If $a,b,c>0, a^2+b^2+c^2=26(a+b+c)$ then:

$$\frac{1}{\sqrt{a+b^2}}+\frac{1}{\sqrt{b+c^2}}+\frac{1}{\sqrt{c+a^2}}\geq\frac{1}{\sqrt{a+b+c}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marjan Milanovic-Nis-Serbia , Solution 2 by Soumava Chakraborty-Kolkata-India, Solution 3 by Seyran Ibrahimov-Maasilli-Azerbaidian

Solution 4 by Shivam Sharma-New Delhi-India, Solution 5 by Abdul Aziz-Semarang-Indonesia , Solution 6 by Eliezer Okeke-Nigeria , Solution 7 by Nguyen Thanh Nho-Tra Vinh-Vietnam

Solution 1 by Marjan Milanovic-Nis-Serbia

By Jensen, since $x^{\left(-\frac{1}{2}\right)}$ is convex,

$$\sum(a+b^2)^{\left(-\frac{1}{2}\right)}\geq 3\left(\frac{a+b+c+a^2+b^2+c^2}{3}\right)^{\left(-\frac{1}{2}\right)}=$$

$$=3\left(\frac{27(a+b+c)}{3}\right)^{\left(-\frac{1}{2}\right)}=(a+b+c)^{\left(-\frac{1}{2}\right)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS\overset{Bergstrom}{\geq}\frac{9}{\sqrt{a+b^2}+\sqrt{b+c^2}+\sqrt{c+a^2}}\quad \textbf{(1)}$$

$$\left(\sqrt{a+b^2}+\sqrt{b+c^2}+\sqrt{c+a^2}\right)^2\leq 3\left(\sum a+\sum a^2\right)\left(\because \left(\sum x\right)^2\leq 3\sum x^2\right)$$

$$=\frac{8}{\left(\sum a\right)}\left(\because \sum a^2=26\sum a\right)$$

$$\Rightarrow\frac{1}{\sqrt{a+b^2}+\sqrt{b+c^2}+\sqrt{c+a^2}}\geq\frac{1}{9\sqrt{\sum a}}\Rightarrow\frac{9}{\sqrt{a+b^2}+\sqrt{b+c^2}+\sqrt{c+a^2}}\geq\frac{1}{\sqrt{\sum a}}\quad \textbf{(2)}$$

$$\textbf{(1), (2)}\Rightarrow LHS\geq\frac{1}{\sqrt{\sum a}}$$

(Proved)

Solution 3 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\frac{1}{\sqrt{a+b^2}}+\frac{1}{\sqrt{b+c^2}}+\frac{1}{\sqrt{c+a^2}}\geq\frac{1}{\sqrt{a+b+c}}$$

$$\left.\begin{array}{r} a+b^2=x^2\\ b+c^2=y^2\\ c+a^2=z^2\end{array}\right| \begin{array}{c} a+b+c+a^2+b^2+c^2=x^2+y^2+z^2\\ 27(a+b+c)=x^2+y^2+z^2\end{array}$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{3\sqrt{3}}{\sqrt{x^2+y^2+z^2}}\geq\frac{9}{x+y+z}\Rightarrow x^2+y^2+z^2\geq\frac{(x+y+z)^2}{3}$$

$$(x+y+z)(xy+yz+xz)\geq 9xyz \quad \textbf{(AM-GM)}$$

Solution 4 by Shivam Sharma-New Delhi-India

Applying A.M ≥ H.M, we get,

$$\frac{1}{\sqrt{a+b^2}}+\frac{1}{\sqrt{b+c^2}}+\frac{1}{\sqrt{c+a^2}}\geq\frac{9}{\left(\sqrt{a+b^2}+\sqrt{b+c^2}+\sqrt{c+a^2}\right)}$$

Applying Cauchy – Schwarz inequality and then put

$a^2+b^2+c^2=26(a+b+c)$, ***we get,***

$$\frac{1}{\sqrt{a+b^2}}+\frac{1}{\sqrt{b+c^2}}+\frac{1}{\sqrt{c+a^2}}\geq\frac{9}{\sqrt{(1+1+1)(a+b+c+a^2+b^2+c^2)}}$$

$$=\frac{9}{\sqrt{3\big(27(a+b+c)\big)}}=\frac{9}{\sqrt{9\times 9\,(a+b+c)}}=\frac{1}{\sqrt{a+b+c}}$$

Hence,

$$\frac{1}{\sqrt{a+b^2}}+\frac{1}{\sqrt{b+c^2}}+\frac{1}{\sqrt{c+a^2}}\geq\frac{1}{\sqrt{a+b+c}}$$

(Proved)

Solution 5 by Abdul Aziz-Semarang-Indonesia

$$a, b, c > 0$$

$$a^2 + b^2 + c^2 = 26(a + b + c)$$

$$\frac{1}{\sqrt{a+b^2}} + \frac{1}{\sqrt{b+c^2}} + \frac{1}{\sqrt{c+a^2}} \geq \frac{9}{\sqrt{a+b^2}+\sqrt{b+c^2}+\sqrt{c+a^2}} \quad \textbf{(AM-HM)}$$

$$\geq \frac{9}{\sqrt{(1^2 + 1^2 + 1^2)(a + b^2 + b + c^2 + c + a^2)}} \quad (cs)$$

$$= \frac{9}{\sqrt{3(a^2 + b^2 + c^2 + a + b + c)}} = \frac{9}{\sqrt{3\big(27(a + b + c)\big)}} = \frac{1}{\sqrt{a + b + c}}$$

Inequality holds when $a = b = c = 2b$

Solution 6 by Eliezer Okeke-Nigeria

Given $\sum a^2 = 2b \sum a \,;\, a, b, c > 0$

Show

$$\frac{1}{\sqrt{a + b^2}} + \frac{1}{\sqrt{b + c^2}} + \frac{1}{\sqrt{c + a^2}} \geq \frac{1}{\sqrt{a + b + c}}$$

$$\sum\left(\frac{1}{\sqrt{a + b^2}}\right)\sum\left(\frac{1}{\sqrt{a + b^2}}\right)\sum(a + b^2) \overset{Hölder}{\geq} (1 + 1 + 1)^3$$

$$\Rightarrow \sum \frac{1}{\sqrt{a + b^2}} \geq \sqrt{\frac{27}{\sum(a + b^2)}} = \frac{1}{\sqrt{\frac{\sum(a + b^2)}{27}}}$$

It suffices to show

$$\sqrt{\frac{\sum(a + b^2)}{27}} \leq \sqrt{a + b + c} = \sqrt{\sum a}$$

$$\sqrt{\frac{\sum(a+b^2)}{27}}=\sqrt{\frac{\sum a+\sum b^2}{27}}=\sqrt{\frac{\sum a+26\sum a}{27}}=\sqrt{\sum a}$$

(Proved)

Solution 7 by Nguyen Thanh Nho-Tra Vinh-Vietnam

$$\frac{1}{\sqrt{a+b^2}}+\frac{1}{\sqrt{b+c^2}}+\frac{1}{\sqrt{c+a^2}}\overset{C-S}{\geq}\frac{(1+1+1)^2}{\sqrt{a+b^2}+\sqrt{b+c^2}+\sqrt{c+a^2}}$$

$$=\frac{9}{1\cdot\sqrt{a+b^2}+1\cdot\sqrt{b+c^2}+1\cdot\sqrt{c+a^2}}\geq$$

$$\underset{C-B-S}{\geq}\frac{9}{\sqrt{(1^2+2^2+3^2)\left[\left(\sqrt{a+b^2}\right)^2+\left(\sqrt{b+c^2}\right)^2+\left(\sqrt{c+a^2}\right)^2\right]}}$$

$$=\frac{9}{\sqrt{3\cdot(a^2+b^2+c^2+a+b+c)}}=\frac{9}{\sqrt{3\cdot 27(a+b+c)}}=\frac{1}{\sqrt{a+b+c}}$$

$$\text{“=”}\Leftrightarrow\begin{cases}a^2+b^2+c^2=26(a+b+c)\\ \frac{1}{\sqrt{a+b^2}}=\frac{1}{\sqrt{b+c^2}}=\frac{1}{\sqrt{c+a^2}}\end{cases}\Leftrightarrow a=b=c=26$$