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PROBLEMS FOR JUNIORS

JP.046. Let a, b, c, d be positive real numbers such that $a+b+c+d=4$. Prove that

$$\frac{a}{b(b+c+d)^2}+\frac{b}{c(c+d+a)^2}+\frac{c}{d(d+a+b)^2}+\frac{d}{a(a+b+c)^2}\geq\frac{4}{9}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.047. Let a, b, c be positive real numbers such that $ab+bc+ca+abc\leq 4$. Prove that

a. $3+a+b+c\geq 2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})$.

b. $3+\frac{5}{3}(a+b+c)\geq(\sqrt[3]{a}+\sqrt[3]{b})(\sqrt[3]{b}+\sqrt[3]{c})(\sqrt[3]{c}+\sqrt[3]{a})$.

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.048. Prove that for any positive real numbers a, b, c

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{1}{3}+\frac{(a+b)(b+c)(c+a)}{a^2b+b^2c+c^2a}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.049. Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that

$$\frac{1}{x_1}+\frac{2}{x_2}+\ldots+\frac{n}{x_n}=\frac{n(n+1)}{2}.$$

Find the minimum possible value of

$$x_1+x_2^2+\ldots+x_n^n.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.050. Let a, b, c and d be distinct positive integers such that

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+d}+\frac{d}{d+a}$$

is a integer. Prove that $a+b+c+d$ is not prime.

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.051. Prove that in any triangle the following relationship holds:

$$\frac{a^2+b^2+c^2}{ab+bc+ca}\le\frac{R}{nR+(1-n)\cdot 2r}$$

where $0\le n\le\frac{1}{2}$.

Proposed by Marin Chirciu - Romania

JP.052. Given $a,b,c>0$ and $a^2+b^2+c^2=6$, prove

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+a+b+c\ge 6$$

Proposed by Nguyen Phuc Tang - Dong Thap - Vietnam

JP.053. If $a,b,c>0$ and $a+b+c=3$ prove that

$$\sum a\left(\frac{1}{b^n}+\frac{1}{c^n}\right)\ge\frac{18}{a^n+b^n+c^n}$$

where $n\ge 0$.

Proposed by Marin Chirciu - Romania

JP.054. Let m_a,m_b,m_c be the lengths of the medians of a triangle ABC. Prove that

$$\frac{9}{4R+r}\le\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\le\frac{1}{r},$$

where R and r are the circumradius and inradius of ABC respectively.

Proposed by Martin Lukarevski - Stip - Macedonia

JP.055. Let $ABCD$ be an inscriptible and circumscriptible quadrilateral, p its semi perimeter. R and r the radii of circumcenter, respectively incenter, a,b,c,d its sides (a and c are the opposite sides). Prove that:

a) $2\frac{R^2}{r^2}\ge\frac{a}{c}+\frac{c}{a}+\frac{b}{d}+\frac{d}{b}\ge 2\sqrt{2}\frac{R}{r}$

b) $\frac{R^2}{r^2}-4\ge\left(\frac{a}{c}+\frac{c}{a}\right)\left(\frac{b}{d}+\frac{d}{b}\right)$

Proposed by Vasile Jiglău - Romania

JP.056. Let s_a is symedian and r_a,r are exradius and inradius triangle of ABC respectively. Prove that

$$\frac{r_a}{s_a+r}+\frac{r_b}{s_b+r}+\frac{r_c}{s_c+r}\ge\left(\frac{3r}{R}\right)^2$$

Proposed by Mehmet Şahin - Ankara - Turkey

JP.057. Let ABC be an arbitrary triangle and I_a, I_b, I_c are excenters of ABC. I_aBC, I_bCA, I_cAB are the extriangles of ABC. Let $h_i (i = 1, 2, 3, \ldots, 9)$ the altitudes of extriangles. Prove that

$$\prod_{i=1}^{9} h_i = \left(\prod_{a,b,c} r_a \right)^3$$

Proposed by Mehmet Şahin - Ankara - Turkey

JP.058. Prove that for all $x \in \mathbb{R}$ we have

$$\cos(\sin x) > |\sin(\cos x)|$$

Proposed by Abdallah El Farisi – Bechar – Algerie

JP.059. Let a, b, c be the side lengths of a triangle ABC with inradius r. Prove that

$$\frac{1}{a^3} \tan \frac{A}{2} + \frac{1}{b^3} \tan \frac{B}{2} + \frac{1}{c^3} \tan \frac{C}{2} \le \frac{R}{48r^4}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

JP.060. Let a, b and c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a} \le \frac{3\sqrt{3}}{2} R$$

Proposed by George Apostolopoulos - Messolonghi - Greece

PROBLEMS FOR SENIORS

SP.046. Prove that for every positive integer n,

$$\ln \frac{n+1}{2} < \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < \log_2 \frac{n+1}{2}.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.047. Evaluate without calculator

$$\sum_{k=1}^{17} \cos^4 \frac{k\pi}{36}.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.048. Prove that the following inequality holds for all non-negative real numbers a, b, c

$$(a^4+b^4+c^4)(ab^3+bc^3+ca^3) \ge (a^3b+b^3c+c^3a)(a^2b^2+b^2c^2+c^2a^2)$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.049. Prove that the following inequality holds for all positive real numbers x, y

$$x^{y-x} \cdot y^{x-y} \leq 1.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.050. Let $a \geq b \geq c > 0$. Prove that

$$a^{a-b}b^{b-c}c^{c-a} \geq 1.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.051. If $a, b, x, y, z \in (0, \infty)$ and $m \in [0, \infty)$ then:

$$\frac{x}{(ay+bz)^{m+1}}+\frac{y}{(az+bx)^{m+1}}+\frac{z}{(ax+by)^{m+1}} \geq \frac{3^{m+1}}{(a+b)^{m+1}(x+y+z)^m}$$

Proposed by D.M. Bătineţu - Giurgiu; Neculai Stanciu - Romania

SP.052. In any triangle ABC the following relationship holds:

$$\frac{1}{(\cos A+\cos B)^2}+\frac{1}{(\cos B+\cos C)^2}+\frac{1}{(\cos C+\cos A)^2} \geq 3$$

Proposed by D.M. Bătineţu - Giurgiu; Neculai Stanciu - Romania

SP.053. If $x, y, z \in (0, \infty)$ then:

$$\frac{x}{(y+z)^3}+\frac{y}{(z+x)^3}+\frac{z}{(x+y)^3} \geq \frac{27}{8(x+y+z)^2}$$

Proposed by D.M. Bătineţu - Giurgiu - Romania

SP.054. Let $a \in \left(0, \frac{\pi}{2}\right), b \in [1, \infty), m, n \in \mathbb{R}_+^*$ and $f, g, h, k : [-a, a] \to \mathbb{R}$ be a continuous functions such that: $f(-x) = -f(x), g(-x) = -g(x), h(-x) = h(x), k(-x) = k(x)$. Evaluate

$$\int_{-a}^{a} \frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n} x} dx$$

Proposed by D.M. Bătineţu - Giurgiu; Neculai Stanciu - Romania

SP.055. Let m_a, m_b, m_c be the lengths of medians of a triangle ABC with inradius r. Prove that

$$\frac{m_a+m_b+m_c}{\sin^2 A+\sin^2 B+\sin^2 C} \geq 4r.$$

Proposed by George Apostolopoulos - Messolonghi - Greece

SP.056. Let ABC be a triangle such that

$$\left(\frac{1}{\sin B}+\frac{1}{\sin C}\right)\cdot\left(-\sin A+\sin B+\sin C\right)=2$$

Prove that $\sphericalangle A\leq\frac{\pi}{3}$.

Proposed by George Apostolopoulos - Messolonghi - Greece

SP.057. If $a,b,c,d\in\mathbb{R}_+^*, a<b$ and $f:\mathbb{R}\to\mathbb{R}_+^*$ is a continuous function such that $f(a+b-x)=f(x),\forall x\in\mathbb{R}$, then evaluate

$$\int_a^b\frac{f(x-a)\Big(c+df(b-x)\Big)}{c\Big(f(x-a)+f(b-x)\Big)+2df(x-a)f(b-x)}dx$$

Proposed by D.M. Bătineţu - Giurgiu; Neculai Stanciu - Romania

SP.058. Compute:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\tan\frac{1}{n+i}-\tan\frac{1}{n+i+1}\right)\left(\cos\frac{1}{n+1}+\cos\frac{1}{n+2}+\ldots+\cos\frac{1}{n+i}\right)$$

Proposed by Daniel Sitaru - Romania

SP.059. Compute:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(1^p+2^p+\ldots+k^p)^2}{n^{p+1}(1^p+2^p+\ldots+n^p)};p\in\mathbb{N}$$

Proposed by Daniel Sitaru - Romania

SP.060. Prove that if $a,b,c,d\in\mathbb{R};a^2+b^2\neq 0;c^2+d^2\neq 0$ then:

$$\frac{(ad-bc)\Big(3(a^2+b^2)(c^2+d^2)-4(ad-bc)^2\Big)}{\Big((a^2+b^2)(c^2+d^2)\Big)^{\frac{3}{2}}}\leq 1$$

Proposed by Daniel Sitaru - Romania

UNDERGRADUATE PROBLEMS

UP.046. Let a,b,c be positive real numbers such that $a+b+c=1$. Prove that

$$a^{a^2}b^{b^2}c^{c^2}\geq(a^2+b^2+c^2)^{a^2+b^2+c^2}.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

UP.047. Let a, b, c be distinct rational numbers such that

$$\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}=0.$$

Prove that

$$\sqrt{\frac{(b-c)^4}{a^2}+\frac{(c-a)^4}{b^2}+\frac{(a-b)^4}{c^2}}$$

is a rational number.

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

UP.048. Let a, b, c be non-negative real numbers such that $a+b+c=1$. Prove that

$$a^4+b^4+c^4+26abc \leq 1$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

UP.049. Prove that the following inequality holds for any triangle ABC,

$$a^2(5m_a-m_b-m_c)+b^2(5m_b-m_c-m_a)+c^2(5m_c-m_a-m_b) \leq 12m_am_bm_c$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

UP.050. Let a, b, c be positive real numbers such that $a^2b+b^2c+c^2a=3$. Prove that

$$\frac{1}{a(a+b)^2}+\frac{1}{b(b+c)^2}+\frac{1}{c(c+a)^2} \geq \frac{3}{4}.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

UP.051. Let be $a \in [0, \infty)$; $f : (0, \infty) \to (0, \infty)$

$$f(x)=\Big(\Gamma(x+1)\Big)^{\frac{1}{x}}. \text{ Find:}$$

$$\Omega=\lim_{x\to\infty}\left(\Big(f(x+1)\Big)^a-\Big(f(x)\Big)^a\right)\cdot x^{1-a}$$

Proposed by D.M. Bătineţu - Giurgiu; Neculai Stanciu - Romania

UP.052. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{a^6}{a^2+b}+\frac{b^6}{b^2+c}+\frac{c^6}{c^2+a} \geq \frac{3}{2}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

UP.053. If $x, y, z \in \mathbb{C}^*; A, B, C \in M_n(\mathbb{C}); n \geq 2$ are such that $x^2A + B = xAB; y^2B + C = yBC; z^2C + A = zCA$ then:

$$\left((y^2+1)x+\frac{x^2+1}{z}\right)A+\left((z^2+1)y+\frac{y^2+1}{x}\right)B+\left((x^2+1)z+\frac{z^2+1}{y}\right)C =$$

$$= (x+y+z)ABC$$

Proposed by D.M. Bătineţu - Giurgiu; Neculai Stanciu - Romania

UP.054. If $x > 0$ then:

$$2\sqrt{x} \leq \left(\frac{\Gamma(\sqrt{x}+1)}{\Gamma(\sqrt{x}+\frac{1}{2})}\right)^2 + \left(\frac{\Gamma(\sqrt[4]{x}+1)}{\Gamma(\sqrt[4]{x}+\frac{1}{2})}\right)^4 \leq 2\sqrt{x} + \sqrt[4]{x} + \frac{1}{4}$$

where Γ denote the Euler's gamma function.

Proposed by Mihály Bencze - Romania

UP.055. Evaluate:

$$I = \int_0^1 \frac{\ln^3 x}{2-x} dx$$

Proposed by Shivam Sharma - New Delhi - India

UP.056. Let ABC and be a triangle and Ω is first Brocard point of ABC. Let D, E, F are on the sides BC, CA, AB of ABC respectively. If $m(B\Omega D) = m(C\Omega E) = m(A\Omega F) = 90^\circ$ then prove that $\frac{|BD|}{|BC|} + \frac{|CE|}{|CA|} + \frac{|AF|}{|AB|} = 2$

Proposed by Mehmet Şahin - Ankara - Turkey

UP.057. Let $a, b \in \mathbb{R}$ such that $a + b > 0$ then

$$\left(\frac{a+b}{2}\right)^n \leq \frac{1}{n+1}\sum_{k=0}^{n} a^k b^{n-k} \leq \frac{a^n+b^n}{2}$$

Proposed by Abdallah El Farisi – Bechar – Algerie

UP.058. Let ABC be an arbitrary triangle and XYZ is the Kiepert triangle of ABC. If $K(\theta)$ is a Kiepert perspector ω is first Brocard angle then prove that

a) $\frac{Area(XYZ)}{Area(ABC)} = \frac{1}{4}(3\tan^2\theta + 2\tan\theta \cdot \cot\omega + 1)$

b) if $\theta = \omega$ then XYZ is Gallatly - Kiepert triangle takes the name.

Prove that

$$\frac{Area(XYZ)}{Area(ABC)} = 3 \cdot \frac{a^2b^2+b^2c^2+c^2a^2}{(a^2+b^2+c^2)^2}$$

Proposed by Mehmet Şahin - Ankara - Turkey

UP.059. Let a, b, c be positive real numbers. Prove that

$$\frac{(a^2-ab+b^2)^2}{(a+b)^4}+\frac{(b^2-bc+c^2)^2}{(b+c)^4}+\frac{(c^2-ca+a^2)^2}{(c+a)^4}\geq\frac{3}{16}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

UP.060. Let a, b, c be positive real numbers with $a+b+c=1$. Prove that

$$\left(1+\frac{1}{2a+b}\right)^c\cdot\left(1+\frac{1}{2b+c}\right)^a\cdot\left(1+\frac{1}{2c+a}\right)^b\geq 2$$

Proposed by George Apostolopoulos - Messolonghi - Greece

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