \frac{z}{x^3 + \lambda x^2} \geq \frac{3}{\lambda + 1}.$$

Proposed by Marin Chirciu-Romania

J.2300 If $a, b, c > 0$ such that $abc = 1$ and $\lambda \geq 0$ then:

$$\frac{a^4}{b^2 + b + \lambda bc} + \frac{b^4}{c^2 + c + \lambda ca} + \frac{c^4}{a^2 + a + \lambda ab} \geq \frac{3}{\lambda + 2}.$$

Proposed by Marin Chirciu-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

PROBLEMS FOR SENIORS

S.2201 Let $(a_n)_{n\geq 1}, a_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$ and $(b_n)_{n\geq 1}$ an sequence of real numbers strictly positive such that $\lim\limits_{n\to\infty} \frac{b_n}{\sqrt{n+1}} \cdot e^{2\sqrt{n}} = b > 0$. Find:

$$\Omega = \lim_{n\to\infty} \left(e^{a_{n+1}} - e^{a_n}\right) \cdot b_n$$

Proposed by D.M. Bătinețu-Giurgiu, Mihaly Bencze-Romania

S.2202 Let $t \in \mathbb{R}$ and $(a_n)_{n\geq 1}$ be sequence of real numbers strictly positive such that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{n\cdot a_n} = a > 0$, then find:

$$\Omega = \lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}^t} - \sqrt[n]{a_n^t}}{\sqrt[n]{a_n^{t-1}}}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

S.2203 If $m \geq 0, x, y > 0, T, U \in Int(\Delta ABC)$ and $t_a = d(T, BC), u_a = d(U, BC)$ and analogous t_b, u_b, t_c, u_c, then:

$$\frac{a^{m+1}b}{(xt_b + yu_b)^m} + \frac{b^{m+1}c}{(xt_c + yu_c)^m} + \frac{c^{m+1}a}{(xt_a + yu_a)^m} \geq \frac{2^{m+2}\left(\sqrt{3}\right)^{m+1}}{(x+y)^m} \cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți-Romania

S.2204 If $t, u, v, x, y, z > 0$ then: $(x^2 + t^2)(y^2 + u^2)(z^2 + v^2) \geq \frac{3}{4}(xuv + ytv + ztu)^2$

Proposed by D.M. Bătinețu-Giurgiu, Mihaly Bencze-Romania

S.2205 Let $x, y > 0$ and m_A, m_B, m_C, m_D be lengths of medians in tetrahedron $ABCD$ and

$$M \in Int(ABCD), d_A = d\big(M, (BCD)\big), d_B = d\big(M, (CDA)\big), d_C = d\big(M, (DAB)\big),$$