SOCIETATEA DE ȘTIINȚE MATEMATICE DIN ROMÂNIA
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Filiala Mehedinți - Mehedinți Branch
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In acute $\Delta\ ABC$ with orthocenter H:

$$HA \cdot HB + HB \cdot HC + HC \cdot HA \le 3R^2.$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Daniel Sitaru – Romania , Solution 2 by Kevin Soto Palacios – Huarmey – Peru , Solution 3 by Soumava Chakraborty-Kolkata-India

Solution 1 by Daniel Sitaru – Romania

$$\sum HA \cdot HB = 4R^2 \sum \cos A \cos B =$$

$$= 4R^2 \cdot \frac{s^2 + r^2 - 4R^2}{4R^2} = s^2 + r^2 - 4R^2 \le \frac{27}{4}R^2 + \frac{R^2}{4} - 4R^2 = 3R^2$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

En un triángulo acutángulo ABC con Ortocentro H:

$$HAHB + HBHC + HCHA \le 3R^2$$

Dado que es un triángulo acutángulo:

$$HA = 2R\cos A > 0, HB = 2R\cos B > 0, HC = 2R\cos C > 0$$

Teniendo en cuenta las siguientes identidades y desigualdades en un triángulo ABC:

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R} \le \frac{3}{2},$$

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C \ge \frac{3}{4}$$

La desigualdad es equivalente:

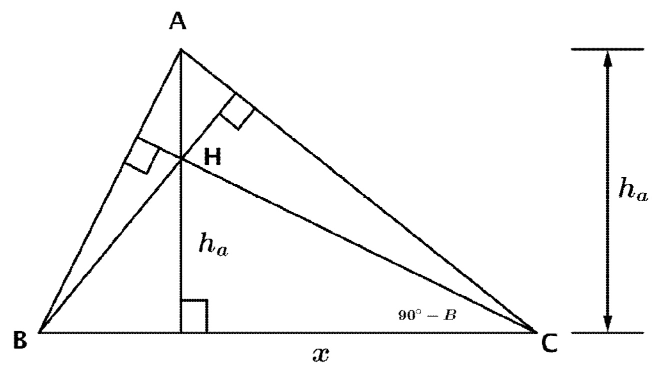
$$2\cos A \cos B + 2\cos B \cos C + 2\cos C \cos A \le \frac{3}{2}$$

Lo cual es cierto ya que

$$2\cos A\cos B + 2\cos B\cos C + 2\cos C\cos A =$$

$$= (\cos A + \cos B + \cos C)^2 - (\cos^2 A + \cos^2 B + \cos C^2) \le \frac{9}{4} - \frac{3}{4} = \frac{3}{2}$$

Solution 3 by Soumava Chakraborty-Kolkata-India



$$x = b\cos C$$

$$\frac{h'_a}{x} = \tan(90^\circ - B) = \cot B$$

$$\Rightarrow h'_a = b\cos C\cot B = 2R\sin B\cos C\frac{\cos B}{\sin B}$$

$$\Rightarrow h'_a = 2R\cos B\cos C$$

$$\therefore AH = h_a - h'_a = b\sin C - 2R\cos B\cos C$$

$$= 2R\sin B\sin C - 2R\cos B\cos C = -2R\cos(B + C) = 2R\cos A$$

$\therefore AH = 2R\cos A$ ***Similarly***, $BH = 2R\cos B$ ***and*** $CH = 2R\cos C$

$$\therefore LHS = 4R^2\sum\cos A\cos B = 2R^2\left(2\sum\cos A\cos B\right)$$

$$= 2R^2\left\{\left(\sum\cos A\right)^2 - \sum\cos^2 A\right\} = 2R^2\left\{\left(1 + \frac{r}{R}\right)^2 - \sum(1 - \sin^2 A)\right\}$$

$$= 2R^2\left\{\frac{(R+r)^2}{R^2} - 3 + \frac{\sum a^2}{4R^2}\right\}$$

$$= 2(R+r)^2 - 6R^2 + (s^2 - 4Rr - r^2) = s^2 - 4R^2 + r^2$$

$\therefore$ ***given inequality*** $\Leftrightarrow s^2 - 4R^2 + r^2 \le 3R^2 - \frac{r^2(R-2r)}{R-r}$

$$\Leftrightarrow s^2 \le 7R^2 - \frac{r^2(2R-3r)}{R-r}$$

$$\Leftrightarrow s^2(R-r) \le 7R^3 - 7R^2r - 2Rr^2 + 3r^3 \quad \textbf{(1)}$$

Gerretsen $\Rightarrow s^2(R-r) \underset{(2)}{\le} (4R^2 + 4Rr + 3r^2)(R-r)$

$$= 4R^3 - Rr^2 - 3r^3$$

(1), (2) $\Rightarrow$ ***it suffices to prove:***

$$4R^3 - Rr^2 - 3r^3 \le 7R^3 - 7R^2r - 2Rr^2 + 3r^3$$

$$\Leftrightarrow 3R^3 - 7R^2r - Rr^2 + 6r^3 \ge 0 \Leftrightarrow (t-2)(3t^2 - t - 3) \ge 0$$

$$\left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\{3t(t-2) + 5(t-2) + 7\} \ge 0$$

which is true $\because t = \frac{R}{r} \ge 2$

(Proved)