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If $0 < x < \frac{\pi}{2}$ then:

$$\left(\left(\frac{\sin x}{x}\right)^2 + \left(\frac{\tan x}{x}\right)^3\right)\left(\left(\frac{\sin x}{x}\right)^3 + \left(\frac{\tan x}{x}\right)^4\right)\left(\left(\frac{\sin x}{x}\right)^4 + \left(\frac{\tan x}{x}\right)^5\right) > 8$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India, Solution 2 by Tran Hon-Dong Thap-Vietnam, Solution 3 by Soumava Chakraborty-Kolkata-India, Solution 4 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Amit Dutta-Jamshedpur-India

$$\because x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \tan x, \sin x > 0$$

$$AM > GM \Rightarrow \left(\frac{\sin x}{x}\right)^2 + \left(\frac{\tan x}{x}\right)^3 > 2\sqrt{\left(\frac{\sin^2 x \tan^3 x}{x^5}\right)}$$

$$\Rightarrow \left(\frac{\sin x}{x}\right)^2 + \left(\frac{\tan x}{x}\right)^3 > 2\left(\frac{\sin x \tan x}{x^2}\right)\sqrt{\left(\frac{\tan x}{x}\right)}$$

$$> \frac{2 \sin x \tan x}{x^2}\left\{\because \frac{\tan x}{x} > 1; \forall x \in \left(0, \frac{\pi}{2}\right)\right\}$$

$$\Rightarrow \left(\frac{\sin x}{x}\right)^2 + \left(\frac{\tan x}{x}\right)^3 > 2\left(\frac{\sin x \tan x}{x^2}\right) \quad \textit{(1)}$$

Again, using $AM > GM$: $\left(\frac{\sin x}{x}\right)^3 + \left(\frac{\tan x}{x}\right)^4 > 2\left(\frac{\sin x \tan x}{x}\right)\sqrt{\left(\frac{\sin x \tan x}{x^2}\right)\left(\frac{\tan x}{x}\right)}$

$$\left(\frac{\sin x}{x}\right)^3+\left(\frac{\tan x}{x}\right)^4>2\left(\frac{\sin x\tan x}{x^2}\right)\sqrt{\left(\frac{\sin x\tan x}{x^2}\right)}\quad (2)$$

$$\left\{\frac{\tan x}{x}>1;\forall x\in\left(0,\frac{\pi}{2}\right)\right\}$$

Again, using $AM>GM$: $\left(\frac{\sin x}{x}\right)^4+\left(\frac{\tan x}{x}\right)^5>2\left(\frac{\sin^2 x\tan^2 x}{x^4}\right)\sqrt{\frac{\tan x}{x}}>2\left(\frac{\sin x\tan x}{x^2}\right)^2\sqrt{\frac{\tan x}{x}}$

$$\left(\frac{\sin x}{x}\right)^4+\left(\frac{\tan x}{x}\right)^5>2\left(\frac{\sin x\tan x}{x^2}\right)^2\quad (3)$$

$$\left\{\frac{\tan x}{x}>1\ \forall x\in\left(0,\frac{\pi}{2}\right)\right\}$$

Now, multiplying (1), (2) and (3):

$$\left[\left(\frac{\sin x}{x}\right)^2+\left(\frac{\tan x}{x}\right)^3\right]\left[\left(\frac{\sin x}{x}\right)^3+\left(\frac{\tan x}{x}\right)^4\right]\left[\left(\frac{\sin x}{x}\right)^4+\left(\frac{\tan x}{x}\right)^5\right]>8\left(\frac{\sin x\tan x}{x^2}\right)^{\frac{9}{2}}\quad (4)$$

Now, let $F(x)=\sin x\tan x-x^2,\forall x\in\left(0,\frac{\pi}{2}\right)$

$$F'(x)=\sin x\sec^2 x+\tan x\cos x-2x$$

$$F'(x)=\tan x\sec x+\sin x-2x$$

$$F''(x)=\tan^2 x\sec x+\sec^2 x+\cos x-2$$

$$F'''(x)=\tan^3 x\sec x+2\sec^3 x\tan x+3\sec^2 x\tan x+\sin x>0$$

$$F'''(x)>0\because F'''(x)>0$$

$\Rightarrow F''(x)$ ***is an increasing function*** $\forall x\in\left(0,\frac{\pi}{2}\right)$

$$x>0;F''(x)>F''(0)>0;F''(x)>0$$

$\Rightarrow F'(x)$ ***is an increasing function*** $\forall x\in\left(0,\frac{\pi}{2}\right)$

$$x>0;F'(x)>F'(0);F'(x)>0$$

$\Rightarrow F(x)$ ***is an increasing function*** $\forall x\in\left(0,\frac{\pi}{2}\right)$

$$x>0;F(x)>F(0);F(x)>0\Rightarrow(\sin x\tan x-x^2)>0$$

$$\left(\frac{\sin x\tan x}{x^2}\right)>1$$

Now, using this result i.e. $\left(\frac{\sin x\tan x}{x^2}\right)>1$

in (4), we have:

$$\left[\left(\frac{\sin x}{x}\right)^2+\left(\frac{\tan x}{x}\right)^3\right]\left[\left(\frac{\sin x}{3}\right)^3+\left(\frac{\tan x}{3}\right)^4\right]\left[\left(\frac{\sin x}{x}\right)^4+\left(\frac{\tan x}{x}\right)^5\right]>8\ \textit{(proved)}$$

Solution 2 by Tran Hon-Dong Thap-Vietnam

For $x>0$ *we have:* $\tan x>x\Rightarrow\frac{\tan x}{x}>1$

$$\because\left(\frac{\sin x}{x}\right)^2+\left(\frac{\tan x}{x}\right)^3>\left(\frac{\sin x}{x}\right)^2+\left(\frac{\tan x}{x}\right)^2>\frac{1}{2}\left[\frac{\sin x}{x}+\frac{\tan x}{x}\right]^2\overset{(1)}{>}2$$

$$\textit{(1)}\Leftrightarrow\left(\frac{\sin x}{x}+\frac{\tan x}{x}\right)^2>4$$

$$\Leftrightarrow(\sin x+\tan x)>4x^2\Leftrightarrow\sin x+\tan x>2x$$

$$\Leftrightarrow\sin x+\tan x-2x>0\quad\textit{(2)}$$

$$\textit{Let } f(x)=\sin x+\tan x-2x\ \left(0<x<\frac{\pi}{2}\right)\Rightarrow f'(t)=\cos x+\frac{1}{\cos^2 x}-2$$

$$=\frac{(\cos x-1)(\cos^2 x-\cos x-1)}{\cos^2 x}>0\ (\because 0<\cos x<1)$$

$$f(x)\nearrow\left(0,\frac{\pi}{2}\right)\Rightarrow f(x)>f(0)\Rightarrow\textit{(2) true}\Rightarrow\textit{(1) true.}$$

$$\left(\frac{\sin x}{x}\right)^3+\left(\frac{\tan x}{x}\right)^4>\left(\frac{\sin x}{x}\right)^3+\left(\frac{\tan x}{x}\right)^3>\frac{1}{2^2}\left[\frac{\sin x}{x}+\frac{\tan x}{x}\right]^3\overset{(3)}{>}2$$

$$\textit{(3)}\Leftrightarrow\left(\frac{\sin x}{x}+\frac{\tan x}{x}\right)^3>2^3\Leftrightarrow(\sin x+\tan x)^3>(2x)^3$$

$$\Leftrightarrow\sin x+\tan x>2x\Leftrightarrow\sin x+\tan x-2x>0\ \textit{(It is true by (2))}\Rightarrow\textit{(3) true.}$$

$$\left(\frac{\sin x}{x}\right)^4+\left(\frac{\tan x}{x}\right)^5>\left(\frac{\sin x}{x}\right)^4+\left(\frac{\tan x}{x}\right)^4>\frac{1}{2^3}\left[\frac{\sin x}{x}+\frac{\tan x}{x}\right]^4\overset{(4)}{>}2$$

$$\Leftrightarrow\left[\frac{\sin x}{x}+\frac{\tan x}{x}\right]^4>2^4\Leftrightarrow(\sin x+\tan x)^4>(2x)^4\Leftrightarrow\sin x+\tan x>2x$$

$$\Leftrightarrow\sin x+\tan x-2x>0\ \textit{(It is true by (2))}\overset{(1).(3).(4)}{\Rightarrow}LHS>2\cdot2\cdot2=8$$

(Proved)

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\left(\left(\frac{\sin x}{x}\right)^2+\left(\frac{\tan x}{x}\right)^3\right)\left(\left(\frac{\sin x}{x}\right)^3+\left(\frac{\tan x}{x}\right)^4\right)\left(\left(\frac{\sin x}{x}\right)^4+\left(\frac{\tan x}{x}\right)^5\right)\overset{(1)}{>}8$$

$$\textit{Wilker}\Rightarrow\left(\frac{\sin x}{2}\right)^2+\frac{\tan x}{x}>2,\forall x\in\left(0,\frac{\pi}{2}\right)\ \textit{and,}$$

$$\textit{Huygens} \Rightarrow 2\frac{\sin x}{x} + \frac{\tan x}{x} > 3$$

$$\textit{Let} \left(\frac{\sin x}{x}\right) = a, \left(\frac{\tan x}{x}\right) = b \therefore \textbf{(1)} \Leftrightarrow (a^2 + b^3)(a^3 + b^4)(a^4 + b^5) \overset{(2)}{>} 8$$

$$\because b > 1 \therefore \textit{LHS of (2)} > (a^2 + b^2)(a^3 + b^3)(a^4 + b^4)$$

$$\overset{\textit{Chebyshev}}{>} (a^2 + b^2)\left\{\frac{1}{2}(a+b)(a^2+b^2)\right\}\left\{\frac{1}{2}(a^2+b^2)^2\right\}$$

$$= \frac{1}{4}(a^2 + b^2)^4(a+b) \overset{b>1}{>} \frac{1}{4}(a^2+b)^4(a+b) \overset{\textit{Wilker}}{>} \frac{1}{4}\{(2)^4\}(a+b) = 4(a+b)$$

$$\overset{\textit{Huygens}}{>} 4(3-a) = 4\left(3 - \frac{\sin x}{x}\right) \overset{\frac{\sin x}{x}<1}{>} 4(3-1) = 8$$

(Proved)

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$\sin x \geq x - \frac{x^3}{6} \textit{ and } \tan x \geq x + \frac{x^3}{3} \textit{ for all } x \geq 0$$

$$\textit{again, } \frac{\pi}{2} > x > 0 \Rightarrow \left(1 + \frac{x^2}{3}\right)^4 > \left(1 + \frac{x^2}{3}\right)^3$$

$$\therefore \left(\left(\frac{\sin x}{x}\right)^2 + \left(\frac{\tan x}{x}\right)^3\right)\left(\left(\frac{\sin x}{x}\right)^3 + \left(\frac{\tan x}{x}\right)^4\right)\left(\left(\frac{\sin x}{x}\right)^4 + \left(\frac{\tan x}{x}\right)^5\right) \overset{\textit{Holder}}{\geq}$$

$$\geq \left(\left(\frac{\sin x}{x}\right)^3 + \left(\frac{\tan x}{x}\right)^4\right)^3 > \left(\left(\frac{\sin x}{x}\right)^3 + \left(\frac{\tan x}{x}\right)^3\right)^3$$

$$= \left(\left(1 - \frac{x^2}{6}\right)^3 + \left(1 + \frac{x^2}{3}\right)^3\right)^3 = \left(1 - \frac{x^2}{2} + \frac{x^4}{12} - \frac{x^6}{216} + 1 + x^2 + \frac{x^4}{9} + \frac{x^6}{27}\right)^3$$

$$= \left(2 + \frac{x^2}{2} + \frac{7x^4}{36} + \frac{7x^6}{216}\right)^3 > 2^3 = 8 \quad \textit{(Proved)}$$