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DANIEL SITARU

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SOLUTIONS

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Proposed by

Daniel Sitaru – Romania

Nguyen Viet Hung – Hanoi – Vietnam,Marin Chirciu – Romania

Nguyen Phuc Tang-Dong Thap – Vietnam

Martin Lukarevski – Stip – Macedonia, Vasile Jiglău – Romania

Mehmet Șahin – Ankara – Turkey

Abdallah El Farissi – Bechar – Algerie

George Apostolopoulos – Messolonghi – Greece

D.M. Bătinețu-Giurgiu – Romania

Neculai Stanciu – Romania

Mihály Bencze – Romania

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Solutions by

Daniel Sitaru – Romania, SK Rejuan-West Bengal-India, Kevin Soto Palacios Huarmey – Peru, Soumava Chakraborty-Kolkata-India, Ravi Prakash-New Delhi-India, Ngo Minh Ngoc Bao-Hanoi-Vietnam, Henry Ricardo-New York-USA. Shreyansh Sharma-India, Nguyen Viet Hung – Hanoi – Vietnam
Imad Zak-Saida-Lebanon, Nguyen Phuc Tang-Dong Thap-Vietnam
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Shafiqur Rahman-Bangladesh, Sanong Hauyrerai – Nakhonpathom – Thailand
Mihály Bencze – Romania
Abdilkadir Altintas – Afyon – Turkey

JP.046. Let a, b, c, d be positive real numbers such that $a+b+c+d=4$. Prove that

$$\frac{a}{b(b+c+d)^2}+\frac{b}{c(c+d+a)^2}+\frac{c}{d(d+a+b)^2}+\frac{d}{a(a+b+c)^2}\geq\frac{4}{9}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by SK Rejuan-West Bengal-India

$$a,b,c,d\in\mathbb{R}^+$$

$$\sum\frac{a}{b(b+c+d)^2}\geq 4\left\{\prod\frac{a}{b(b+c+d)^2}\right\}^{\frac{1}{4}}=4\left\{\prod\frac{1}{(b+c+d)^2}\right\}^{\frac{1}{4}}$$

AM ≥ GM

$$\Rightarrow\sum\frac{a}{b(b+c+d)^2}\geq 4\left\{\prod\frac{1}{(b+c+d)}\right\}^{\frac{1}{2}}\quad (1)$$

Again by GM ≥ H.M we get, $\prod\frac{1}{b+c+d}\geq\left\{\frac{9}{\sum(b+c+d)}\right\}^4=\left\{\frac{9}{3\sum a}\right\}^4$

$$\Rightarrow\prod\left(\frac{1}{b+c+d}\right)\geq\left\{\frac{9}{3\sum a}\right\}^4=\left\{\frac{1}{3}\right\}^4\quad [as\sum a=9]\Rightarrow\prod\left(\frac{1}{b+c+d}\right)^{\frac{1}{2}}\geq\frac{1}{9}\quad (2)$$

Combining (1) & (2) we get, $\sum\frac{a}{b(b+c+d)^2}\geq 4\left\{\prod\frac{1}{(b+c+d)}\right\}^{\frac{1}{2}}\geq 4\cdot\frac{1}{9}\Rightarrow\sum\frac{a}{b(b+c+d)^2}\geq\frac{4}{9}$

(Proved)

Solution 2 by Kevin Soto Palacios Huarmey – Peru

Siendo: a, b, c, d ***números*** R^+***, de tal manera que:*** $a+b+c+d=4$

Probar que: $\frac{a}{b(b+c+d)^2}+\frac{b}{c(c+d+a)^2}+\frac{c}{d(d+a+b)^2}+\frac{d}{a(a+b+c)^2}\geq\frac{4}{9}$

Acomodando convenienenmente: $\frac{\left(\frac{a}{b+c+d}\right)^2}{ab}+\frac{\left(\frac{b}{c+d+a}\right)^2}{bc}+\frac{\left(\frac{c}{d+a+b}\right)^2}{cd}+\frac{\left(\frac{d}{a+b+c}\right)^2}{da}$

1) Siendo: $a, b, c, d>0$***, se cumple la siguiente desigualdad:***

$$a^2+b^2+c^2+d^2\geq\frac{1}{4}(a+b+c+d)^2\Leftrightarrow$$

$$\Leftrightarrow 3(a^2+b^2+c^2+d^2)\geq 2(ab+ac+ad+bc+bd+cd)$$

$$\Leftrightarrow 3(a+b+c+d)^2\geq 8(ab+ac+ad+bc+bd+cd)\Leftrightarrow$$

$$\Leftrightarrow (a+b+c+d)^2\geq\frac{8}{3}(ab+ac+ad+bc+bd+cd)$$

$$\textit{2)}\ ab+bc+cd+da=(a+b)(c+d)\le\frac{\left((a+b)+(c+d)\right)^2}{4}=4$$

Aplicando la desigualdad de Cauchy: $\sum\frac{\left(\frac{a}{b+c+d}\right)^2}{ab}\ge\frac{\sum\left(\frac{a}{b+c+d}\right)^2}{ab+bc+cd+da}\ge\frac{\sum\left(\frac{a^2}{ab+ac+ad}\right)^2}{(a+b)(c+d)}\ge$

$$\ge\frac{\left(\frac{(a+b+c+d)^2}{2(ab+ac+ad+bc+bd+cd)}\right)^2}{(a+b)(c+d)}\ge\frac{\frac{16}{9}}{4}=\frac{4}{9}\ \textit{... (LQQD)}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$LHS\underset{(1)}{\overset{A-G}{\ge}}4\sqrt[4]{\frac{1}{(b+c+d)^2(c+d+a)^2(d+a+b)^2(a+b+c)^2}}$$

$$=\frac{4}{\sqrt{(b+c+d)(c+d+a)(d+a+b)(a+b+c)}}$$

$$\sqrt[4]{(b+c+d)(c+d+a)(d+a+b)(a+b+c)}\overset{G-A}{\le}\frac{3(a+b+c+d)}{4}=3$$

$$\therefore\sqrt{(b+c+d)(c+d+a)(d+a+b)(a+b+c)}\le 9$$

$$\Rightarrow\frac{4}{\sqrt{(b+c+d)(c+d+a)(d+a+b)(a+b+c)}}\ge\frac{4}{9}\quad\textit{(2)}$$

$$\textit{(1), (2)}\Rightarrow LHS\ge\frac{4}{9}\ \textit{(Proved)}$$

JP.047. Let a,b,c be positive real numbers such that

$ab+bc+ca+abc\le 4$. Prove that

a. $3+a+b+c\ge 2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$.

b. $3+\frac{5}{3}(a+b+c)\ge\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\left(\sqrt[3]{b}+\sqrt[3]{c}\right)\left(\sqrt[3]{c}+\sqrt[3]{a}\right)$.

Proposed by Nguyen Hung Viet – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo a,b,c *números* R^+ *de tal manera que* $ab+bc+ca+abc\le 4$

Probar que $a+b+c+3\ge 2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$

De la condición $ab+bc+ca+abc\le 4$ *es equivalente*

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}\ge 1\Leftrightarrow$$

$$\Leftrightarrow\left(\frac{1}{a+2}-\frac{1}{2}\right)+\left(\frac{1}{b+2}-\frac{1}{2}\right)+\left(\frac{1}{c+2}-\frac{1}{2}\right)\geq 1-\frac{3}{2}=-\frac{1}{2}$$

$$\Leftrightarrow-\frac{a}{a+2}-\frac{b}{b+2}-\frac{c}{c+2}\leq -1\Leftrightarrow 1\geq\frac{a}{a+2}+\frac{b}{b+2}+\frac{c}{c+2}$$

Por la desigualdad de Cauchy $1\geq\frac{a}{a+2}+\frac{b}{b+2}+\frac{c}{c+2}\geq\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}{a+2+b+2+c+2}$

$$\Leftrightarrow a+b+c+6\geq a+b+c+2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

$\Leftrightarrow 3\geq\sqrt{ab}+\sqrt{bc}+\sqrt{ca}$ ***(A)*** $\wedge\ a+b+c\geq\sqrt{ab}+\sqrt{bc}+\sqrt{ca}$ ***(B)***

Sumando (A) + (B) se obtiene la desigualdad pedida

$a+b+c+3\geq 2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$ ***(LQQD)***

Siendo a,b,c ***números*** R^{+} ***de tal manera que*** $ab+bc+ca+abc\leq 4$

Probar que $3+\frac{5}{3}(a+b+c)\geq\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\left(\sqrt[3]{b}+\sqrt[3]{c}\right)\left(\sqrt[3]{c}+\sqrt[3]{a}\right)$

$$\Leftrightarrow 9+5(a+b+c)\geq 3\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\left(\sqrt[3]{b}+\sqrt[3]{c}\right)\left(\sqrt[3]{c}+\sqrt[3]{a}\right)$$

$$\Leftrightarrow 9+6(a+b+c)\geq(a+b+c)+3\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\left(\sqrt[3]{b}+\sqrt[3]{c}\right)\left(\sqrt[3]{c}+\sqrt[3]{a}\right)$$

$\Leftrightarrow 9+6(a+b+c)\geq\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^3$***. Por la desigualdad de Holder***

$$\left((2a+1)+(2b+1)+(2c+1)\right)\left(\frac{a}{2a+1}+\frac{b}{2b+1}+\frac{c}{2c+1}\right)(1+1+1)\geq$$

$\geq\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^3$***. Es necesario demostrar lo siguiente***

$$A=\frac{a}{2a+1}+\frac{b}{2b+1}+\frac{c}{2c+1}\leq 1\Leftrightarrow\frac{2a}{2a+1}+\frac{2b}{2b+1}+\frac{2c}{2c+1}\leq 2$$

$$\Leftrightarrow\left(\frac{2a}{2a+1}+\frac{1}{2a+1}\right)+\left(\frac{2b}{2b+1}+\frac{1}{2b+1}\right)+\left(\frac{2c}{2c+1}+\frac{1}{2c+1}\right)\geq$$

$\geq 2+\frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1}$***. Lo cual nos resulta lo siguiente***

$$\frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1}\geq 1$$

$$\Leftrightarrow(2b+1)(2c+1)+(2c+1)(2a+1)+(2a+1)(2b+1)\geq$$

$$\geq(2a+1)(2b+1)(2c+1)\Leftrightarrow 4(ab+bc+ca)+4(a+b+c)+3\geq$$

$$\geq 1+2(a+b+c)+4(ab+bc+ca)+8abc$$

$\Leftrightarrow 2(a+b+c)+2\geq 8abc\Leftrightarrow a+b+c+1\geq 4abc$ ***(B)***

Por la desigualdad de Cauchy

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}+\frac{1}{abc}\geq\frac{16}{ab+bc+ca+abc}\geq\frac{16}{4}=4$$

Por transitividad se tiene lo siguiente

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}+\frac{1}{abc}\geq 4\Leftrightarrow a+b+c+1\geq 4abc\ \ \textbf{(LQQD)}$$

Por ultimo $\rightarrow 9+6(a+b+c)\geq\frac{\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^3}{A}\geq$

$$\geq\frac{\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^3}{1}=\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^3$$

JP.048. Prove that for any positive real numbers a,b,c

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{1}{3}+\frac{(a+b)(b+c)(c+a)}{a^2b+b^2c+c^2a}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los numerous R^+, la siguiente desigualdad:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{1}{3}+\frac{(a+b)(b+c)(c+a)}{a^2b+b^2c+c^2a}$$

A lo que es equivalente:

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)(a^2b+b^2c+c^2a)\geq a^2b+b^2c+c^2a+3(a+b)(b+c)(c+a)$$

Cómo: $a,b,c>0$. Por la desigualdad de Holder:

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)(a^2b+b^2c+c^2a)(1+1+1)\geq(a+b+c)^3$$

Es suficiente demostrar que:

$$(a+b+c)^3\geq a^2b+b^2c+c^2a+3(a+b)(b+c)(c+a)$$

$$a^3+b^3+c^3+3(a+b)(b+c)(c+a)+a^2b+b^2c+c^2a+3(a+b)(b+c)(c+a)$$

Lo cual se reduce: $a^3+b^3+c^3\geq a^2b+b^2c+c^2a$

Por: $MA\geq MG$

$a^3+a^3+b^3\geq 3a^2b$ ... **(1),**

$b^3+b^3+c^3\geq 3b^2c$ ... **(2),**

$$c^3 + c^3 + a^3 \geq 3c^2a \dots \textit{(3)},$$

$$\textit{Sumando: (1) + (2) + (3): } a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a \dots \textit{(LQQD)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\textit{Given inequality} \Leftrightarrow \frac{\sum ab^2}{abc} \geq \frac{1}{3} + \frac{2abc + \sum a^2b + \sum ab^2}{\sum a^2b} \Leftrightarrow \frac{\sum ab^2}{abc} \geq \frac{4\sum a^2b + 3\sum ab^2 + 6abc}{3\sum a^2b}$$

$$\Leftrightarrow 3\left(\sum ab^2\right)\left(\sum a^2b\right) \geq 4abc\left(\sum a^2b\right) + 3abc\left(\sum ab^2\right) + 6a^2b^2c^2 \quad \textit{(i)}$$

$$\text{Now, } \sum ab^2 \overset{A-G}{\geq} 3abc, \text{ Also } \sum a^2b \overset{A-G}{\geq} 3abc$$

$$\therefore \left(\sum ab^2\right)\left(\sum a^2b\right) \geq 3abc\left(\sum ab^2\right) \textit{ (1)}$$

$$\therefore \textit{it remains to prove: } 2\left(\sum ab^2\right)\left(\sum a^2b\right) \geq 4abc\left(\sum a^2b\right) + 6a^2b^2c^2 \textit{ (from (i), (1))}$$

$$\Leftrightarrow \left(\sum ab^2\right)\left(\sum a^2b\right) \geq 2abc\left(\sum a^2b\right) + 3a^2b^2c^2 \textit{ (ii)}$$

$$\text{Now, } \frac{\left(\sum ab^2\right)\left(\sum a^2b\right)}{3} \underset{(2)}{\geq} \frac{3abc \cdot 3abc}{3} = 3a^2b^2c^2$$

$$\therefore \textit{it remains to prove: } \frac{2}{3}\left(\sum ab^2\right)\left(\sum a^2b\right) \geq 2abc\left(\sum a^2b\right) \textit{ (from (ii), (2))}$$

$$\Leftrightarrow \left(\sum ab^2\right)\left(\sum a^2b\right) \geq 3abc\left(\sum a^2b\right) \Leftrightarrow \sum ab^2 \geq 3abc \rightarrow \textit{which is true (Proved)}$$

JP.049. Let $x_1, x_2, \dots, x_n$ be positive real numbers such that

$$\frac{1}{x_1} + \frac{2}{x_2} + \cdots + \frac{n}{x_n} = \frac{n(n+1)}{2}.$$

Find the minimum possible value of $x_1 + x_2^2 + \cdots + x_n^n$.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by SK Rejuan-West Bengal-India

$$\textit{Given } x_i \in \mathbb{R}^+ \quad \forall\, i \in \{1, 2, \dots, n\}$$

$$\textit{and } \frac{1}{x_1} + \frac{2}{x_2} + \cdots + \frac{n}{x_n} = \frac{n(n+1)}{2} \quad \textit{(1)}$$

Let us take $x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}$ with the associated weights $1, 2, \dots, n$ respectively, by applying AM $\geq$ GM we get,

$$\frac{1 \cdot x_1^{-1} + 2 \cdot x_2^{-1} + \cdots + n x_n^{-1}}{1 + 2 + \cdots + n} \geq \left\{x_1^{-1}, x_2^{-2}, \dots, x_n^{-n}\right\}^{\frac{1}{1+2+\cdots+n}}$$

$$\Rightarrow \frac{\frac{1}{x_1}+\frac{2}{x_2}+\cdots+\frac{n}{x_n}}{1+2+\cdots+n} \ge \left(x_1^{-1}x_2^{-2}\dots x_n^{-n}\right)^{\frac{2}{n(n+1)}}$$

$$\Rightarrow \frac{\frac{n(n+1)}{2}}{\frac{n(n+1)}{2}} \ge \left\{x_1^{-1}x_2^{-2}\dots x_n^{-n}\right\}^{\frac{2}{n(n+1)}} \Rightarrow 1 \ge x_1^{-1}x_2^{-2}\dots x_n^{-n}$$

$$\Rightarrow x_1^1x_2^2\dots x_n^n \ge 1 \quad (2)$$

Applying AM $\ge$ ***GM with the*** n ***term*** > 1,

$$\frac{x_1+x_2^2+x_3^3+\cdots+x_n^n}{n} \ge \left\{x_1^1x_2^2\dots x_n^n\right\}^{\frac{1}{n}} \Rightarrow \sum_{i=1}^n x_i^i \ge n\cdot(1)^{\frac{1}{n}} \text{ [from (2)]} \Rightarrow \sum_{i=1}^1 x_i^i \ge n$$

$$\therefore \min\left(x_1^1+x_2^2+\cdots+x_n^n\right) = n.$$

Solution 2 by Ravi Prakash-New Delhi-India

$$1 = \frac{2}{n(n+1)}\left[\frac{1}{x_1}+2\left(\frac{1}{x_2}\right)+3\left(\frac{1}{x_3}\right)+\cdots+n\left(\frac{1}{x_n}\right)\right]$$

$$\ge \left[\frac{1}{x_1}\left(\frac{1}{x_2}\right)^2\left(\frac{1}{x_3}\right)^3\dots\left(\frac{1}{x_n}\right)^n\right]^{\frac{2}{n(n+1)}} \quad [AM \ge GM]$$

$$\Rightarrow 1 \ge \left[\frac{1}{x_1}\cdot\frac{1}{x_2^2}\cdot\frac{1}{x_3^3}\dots\frac{1}{x_n^n}\right]^{\frac{1}{n}} \ge \frac{n}{x_1+x_2^2+\cdots+x_n^n} \quad [GM \ge HM]$$

$$\Rightarrow x_1+x_2^2+\cdots+x_n^n \ge n$$

Equality when $x_1 = x_2 = \cdots = x_n = 1$

Solution 3 by Ngo Minh Ngoc Bao-Hanoi-Vietnam

Use AM – GM inequalities:

$$\frac{n(n+1)}{2} = \sum_{k=1}^n \frac{k}{x_k} \ge \frac{n(n+1)}{2\sqrt[\frac{n(n+1)}{2}]{\prod_{k=1}^n x_k^k}} \Rightarrow \sqrt[\frac{n(n+1)}{2}]{\prod_{k=1}^n x_k^k} \ge 1 \Rightarrow \prod_{k=1}^n x_k^k \ge 1.$$

We know:

$$x^n \ge nx-(n-1) \Rightarrow x_1+x_2^2+\cdots+x_n^n \ge \sum_{k=1}^n kx_k-(1+2+\cdots+n)+1$$

$$\Rightarrow P \geq \sum_{k=1}^{n} kx_k - (1+2+\cdots+n) + 1 \geq \frac{n(n+1)}{2}\sqrt[\frac{n(n+1)}{2}]{\prod_{k=1}^{n} x_k^k} - \frac{n(n-1)}{2} \geq n$$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$x_1, x_2, \dots, x_n > 0 \,|\, \frac{1}{x_1} + \frac{2}{x_2} + \cdots + \frac{n}{x_n} = \frac{n(n+1)}{2}$$

$$\left(x_1 + x_2^2 + \cdots + x_n^n\right)_{min} = ?$$

$$\frac{1+2+\cdots+n}{\frac{1}{x_1} + \frac{2}{x_2} + \cdots + \frac{n}{x_n}} \overset{weighted\ HM-GM}{\leq} \sqrt[(1+2+\cdots+n)]{x_1 \cdot x_2^2 \cdot \ldots \cdot x_n^n}$$

$$\Rightarrow \frac{\frac{n(n+1)}{2}}{\frac{n(n+1)}{2}} \leq \sqrt[\frac{n(n+1)}{2}]{x_1 x_2^2 \dots x_n^n}$$

$$\Rightarrow \left(x_1 x_2^2 \dots x_n^n\right)^{\frac{2}{n(n+1)}} \geq 1 \Rightarrow x_1 x_2^2 \dots x_n^n \geq 1 \quad \textbf{(1)}$$

Now, $x_1 + x_2^2 + \cdots + x_n^n \overset{A-G}{\geq} n\sqrt[n]{x_1 x_2^2 \dots x_n^n} \geq n$ ***(using (1))***

$$\therefore \left(x_1 + x_2^2 + \cdots + x_n^n\right)_{min} = n$$

Solution 5 by Henry Ricardo-New York-USA

We use the weighted harmonic – geometric – arithmetic means inequality:

$$\left(\sum_{i=1}^{n} \frac{\alpha_i}{x_i}\right)^{-1} \leq \prod_{i=1}^{n} x_i^{\alpha_i} \leq \sum_{i=1}^{n} \alpha_i x_i,$$

where $\sum_i \alpha_i = 1$. ***In this problem, we let*** $a_i = \frac{2i}{n(n+1)}$ ***first and then*** $\alpha_i = \frac{1}{n}$ ***for all*** i.

Now we have $1 = \frac{\frac{n(n+1)}{2}}{\frac{1}{x_1} + \frac{2}{x_2} + \cdots + \frac{n}{x_n}} \leq \left(x_1 \cdot x_2^2 \cdot \ldots \cdot x_n^n\right)^{\frac{2}{n(n+1)}}$

$$= \left[\left(x_1 \cdot x_2^2 \cdot \ldots \cdot x_n^n\right)^{\frac{1}{n}}\right]^{\frac{2}{n+1}} \leq \left[\frac{x_1 + x_2^2 + \cdots + x_n^n}{n}\right]^{\frac{2}{n+1}}$$

which implies that $x_1 + x_2^2 + \cdots + x_n^n \geq n$.

Equality holds if and only if $x_1 = x_2 = \cdots = x_n = 1$.

Solution 6 by Shreyansh Sharma-India

Given $\frac{1}{x_1}+\frac{2}{x_2}+\cdots+\frac{n}{x_n}=\frac{n(n+1)}{2}$. *To find* $\min\left(x_1+x_2^2+\cdots+x_n^n\right)$

Using AM ≥ GM

$$\frac{1}{x_1}+\left(\frac{1}{x_2}+\frac{1}{x_2}\right)+\left(\frac{1}{x_3}+\frac{1}{x_3}+\frac{1}{x_3}\right)+\cdots+\underbrace{\left(\frac{1}{x_n}+\cdots+\frac{1}{x_n}\right)}_{n\ times}\geq\left(\frac{1}{x_1}\cdot\frac{1}{x_2^2}\cdot\ldots\cdot\frac{1}{x_n^n}\right)^{\frac{n(n+1)}{2}}$$

$$\frac{2}{n(n+1)}\leq\left(x_1\cdot x_2^2\cdot x_3^3\cdot\ldots\cdot x_n^n\right)^{\frac{n(n+1)}{2}}$$

Now again using AM ≥ GM $x_1+x_2^2+\cdots+x_n^n\geq\left(x_1\cdot x_2^2\cdot\ldots\cdot x_n^n\right)^n\geq\frac{2}{n(n+1)}$

Hence $x_1+x_2^2+\cdots+x_n^n\geq\frac{2}{n(n+1)}$. *Equality holds if* $n=1$.

JP.050. Let a,b,c be non – negative real numbers such that $a+b+c=3$. Prove that

$$a^3+b^3+c^3+8(ab+bc+ca)\leq 27$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Nguyen Viet Hung – Hanoi – Vietnam

Using the known inequality: $9(a+b)(b+c)(c+a)\geq 8(a+b+c)(ab+bc+ca)$

we have

$$a^3+b^3+c^3+8(ab+bc+ca)=a^3+b^3+c^3+\frac{8}{3}(a+b+c)(ab+bc+ca)$$

$$\leq a^3+b^3+c^3+3(a+b)(b+c)(c+a)=(a+b+c)^3=27$$

and we are done! The equality occurs if and only if $a=b=c=1$.

Solution 2 by Imad Zak-Saida-Lebanon

$a,b,c\geq 0;\ a+b+c=3$, *prove that:* $\sum a^3+8\sum ab\leq 27$ *... (E)*

we know that $pq\geq 9r$ *where* $p=\sum a=3, q=\sum ab$

$r=abc\Rightarrow q\geq 3r$ *... (1)*

$$\sum ab(a+b)=(a+b+c)(ab+bc+ca)-3abc\Rightarrow$$

$$\sum ab(a+b)=3q-3r\ \ldots\ (2)$$

$$(p) \Leftrightarrow 27 - \sum a^3 - 8\sum ab \overset{?}{\geq} 0 \Leftrightarrow \left(\sum a\right)^3 - \sum a^3 - 8r \overset{??}{\geq} 0 \Leftrightarrow$$

$$3\sum ab(a+b) + 6r - 8q \overset{?}{\geq} 0 \Leftrightarrow (9q - 9r) + 6r - 8q \overset{??}{\geq} 0 \Leftrightarrow$$

$q - 3r \geq 0$ ***True by (1)***

$\ll = \gg$ ***at*** $a = b = c = 1$

Solution 3 by Nguyen Phuc Tang-Hanoi-Vietnam

$$p = a + b + c = 3; q = ab + bc + ca; r = abc$$

We have $q \leq \frac{p^2}{3} = 3$

$a^3 + b^3 + c^3 = p^3 - 3pq + 3r$ ***and*** $q^2 \geq 3pr \Rightarrow a^3 + b^3 + c^3 \leq 27 - 9q + \frac{q^2}{3}$

We need prove that $27 - q + \frac{q^2}{3} \leq 27 \Leftrightarrow q(q-3) \leq 0$ ***is true***

Equality hold if $a = b = c = 1$ ***or*** $a = 3, b = c = 0$ ***or permutations***

Solution 4 by Ngo Minh Ngoc Bao-Vietnam

$a^3 + b^3 + c^3 + 8(ab + bc + ca) \leq 27$ ***(*)***

The inequality (&) $\Leftrightarrow a^3 + b^3 + c^3 + \frac{8}{3}(a+b+c)(ab+bc+ca) \leq (a+b+c)^3$

$$\Leftrightarrow \frac{8}{3}(a+b+c)(ab+bc+ca) \leq 3(a+b)(b+c)(c+a)$$

$$\Leftrightarrow \frac{8}{3}(a^2b + abc + ca^2 + ab^2 + b^2c + abc + abc + bc^2 + c^2a) \leq 9(ab + ac + b^2 + bc)(c+a)$$

$$\Leftrightarrow 8\left(\sum a^2b + \sum ab^2 + 3abc\right) \leq 9\left(\sum a^2b + \sum ab^2 + 2abc\right) \Leftrightarrow$$

$\Leftrightarrow \sum a^2b + \sum ab^2 - 6abc \geq 0$ ***(**)***

Consider the third – order symmetry polynomial

$$P(a,b,c) = \sum a^2b + \sum ab^2 - 6abc$$

$P(1,1,1) = 3 + 3 - 6 \geq 0. P(a,b,0) = a^2b + ab^2 - 0 \geq 0 \Rightarrow P(a,b,c) \geq 0$ ***(!).***

Equality when $a = b = c = 1$ ***or*** $a = 3, b = c = 0$ ***or*** $a = b = 0, c = 3$ ***or*** $a = c = 0, b = 3$

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JP.051. Prove that in any triangle the following relationship holds:

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \le \frac{R}{nR+(1-n)\cdot 2r}$$

where $0 \le n \le \frac{1}{2}$.

Proposed by Marin Chirciu – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

In any $\Delta\, ABC, \frac{\sum a^2}{\sum ab} \le \frac{R}{nR+(1-n)2r}\ \ \forall n \in \left[0, \frac{1}{2}\right]$. *Given inequality* $\Leftrightarrow \frac{\sum a^2}{\sum ab} \le \frac{R}{n(R-2r)+2r}$

$$\Leftrightarrow \{n(R-2r)+2r\}\left(\sum a^2\right) \le R\left(\sum ab\right)\ (\because n(R-2r)+2r \ge 2r > 0\ as\ n \ge 0)$$

$$\Leftrightarrow n(R-2r)\left(\sum a^2\right)+2r\left(\sum a^2\right) \le R\left(\sum ab\right)\quad (1)$$

$$\because 0 \le n \le \frac{1}{2}\ \text{ and } R-2r \ge 0, \therefore n(R-2r)\left(\sum a^2\right)+2r\left(\sum a^2\right)$$

$$\overset{(2)}{\le} \frac{R-2r}{2}\left(\sum a^2\right)+2r\left(\sum a^2\right) = \frac{R}{2}\left(\sum a^2\right)+r\left(\sum a^2\right)$$

(1), (2) $\Rightarrow$ *it suffices to prove:* $\frac{R}{2}\left(\sum a^2\right)+r\left(\sum a^2\right) \le R\left(\sum ab\right)$

$$\Leftrightarrow (R+2r)\left(\sum a^2\right) \le 2R\left(\sum ab\right) \Leftrightarrow 2(R+2r)(s^2-4Rr-r^2) \le 2R(s^2+4Rr+r^2)$$

$$\Leftrightarrow Rs^2-R(4Rr+r^2)+2rS^2-2r(4Rr+r^2) \le Rs^2+R(4Rr+r^2)$$

$$\Leftrightarrow s^2-4Rr-r^2 \le R(4R+r) = 4R^2+Rr \Leftrightarrow s^2 \le 4R^2+5Rr+r^2\quad (3)$$

Now, $s^2 \le 4R^2+4Rr+3r^2$ *(Gerretsen) (4)*

It suffices to prove: (3), (4) $\Rightarrow 4Rr+3r^2 \le 5Rr+r^2$

$\Leftrightarrow Rr \ge 2r^2 \Leftrightarrow R \ge 2r \rightarrow$ *true, by Euler (Proved)*

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$0 \le n \le \frac{1}{2} \Rightarrow \frac{2R}{R+2r} \le \frac{R}{n\cdot R+(1-n)\cdot 2r} \le \frac{R}{2r}$$

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \le \frac{2R}{R+2r};\ (R+2r)\cdot(a^2+b^2+c^2) \le 2R\cdot(ab+bc+ca)$$

$$2\cdot(p^2-4Rr-r^2)\cdot(R+2r) \le 2R\cdot(p^2+4Rr+r^2);$$

$$2r\cdot p^2 \le 8R^2\cdot r+10R\cdot r^2+2r^3$$

$$p^2 \le 4R^2+5Rr+r^2 = 4R^2+4Rr+R\cdot r+r^2 \overset{Euler}{\ge} 4R^2+4Rr+3r^2$$

$$p^2 \overset{GERRETSEN}{\leq} 4R^2+4Rr+3r^2 \leq 4R^2+5Rr+r^2$$

JP.052. Given $a,b,c>0$ and $a^2+b^2+c^2=6$, prove

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+a+b+c\geq 6$$

Proposed by Nguyen Phuc Tang-Dong Thap – Vietnam

Solution by Dang Thanh Tung-– Vietnam

$$2(c-a)^2+2(c-b)(a-b)$$

$$b=\min\{a,b,c\}\Rightarrow (a-b)(c-b)\geq 0$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3=\frac{a-b}{b}+\frac{b-c}{c}+\frac{c-a}{a}=\frac{(c-a)^2}{ca}+\frac{(a-b)(c-b)}{bc}$$

$$a+b+c-3=a+b+c-\sqrt{3(a^2+b^2+c^2)}=-2\cdot\frac{a^2+b^2+c^2-ab-bc-ca}{a+b+c+\sqrt{3(a^2+b^2+c^2)}}$$

$$=-2\cdot\frac{(c-a)^2+(a-b)(c-b)}{a+b+c+3}$$

We have: $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+a+b+c-6$

$$=(c-a)^2\left(\frac{1}{ca}-\frac{2}{a+b+c+3}\right)+(a-b)(c-b)\left(\frac{1}{bc}-\frac{2}{a+b+c+3}\right)$$

$$+,\frac{1}{ca}-\frac{2}{a+b+c+3}=\frac{(a-c)^2+b^2+a+b+c}{ca(a+b+c+3)}>0$$

$$+,\frac{1}{bc}-\frac{2}{a+b+c+3}=\frac{(b-c)^2+a^2+a+b+c}{bc(a+b+c+3)}>0\Rightarrow\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+a+b+c\geq 6$$

Equality when $a=b=c=1$.

JP.053. If $a,b,c>0$ and $a+b+c=3$ prove that:

$$\sum a\left(\frac{1}{b^n}+\frac{1}{c^n}\right)\geq\frac{18}{a^n+b^n+c^n}$$

where $n\geq 0$.

Proposed by Marin Chirciu – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

WLOG, we may assume $a \geq b \geq c$

$$\sum a\left(\frac{1}{b^n}+\frac{1}{c^n}\right) \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum a\right)\sum\left(\frac{1}{b^n}+\frac{1}{c^n}\right) = 2\sum\frac{1}{a^n} \overset{Bergstrom}{\geq} \frac{2\cdot 3^2}{\sum a^n} = \frac{18}{\sum a^n}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc} a\left(\frac{1}{b^n}+\frac{1}{c^n}\right) \geq \frac{18}{a^n+b^n+c^n}$$

Let $a \geq b \geq c$ then $\frac{1}{b^n+c^n} \geq \frac{1}{c^n+a^n} \geq \frac{1}{a^n+b^n}$

$$\sum_{cyc} a\left(\frac{1}{b^n}+\frac{1}{c^n}\right) \geq 4\sum_{cyc}\frac{a}{b^n+c^n} \overset{CHEBYSHEV'S\ INEQUALITY}{\geq} \frac{4}{3}\left(\sum_{cyc} a\right)\left(\sum_{cyc}\frac{1}{a^n+b^n}\right)$$

$\geq 4\frac{9}{\sum_{cyc}(a^n+b^n)} \geq \frac{18}{a^n+b^n+c^n}$ *(proved) equality* $a = b = c = 1.$

JP.054. Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC. Prove that

$$\frac{9}{4R+r} \leq \frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c} \leq \frac{1}{r},$$

where R and r are the circumradius and inradius of ABC respectively.

Proposed by Martin Lukarevski – Stip – Macedonia

Solution by Soumava Chakraborty-Kolkata-India

In any $\Delta\ ABC, \frac{9}{4R+r} \leq \sum\frac{1}{m_a} \leq \frac{1}{r}$

$\sum\frac{1}{m_a} \geq \frac{(1+1+1)^2}{\sum m_a}$ *(Bergstrom)* $\geq \frac{9}{4R+r}$ $(\because \sum m_a \leq 4R+r)$

Tereshin $\Rightarrow m_a \geq \frac{b^2+c^2}{4R} \overset{A-G}{\geq} \frac{2bc}{4R} = \frac{bc}{2R}$

$$\therefore \sum\frac{1}{m_a} \leq 2R\sum\frac{1}{bc} = 2R\frac{(a+b+c)}{abc} = \frac{2R\cdot 2s}{4Rrs} = \frac{1}{r}$$

(Proved)

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JP.055. Let $ABCD$ be an inscriptible and circumscriptible quadrilateral, p its semi perimeter. R and r the radii of circumcenter, respectively incenter, a, b, c, d its sides (a and c are the opposite sides). Prove that:

a) $2\frac{R^2}{r^2} \geq \frac{a}{c} + \frac{c}{a} + \frac{b}{d} + \frac{d}{b} \geq 2\sqrt{2}\frac{R}{r}$

b) $4\frac{R^2}{r^2} - 4 \geq \left(\frac{a}{c} + \frac{c}{a}\right)\left(\frac{b}{d} + \frac{d}{b}\right)$

Proposed by Vasile Jiglău – Romania

Solution by Vasile Jiglău – Romania

a) We will use the following formulas (see [1]):

$$a + c = b + d = p; \quad ef = ac + bd \text{ (first theorem of Ptolemy)}$$

$$abcd = p^2r^2; \quad ef = 2r\left(\sqrt{4R^2 + r^2} + r\right);$$

Let's simplify the expression $\frac{a}{c} + \frac{c}{a} + \frac{b}{d} + \frac{d}{b}$. ***We have:*** $\frac{a}{c} + \frac{c}{a} + \frac{b}{d} + \frac{d}{b} = \frac{(a+c)^2}{ac} + \frac{(b+d)^2}{bd} - 4$

$$= \frac{bd(a+c)^2 + ac(b+d)^2}{abcd} - 4 = \frac{p^2ef}{p^2r^2} - 4 =$$

$$= \frac{ef}{r^2} - 4 = \frac{2r\left(\sqrt{4R^2 + r^2} + r\right)}{r^2} - 4 = \frac{2\sqrt{4R^2 + r^2}}{r} - 2$$

$$\text{So } 2\frac{R^2}{r^2} \geq \frac{a}{c} + \frac{c}{a} + \frac{b}{d} + \frac{d}{b}$$

$$\Leftrightarrow 2\frac{R^2}{r^2} \geq \frac{2\sqrt{4R^2 + r^2}}{r} - 2 \Leftrightarrow R^2 \geq r\sqrt{4R^2 + r^2} - r^2 \Leftrightarrow R^4 + r^4 + 2R^2r^2 \geq 4R^2r^2 + r^4 \Leftrightarrow R^2 \geq 2r^2$$

We now prove the second inequality

$$\frac{a}{c} + \frac{c}{a} + \frac{b}{d} + \frac{d}{b} \geq 2\sqrt{2}\frac{R}{r} \Leftrightarrow \frac{2\sqrt{4R^2 + r^2}}{r} - 2 \geq 2\sqrt{2}\frac{R}{r} \Leftrightarrow$$

$$\sqrt{2}R \leq \sqrt{4R^2 + r^2} - r \Leftrightarrow \left(R\sqrt{2} + r\right)^2 \leq 4R^2 + r^2 \Leftrightarrow$$

$$2R^2 + r^2 + 2\sqrt{2}Rr \leq 4R^2 + r^2 \Leftrightarrow 2\sqrt{2}Rr \leq 2R^2 \Leftrightarrow r\sqrt{2} \leq R$$

namely Euler's inquality, true in every bicentric quadrilateral. Equality holds when $ABCD$ is a square.

b) We have:

$$\left(\frac{a}{c}+\frac{c}{a}\right)\left(\frac{b}{d}+\frac{d}{b}\right)=\frac{[(a+c)^2-2ac][(b+d)^2-2bd]}{abcd}=$$

$$=\frac{p^4-2p^2(ac+bd)+4abcd}{abcd}=\frac{p^2-4R\sqrt{4R^2+r^2}}{r^2}$$

So the inequality form enunciation becomes equivalent with:

$$4\frac{R^2}{r^2}-4\geq\frac{p^2-4r\sqrt{4R^2+r^2}}{r^2}\Leftrightarrow 4R^2-4r^2\geq p^2-4r\sqrt{4R^2+r^2}\Leftrightarrow$$

$$\Leftrightarrow 4R^2+4r\sqrt{4R^2+r^2}-4r^2\geq p^2$$

Taking into account that $p^2\leq\left(\sqrt{4R^2+r^2}+r\right)^2$*, its enough to prove that:*

$$4R^2+4r\sqrt{4R^2+r^2}-4r^2\geq\left(\sqrt{4R^2+r^2}+r\right)^2=4R^2+2r^2+2r\sqrt{4R^2+r^2}$$

$$\Leftrightarrow 2r\sqrt{4R^2+r^2}\geq 6r^2\Leftrightarrow 4R^2\geq 8r^2\Leftrightarrow R\geq r\sqrt{2}$$

So the inequality from enunciation is true, the equality is being obtained also, when $ABCD$ *is a square.Reference:*

[1] OT Pop , N Minculete , M Bencze – *An introduction to quadrilateral geometry* , EDP , 2013

JP.056. Let s_a is symedian and r_a, r are exradius and inradius triangle of ABC respectively. Prove that

$$\frac{r_a}{s_a+r}+\frac{r_b}{s_b+r}+\frac{r_c}{s_c+r}\geq\left(\frac{3r}{R}\right)^2$$

Proposed by Mehmet Şahin – Ankara – Turkey

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Let r, r_a, s_a *be the inradius,* A *- exradius and* A *- symmedian of triangle of* ABC *respectively. Prove that:* $\frac{r_a}{s_a+r}+\frac{r_b}{s_b+r}+\frac{r_c}{s_c+r}\geq\left(\frac{3r}{R}\right)^2$ *(A)*

Teniendo en cuenta las siguientes identidades y desigualdades en un triángulo ABC*:*

$$s_a=\frac{2bc}{b^2+c^2}m_a\leq m_a, s_b=\frac{2ca}{c^2+a^2}m_b\leq m_b, s_c=\frac{2ab}{a^2+b^2}m_c\leq m_c$$

$$r_ar_br_c=Sp=p^2r\geq\left(3\sqrt{3}\right)^2r=27r^3, R\geq 2r, m_a+m_b+m_c\leq 4R+r\leq\frac{9R}{2}$$

Como

$$m_a+m_b+m_c\le 4R+r\le\frac{9R}{2}\Leftrightarrow(m_a+r)+(m_b+r)+(m_c+r)\le\frac{9R}{2}+\frac{3R}{2}=6R$$

Luego, aplicando MA ≥ MG en (A)

$$\sum\frac{r_a}{s_a+r}\ge\sum\frac{r_a}{m_a+r}\ge 3\sqrt[3]{\frac{r_ar_br_c}{\prod(m_a+r)}}\ge\frac{3\sqrt[3]{Sp}}{\frac{\sum(m_a+r)}{3}}\ge\frac{9r\cdot R}{2R\cdot R}\ge\frac{9r\cdot 2r}{2R^2}=\left(\frac{3r}{R}\right)^2 \text{ ... (LQQD)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

WLOG, we may assume $a\ge b\ge c$; $r_a\ge r_b\Leftrightarrow\frac{\Delta}{s-a}\ge\frac{\Delta}{s-b}\Leftrightarrow -b\ge -a\Leftrightarrow a\ge b\to$ ***true***

$\therefore r_a\ge r_b$ ***Similarly,*** $r_b\ge r_c\ \therefore r_a\ge r_b\ge r_c$; $\frac{1}{s_a+r}\ge\frac{1}{s_b+r}\Leftrightarrow s_b\ge s_a\Leftrightarrow s_b^2\ge s_a^2$

$$\Leftrightarrow\frac{c^2a^2}{(c^2+a^2)^2}(2c^2+2a^2-b^2)\ge\frac{b^2c^2}{(b^2+c^2)^2}(2b^2+2c^2-a^2)$$

$$\Leftrightarrow a^6b^2+4a^4b^2c^2+2a^4c^4+2a^2c^6\ge a^2b^6+4a^2b^4c^2+2b^4c^4+2b^2c^6\quad \textbf{(a)}$$

$$\because a^4\ge b^4,\therefore a^4\cdot a^2b^2\ge b^4\cdot a^2b^2\Rightarrow a^6b^2\overset{(1)}{\ge}a^2b^6$$

$$\because a^2\ge b^2,\therefore 4a^2b^2c^2\cdot a^2\ge 4a^2b^2c^2\cdot b^2\Rightarrow 4a^4b^2c^2\overset{(2)}{\ge}4a^2b^4c^2$$

$$\because a^4\ge b^4,\therefore 2a^4c^4\overset{(3)}{\ge}2b^4c^4\quad\because a^2\ge b^2,\therefore 2a^2c^6\overset{(4)}{\ge}2b^2c^6$$

(1) + (2) + (3) + (4) ⇒ (a) is true ⇒ $\frac{1}{s_a+r}\ge\frac{1}{s_b+r}$

Similarly, $\frac{1}{s_b+r}\ge\frac{1}{s_c+r}\ \therefore\frac{1}{s_a+r}\ge\frac{1}{s_b+r}\ge\frac{1}{s_c+r}$

$\therefore$ ***applying Chebyshev,*** $LHS\ge\frac{1}{3}\left(\sum r_a\right)\left(\sum\frac{1}{s_a+r}\right)$

$$\overset{Bergstrom}{\ge}\frac{1}{3}(4R+r)\frac{9}{\sum s_a+3r}=\frac{3(4R+r)}{4R+4r}$$

$$\left(\because\sum s_a\le\sum m_a\le\sum r_a\le 4R+r\right)$$

$\therefore$ ***it suffices to prove:*** $\frac{3(4R+r)}{4(R+r)}\ge\frac{9r^2}{R^2}$

$$\Leftrightarrow 4R^3+R^2r-12Rr^2-12r^3\ge 0\Leftrightarrow 4t^3+t^2-12t-12\ge 0$$

$$\left(t=\frac{R}{r}\right)$$

$\Leftrightarrow(t-2)(4t^2+9t+6)\ge 0\to$ ***true*** $\because t=\frac{R}{r}\ge 2$ ***(Euler) (Proved)***

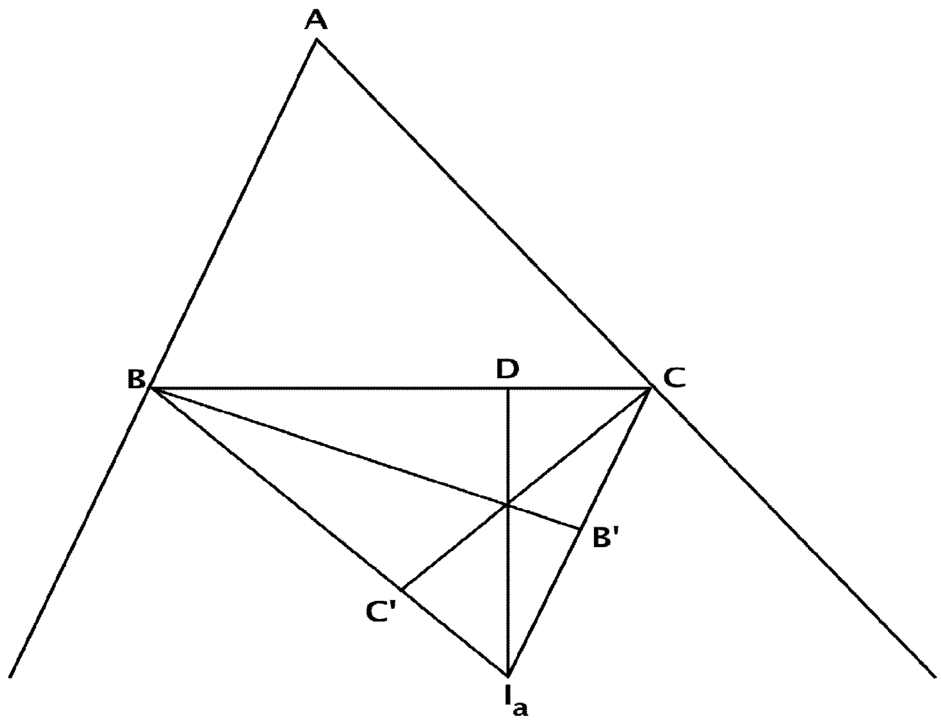
JP.057. Let ABC be an arbitrary triangle and I_a, I_b, I_c are excenters of ABC. I_aBC, I_bCA, I_cAB are the extriangles of ABC. Let h_i $(i = 1, 2, 3, \dots, 9)$ the altitudes of extriangles. Prove that

$$\prod_{i=1}^{9} h_i = \left(\prod_{a,b,c} r_a\right)^3$$

where r_a, r_b, r_c are exradii of ABC.

Proposed by Mehmet Şahin – Ankara – Turkey

Solution 1 by Sayak Chatterjee-Howrah-India



Let us assume, $I_aD = h_1, BB' = h_2, CC' = h_3, h_1 = r_a$

$$\angle I_aBD = 90° - \frac{B}{2}, \angle I_aCD = 90° - \frac{C}{2} \ \therefore h_2 = a\sin\left(90° - \frac{C}{2}\right) = a\cos\frac{C}{2}$$

$[BC = a, CA = b, AB = c]$ *[s is semiperimeter of ABC]. Similarly,* $h_3 = a\cos\frac{B}{2}$

So, $h_1h_2h_3 = r_a a^2 \cos\frac{B}{2}\cos\frac{C}{2}$*. Hence, it will be sufficient to prove that*

$$a^2b^2c^2\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2} = (r_ar_br_c)^2.$$

Now, $BD = s - c$***. So,*** $r_a = (s-c)\tan\left(90^\circ - \frac{B}{2}\right) \Rightarrow r_a = (s-c)\cot\frac{B}{2}$

$$\Rightarrow r_a = (s-c)\sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = \sqrt{\frac{s(s-b)(s-c)}{(s-a)}}$$

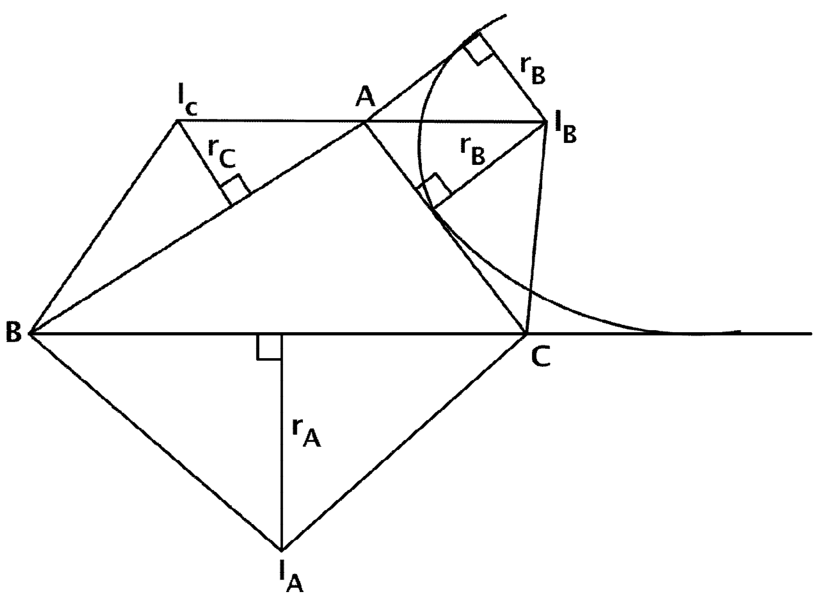
Similarly, $r_b = \sqrt{\frac{s(s-c)(s-a)}{(s-b)}}, r_c = \sqrt{\frac{s(s-a)(s-b)}{(s-c)}}$

$$\therefore a^2b^2c^2\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2} = a^2b^2c^2\cdot\frac{s(s-a)}{bc}\cdot\frac{s(s-b)}{ca}\cdot\frac{s(s-c)}{ab}$$

$$= s^3(s-a)(s-b)(s-c)$$

$$= \left[\frac{s(s-b)(s-c)}{(s-a)}\right]\cdot\left[\frac{s(s-c)(s-a)}{(s-b)}\right]\cdot\left[\frac{s(s-a)(s-b)}{(s-c)}\right] = r_a^2r_b^2r_c^2$$ ***[Proved]***

Solution 2 by Saptak Bhattacharya-Kolkata-India



Observe that for any triangle, given h_1, h_2 ***as altitudes, their ratio is the ratio of corresponding bases. Using this;***

$$\prod_{i=1}^{9} h_i = r_a^3r_b^3r_c^3\frac{a^2b^2c^2}{\prod(AI_B\cdot CI_B)}$$

Remains to show that: $\frac{a^2b^2c^2}{\prod(AI_b\cdot CI_B)} = 1$***. Thus; ratio becomes***

$\frac{a^2b^2c^2}{8\prod r_B\cdot\prod R_B} = \frac{16\Delta^2R^2}{8\prod r_B\prod R_B}$. *Now,* $r\prod r_B = \Delta^2$*, so, we have;* $\frac{2R^2r}{\prod R_B}$

Now, in $\Delta\ AI_BC$*; be sine rule, and using* $\sin 2B = 2\sin\frac{B}{2}\sin\frac{B}{2}$*, we have*

$R_B = 2R\sin\frac{B}{2}$. *Thus,* $\frac{2R^2r}{\prod R_B} = \frac{r}{R\left(4\prod\sin\frac{B}{2}\right)} \Leftrightarrow \frac{a^3b^3c^3}{\prod(AI_B\cdot CI_B b)} = 1$

Now, $\prod(AI_B\cdot CI_B\cdot b) = \prod\left(4\cdot\frac{1}{2}\cdot r_B\cdot b\cdot R_B\right)$

[$\because abc = 4\Delta R$ *is a triangle* R_b *= circumradius of* $\Delta\ AI_BC$*]*

$= 8\prod r_B\cdot\prod R_B\cdot abc$. *Now, we know,* $\sum\cos A = 1 + 4\prod\sin\frac{B}{2} = 1 + \frac{r}{R}$;

Thus, $4\prod\sin\frac{B}{2} = \frac{r}{R}$ *and hence;* $\frac{a^2b^2c^2}{\prod(AI_B\cdot CI_B)} = 1$ *(Proved)*

JP.058. Prove that for all $x \in \mathbb{R}$ we have

$$\cos(\sin x) > |\sin(\cos x)|$$

Proposed by Abdallah El Farissi – Bechar – Algerie

Solution 1 by Ravi Prakash-New Delhi-India

$|\sin(\cos x)| < \cos(\sin x)$. *For* $x = 0, \sin(\cos 0) = \sin 1 < 1 = \cos(\sin 0)$

For $0 < x \le \frac{\pi}{2}, \sin x < x \Rightarrow \cos x < \cos(\sin x)$. *Also,* $\sin(\cos x) \le \cos x$

$|\sin(\cos x)| = \sin(\cos x) < \cos(\sin x)$ *for* $0 \le x \le \frac{\pi}{2}$. *For* $\frac{\pi}{2} \le x \le \pi \Rightarrow 0 \le \pi - x \le \frac{\pi}{2}$

$$|\sin(\cos(\pi - x))| < \cos(\sin(\pi - x))$$

$$\Rightarrow |\sin(-\cos x)| < \cos(\sin x) \Rightarrow |\sin(\cos x)| < \cos(\sin x)$$

Thus, $|\sin(\cos x)| < \cos(\sin x)$. *For* $0 \le x \le \pi$. *For* $\pi \le x \le 2\pi, 0 \le x - \pi \le \pi$

$$\Rightarrow |\sin(\cos(x - \pi))| < \cos(\sin(x - \pi)) \Rightarrow |\sin(\cos x)| < \cos(-\sin x) = \cos(\sin x)$$

Hence, $|\sin(\cos x)| < \cos(\sin x), 0 \le x \le 2\pi$. *Now, let* $x \in \mathbb{R}$*, then*

$x = 2k\pi + \theta$ *for some integer* k *and* $0 \le \theta < 2\pi \Rightarrow \theta = 2k\pi - x$

We have $|\sin(\cos\theta)| < \cos(\sin\theta) \Rightarrow |\sin\{\cos(2k\pi - x)\}| < \cos(\sin(2k\pi - x))$

$$\Rightarrow |\sin(\cos x)| < \cos(\sin x)$$

Solution 2 by Imad Zak-Saida-Lebanon

Note that for $x \in \left[0, \frac{\pi}{3}\right], x \ge \sin x$ *(1)*

$$\begin{matrix} 0 \le x \le \frac{\pi}{2} \\ -1 \le -\sin x \le 0 \end{matrix} \Rightarrow x - \sin x \le \frac{\pi}{2} \Rightarrow x - \sin x \in \left[0, \frac{\pi}{2}\right] \quad (2)$$

$$0 \le \frac{x + \sin x}{2} \le \frac{\pi}{4} + \frac{1}{2} \le \frac{\pi}{2} \Rightarrow$$

$$\frac{x+\sin x}{2} \in \left[0, \frac{\pi}{2}\right] \quad (3)$$

$$\cos(p) - \cos(q) = -2\sin\frac{p+q}{2}\sin\frac{p-q}{2} \quad (4)$$

***Not let** $f(x) = \cos(\sin x) - |\sin(\cos x)|$ **we have** $f(x + \pi) = f(x) \Rightarrow f$ **is periodioc of period** π**. So let's study it over** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$**; Moreover** $f(-x) = f(x) \Rightarrow (y'y)$ **is an axis of symmetry** $f(x) = \cos(\sin x) - \sin(\cos x)$ **over** $\left[0, \frac{\pi}{2}\right]$*

$$= 2\sin\left(\frac{\sin x + x}{2}\right)\sin\left(\frac{x - \sin x}{2}\right) \textit{ by (4)}$$

$$\ge 0 \textit{ by (3) \& (2)}$$

$$\Rightarrow f(x) \ge 0 \Rightarrow \cos(\sin x) \ge \sin(\cos x) \textit{ Q.E.D.}$$

JP.059. Let a, b, c be the side lengths of a triangle ABC with inradius r.

Prove that

$$\frac{1}{a^3}\tan\frac{A}{2} + \frac{1}{b^3}\tan\frac{B}{2} + \frac{1}{c^3}\tan\frac{C}{2} \le \frac{R}{48r^4}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo:** a, b, c **lados de un triángulo. Probar la siguiente desigualdad:

$$\frac{1}{a^3}\tan\frac{A}{2} + \frac{1}{b^3}\tan\frac{B}{2} + \frac{1}{c^3}\tan\frac{C}{2} \le \frac{R}{48r^4}$$

1) Teniendo en cuenta las siguientes identidades en un triángulo** ABC**:

$$\tan\frac{A}{2} = \frac{(s-b)(s-c)}{S}, \tan\frac{B}{2} = \frac{(s-c)(s-a)}{S}, \tan\frac{C}{2} = \frac{(s-a)(s-b)}{S}$$

2) Recordar las siguientes desigualdades en un triángulo** ABC**:

$$R \ge 2r, s \ge 3\sqrt{3}r, S = sr \ge 3\sqrt{3}r^2, \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \le \frac{\sqrt{3}}{2r} \textit{... (Anteriormente demostrado)}$$

$$(s-b)(s-c)\le\frac{a^2}{4},(s-c)(s-a)\le\frac{b^2}{4},(s-a)(s-b)\le\frac{c^2}{4}$$

La desigualdad propuesta es equivalente:

$$\frac{(s-b)(s-c)}{a^3S}+\frac{(s-c)(s-a)}{b^3S}+\frac{(s-a)(s-b)}{c^3S}\le\frac{1}{4S}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\le$$

$$\le\frac{1}{12\sqrt{3}r^2}\times\frac{\sqrt{3}}{2r}=\frac{R}{24\,R\times r^3}\le\frac{R}{48r^4}$$ ***... (LQQD)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a^3}\tan\frac{A}{2}=\frac{1}{a^2}\cdot\frac{1}{2R\sin A}\cdot\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\frac{1}{a^2}\cdot\frac{\sin\frac{A}{2}}{4R\sin\frac{A}{2}\cos^2\frac{A}{2}}=\frac{1}{4Ra^2}\sec^2\frac{A}{2}\quad (1)$$

WLOG, we may assume $a\ge b\ge c$

$$\therefore\cos\frac{A}{2}\le\cos\frac{B}{2}\le\cos\frac{C}{2}\Rightarrow\sec^2\frac{A}{2}\ge\sec^2\frac{B}{2}\ge\sec^2\frac{C}{2}\text{ and }\frac{1}{a^2}\le\frac{1}{b^2}\le\frac{1}{c^2}$$

$$\therefore LHS=\sum\frac{1}{a^3}\tan\frac{A}{2}=\frac{1}{4R}\sum\left(\frac{1}{a^2}\cdot\sec^2\frac{A}{2}\right)\text{ (from (1))}$$

$$\overset{Chebyshev}{\le}\frac{1}{12R}\sum\frac{1}{a^2}\cdot\sum\sec^2\frac{A}{2}=\frac{1}{12R}\cdot\frac{(\sum a^2b^2)}{a^2b^2c^2}\cdot\sum\sec^2\frac{A}{2}$$

$$\le\frac{1}{12R}\cdot\frac{(4R^2s^2)}{16R^2r^2s^2}\sum\sec^2\frac{A}{2}\text{ (by Goldstone's inequality)}$$

$$=\frac{1}{48Rr^2}\sum\sec^2\frac{A}{2}\therefore\text{ it suffices to prove: }\frac{1}{48Rr^2}\sum\sec^2\frac{A}{2}\le\frac{R}{48r^4}\Leftrightarrow\sum\sec^2\frac{A}{2}\le\frac{R^2}{r^2}$$

$$\Leftrightarrow\frac{bc}{s(s-a)}+\frac{ca}{s(s-b)}+\frac{ab}{s(s-c)}\le\frac{R^2}{r^2}$$

$$\Leftrightarrow\frac{1}{s\cdot rs}\left(bc\cdot\frac{\Delta}{s-a}+ca\cdot\frac{\Delta}{s-b}+ab\cdot\frac{\Delta}{s-c}\right)\le\frac{R^2}{r^2}$$

$$\Leftrightarrow\frac{1}{s^2}(bc\cdot r_a+ca\cdot r_b+ab\cdot r_c)\le\frac{R^2}{r}\quad (2)$$

$$\because a\ge b\ge c,\therefore bc\le ca\le ab\text{ and }r_a\ge r_b\ge r_c$$

$$\therefore LHS\text{ of (2)}\overset{Chebyshev}{\le}\frac{1}{3s^2}\left(\sum ab\right)\left(\sum r_a\right)\le\frac{1}{3s^2}\cdot\sum a^2\cdot(4R+r)$$

$$\overset{Leibnitz}{\le}\frac{1}{3s^2}\cdot 9R^2(4R+r)=\frac{3R^2(4R+r)}{s^2}\therefore\text{ it suffices to prove: }\frac{3R^2(4R+r)}{s^2}\le\frac{R^2}{r}$$

$$\Leftrightarrow s^2\ge 12Rr+3r^2\text{ Now, Gerretsen}\Rightarrow s^2\ge 16Rr-5r^2$$

$$\therefore\text{ it suffices to prove: }16Rr-5r^2\ge 12Rr+3r^2$$

$$\Leftrightarrow 4Rr\ge 8r^2\Leftrightarrow R\ge 2r\to\text{ true (Euler) (Proved)}$$

JP.060. Let a, b and c be the lengths of the sides of a triangle with circumradius R. Prove that

$$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\le\frac{3\sqrt{3}}{2}R.$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo: a, b, c los lados de un triángulo ABC y circunradio R. Probar que:

$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\le\frac{3\sqrt{3}R}{2}$. *Es bien conocido la siguiente desigualdad en un triángulo*

$$ABC: \sin A+\sin B+\sin C\le\frac{3\sqrt{3}}{2}\Leftrightarrow\frac{a+b+c}{2R}\le\frac{3\sqrt{3}}{2}\to p\le\frac{3\sqrt{3}R}{2}$$

Desde que: $abc>0$, utilizamos la siguiente desigualdad MH ≤ MA:

$$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\le\frac{a+b}{4}+\frac{b+c}{4}+\frac{c+a}{4}=\frac{a+b+c}{2}=p\le\frac{3\sqrt{3}R}{2}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let a, b, c be the sides of a $\Delta\ ABC$ and $R=$ circum – radius, then

$$\sum_{cyc}\frac{ab}{a+b}\le\frac{3\sqrt{3}}{2}R$$

We know, $ab\le\frac{(a+b)^2}{4}$, $bc\le\frac{(b+c)^2}{4}$ and $ca\le\frac{(c+a)^2}{4}$

We know, $9R^2\ge a^2+b^2+c^2\ge\frac{1}{3}(a+b+c)^2\Rightarrow 27R^2\ge(a+b+c)^2$

$3\sqrt{3}R\ge a+b+c$. *Now,*

$$\sum_{cyc}\frac{ab}{a+b}\le\frac{1}{4}\sum_{cyc}(a+b)=\frac{a+b+c}{2}\le\frac{3\sqrt{3}}{2}R$$

Solution 3 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\le\frac{3\sqrt{3}R}{2}$$

$$a+b\ge 2\sqrt{ab};\ b+c\ge 2\sqrt{bc};\ c+a\ge 2\sqrt{ca}$$

$$RHS\le\frac{\sqrt{ab}}{2}+\frac{\sqrt{bc}}{2}+\frac{\sqrt{ca}}{2}\le\frac{a+b}{4}+\frac{b+c}{4}+\frac{c+a}{4}\overset{?}{\le}\frac{3\sqrt{3}R}{2};\ S\le\frac{3\sqrt{3}R}{2}$$ *(Proved)*

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\sum \frac{ab}{a+b} \overset{H\le G}{\le} \frac{1}{2}\sum \sqrt{ab} \overset{C-B-S}{\le} \frac{1}{2}\sqrt{\sum a}\sqrt{\sum a} = \frac{\sum a}{2} = S \overset{Mitrinovic}{\le} \frac{3\sqrt{3}R}{2}$$

Solution 5 by Evgenidis Nikolaos-Larissa-Greece

By AM-HM inequality we deduce that:

$$\frac{2ab}{a+b} \le \frac{a+b}{2}, \frac{2bc}{b+c} \le \frac{b+c}{2}, \frac{2ca}{c+a} \le \frac{c+a}{2}$$

Therefore, it suffices to prove that $a+b+c \le 3\sqrt{3}R$ or, if we denote the semiperimeter of the triangle by s, it suffices to show that $2s \le 3\sqrt{3}R$. Blundon's inequality states that $s \le (3\sqrt{3}-4)r + 2R$. Then, it suffices to prove that

$$(3\sqrt{3}-4)2r + 4R \le 3\sqrt{3}R \Leftrightarrow (3\sqrt{3}-4)2r \le (3\sqrt{3}-4)R,$$

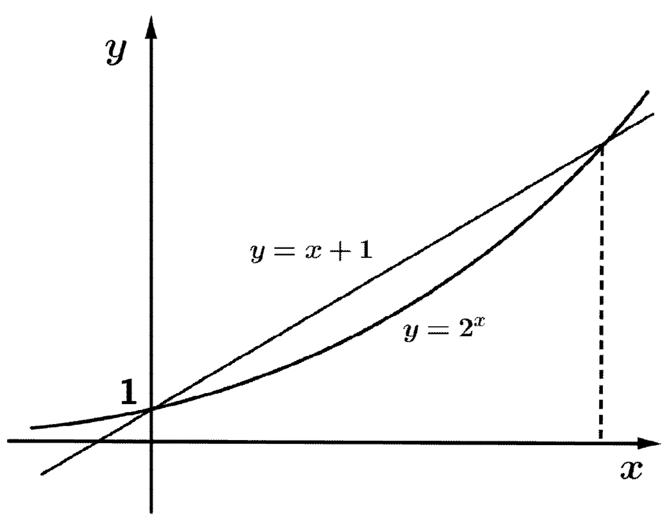
which obviously holds by Euler's inequality $R \ge 2r$. Equality holds if and only if the given triangle is equilateral, i.e. $a = b = c$.

SP.046. Prove that for every positive integer n,

$$\ln\frac{n+1}{2} < \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \log_2\frac{n+1}{2}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Ravi Prakash-New Delhi-India



Let $f(x)=2^x-(1+x);\ 0\le x\le\frac{1}{2};\ f'(x)=2^x\ln 2-1$

For $0<x<\frac{1}{2},\ 2^x<2^{\frac{1}{2}}<\frac{1}{\ln 2}\Rightarrow 2^x\ln 2-1<0$ *for* $0<x<\frac{1}{2}$

$\therefore f'(x)<0$ *for* $0<x<\frac{1}{2}$

$\Rightarrow f(x)$ *is strictly decreasing on* $\left[0,\frac{1}{2}\right]\Rightarrow f(x)<f(0)=0$ *for* $0<x\le\frac{1}{2}$

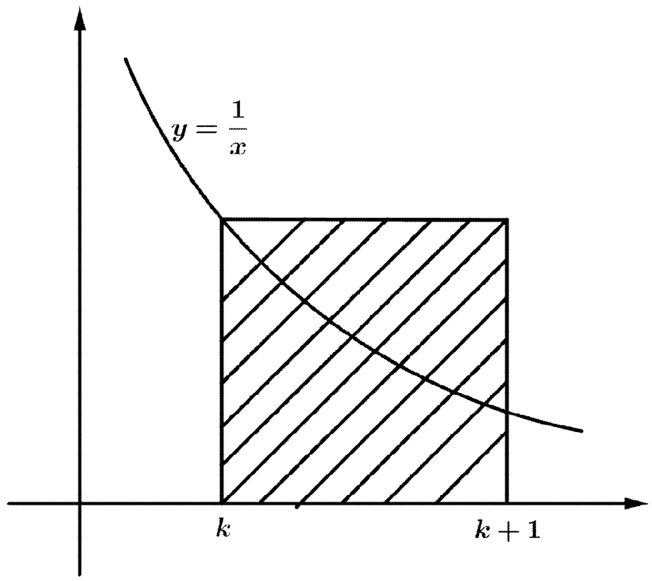
$\Rightarrow 2^x<(1+x)$ *for* $0<x\le\frac{1}{2}$.

$$\therefore 2^{\frac{1}{2}}2^{\frac{1}{3}}2^{\frac{1}{4}}\dots 2^{\frac{1}{n}}<\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\dots\left(1+\frac{1}{n}\right)$$

$$\Rightarrow 2^{\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}}<\frac{n+1}{2}\Rightarrow\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}<\log_2\left(\frac{n+1}{2}\right)\quad (1)$$

Also, for $k\ge 1$,

$$\frac{1}{k}(1)>\int_k^{k+1}\frac{1}{x}dx$$

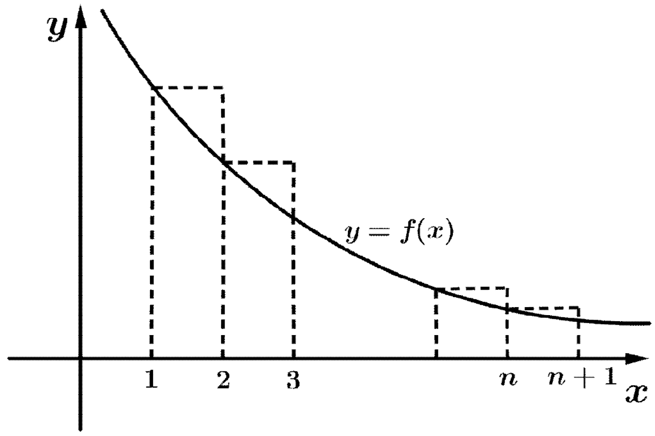


$$\Rightarrow\frac{1}{k}>\ln(k+1)-\ln k\Rightarrow\sum_{k=2}^{n}\frac{1}{k}>\ln(n+1)-\ln(2)$$

$$\Rightarrow\ln\left(\frac{n+1}{2}\right)<\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\quad (2)$$

From (1) and (2), we get the desired inequality.

Solution 2 by Shahlar Maharrahmov-Jebrail-Azerbaidian



Let us use figure. Take $f(x) = \frac{1}{x}$ *and partition* $a_k = \frac{1}{k}$, *then we obtain*

$$\int_1^{n+1} \frac{1}{x}dx \le 1 + \frac{1}{2} + \cdots + \frac{1}{n} \le 1 + \int_1^{n} f(x)\,dx \Rightarrow \ln(n+1) \le$$

$$\le 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \le 1 + \ln n \quad (*)$$

since $\ln\frac{n+1}{2} < \ln(n+1)$ *and* $\log_2\frac{n+1}{2} > 1 + \ln n$

then from (*) $\Rightarrow \ln\frac{n+1}{2} < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \log_2\frac{n+1}{2}$

SP.047. Evaluate without calculator

$$\sum_{k=1}^{17} \cos^4\frac{k\pi}{36}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Evaluar la siguiente suma trigonométrica:

$$M = \sum_{k=1}^{17} \cos^4\frac{k\pi}{36}$$

Para ello utilizaremos las siguientes formulas:

$$\sin^4 x+\cos^4 x=\frac{3+\cos 4x}{3}\wedge \cos x+\cos y=0\Leftrightarrow x+y=\pi$$

Lo pedido es equivalente:

$$M=\cos^4 5+\cos^4 10+\cos^4 15+\cdots+\cos^4 40+\cos^4 45+\cos^4 50\ldots+\cos^4 75+\cos^4 80+\cos^4 85$$

$$M=\cos^4 5+\cos^4 10+\cos^4 15+\cdots+\cos^4 40+\cos^4 45+\sin^4 40\ldots+\sin^4 15+\sin^4 10+\sin^4 5$$

Agrupando convenientemente:

$$(\sin^4 5+\cos^4 5)+(\sin^4 40+\cos^4 40)=\frac{3+\cos 20}{4}+\frac{3+\cos 160}{4}=\frac{3}{2}\ldots \textit{(A)}$$

$$(\sin^4 10+\cos^4 10)+(\sin^4 35+\cos^4 35)=\frac{3+\cos 40}{4}+\frac{4+\cos 140}{4}=\frac{3}{2}\ldots \textit{(B)}$$

$$(\sin^4 15+\cos^4 15)+(\sin^4 30+\cos^4 30)=\frac{3+\cos 60}{4}+\frac{3+\cos 120}{4}=\frac{3}{2}\ldots \textit{(C)}$$

$$(\sin^4 20+\cos^4 20)+(\sin^4 25+\cos^4 25)+\cos^4 45=\frac{3+\cos 80}{4}+\frac{3+\cos 100}{4}+\frac{1}{4}=\frac{3}{2}+\frac{1}{4}\ldots \textit{(D)}$$

Sumando: (A) + (B) + (C) + (D):

$$M=\sum_{k=1}^{17}\cos^4\frac{k\pi}{36}=\frac{3}{2}\cdot 4+\frac{1}{4}=\frac{25}{4}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\sum_{k=1}^{17}\cos^4\left(\frac{k\pi}{36}\right)$$

$$=\sum_{k=1}^{8}\left[\cos^4\left(\frac{k\pi}{36}\right)+\left(\cos\left(\frac{\pi}{2}-\frac{k\pi}{36}\right)\right)^4\right]+\cos^4\left(\frac{\pi}{4}\right)=\sum_{k=1}^{8}\left[\cos^3\left(\frac{k\pi}{36}\right)+\sin^3\left(\frac{k\pi}{36}\right)\right]+\frac{1}{4}$$

$$=\frac{1}{2}\sum_{k=1}^{8}\left[\left(\cos^2\frac{k\pi}{36}+\sin^2\frac{k\pi}{36}\right)^2+\left(\cos^2\frac{k\pi}{36}-\sin^2\frac{k\pi}{36}\right)^2\right]$$

$$=\frac{1}{2}\sum_{k=1}^{8}\left[1+\cos^2\left(\frac{k\pi}{18}\right)\right]+\frac{1}{4}=\frac{1}{2}\sum_{k=1}^{8}\left[1+\frac{1+\cos\left(\frac{k\pi}{a}\right)}{2}\right]+\frac{1}{4}$$

$$=\frac{1}{4}\sum_{k=1}^{8}\left[3+\cos\left(\frac{k\pi}{9}\right)\right]+\frac{1}{4}=\frac{25}{4}+\frac{1}{2}S_1$$

where $S_1=\sum_{k=1}^{8}\cos\left(\frac{k\pi}{9}\right)$

$$= \sum_{k=1}^{4}\left[\cos\left(\frac{k\pi}{9}\right) + \cos\left(\pi - \frac{k\pi}{9}\right)\right] = \sum_{k=1}^{4}\left[\cos\left(\frac{k\pi}{9}\right) - \cos\left(\frac{k\pi}{9}\right)\right] = 0$$

$$\sum_{k=1}^{17}\cos^4\left(\frac{k\pi}{36}\right) = \frac{25}{4}$$

SP.048. Prove that the following inequality holds for all non-negative real numbers a, b, c

$$(a^4 + b^4 + c^4)(ab^3 + bc^3 + ca^3) \geq (a^3b + b^3c + c^3a)(a^2b^2 + b^2c^2 + c^2a^2)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los reales no negativos: a, b, c ***la siguiente desigualdad:***

$$(a^4 + b^4 + c^4)(ab^3 + bc^3 + c^3a) \geq (a^3b + b^3c + a^3c)(a^2b^2 + b^2c^2 + c^2a^2)$$

Siendo: $a, b, c \geq 0$***. Por la desigualdad de Cauchy:***

$$(a^4 + b^4 + c^4)(a^2b^2 + b^2c^2 + c^2a^2) \geq (a^3b + b^3c + a^3c)^2 \ \dots \ \textit{(A)}$$

$$(ab^3 + bc^3 + c^3a)(a^3b + b^3c + a^3) \geq (a^2b^2 + b^2c^2 + c^2a^2)^2 \ \dots \ \textit{(B)}$$

Multiplicando, se obtiene: (A) × (B):

$$(a^4 + b^4 + c^4)(ab^3 + bc^3 + c^3a) \geq (a^3b + b^3c + a^3c)(a^2b^2 + b^2c^2 + c^2a^2)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Case 1: Exactly 1 of $a, b, c = 0$

WLOG, let $c = 0. LHS \geq RHS \Leftrightarrow (a^4 + b^4)ab^3 \geq a^5b^3$

$\Leftrightarrow b^4 \geq 0 \rightarrow$ ***true*** $\Rightarrow$ ***inequality is valid in this case***

Case 2: Exactly 2 of $a, b, c = 0$***. Then*** $LHS = RHS = 0$

And $\because 0 \geq 0, \therefore$ ***inequality is valid in this case.***

Case 3: $a = b = c = 0. \therefore LHS = RHS = 0 \Rightarrow LHS \geq RHS$

Case 4: $a, b, c > 0$***;*** $LHS - RHS \geq 0 \Leftrightarrow$

$$\Leftrightarrow ab^7 + bc^7 + ca^7 + a^4bc^3 + b^4ca^3 + c^4ab^3 \geq$$

$$\geq a^5bc^2 + b^5ca^2 + c^5ab^2 + a^3b^3c^2 + b^3c^2a^2 + c^3a^3b^2 \quad \textit{(1)}$$

$$a^4bc^3 + a^4bc^3 + ab^7 \overset{A-G}{\geq} 3a^3b^3c^2 \quad | \quad ab^7 + b^3a^3c^2 \overset{A-G}{\geq} 2b^5a^2c$$

$$b^4ca^3 + b^4ca^3 + bc^7 \overset{A-G}{\geq} 3b^3c^3a^2 \quad | \quad bc^7 + c^3b^3a^2 \overset{A-G}{\geq} 2c^5b^2a$$

$$c^4ab^3 + c^4ab^3 + ca^7 \overset{A-G}{\geq} 3c^3a^3b^2 \quad | \quad ca^7 + c^3a^3b^2 \overset{A-G}{\geq} 2a^5c^2b$$

Adding the above 6 inequalities, we get:

$$2(ab^7 + bc^7 + ca^7 + a^4bc^3 + b^4ca^3 + c^4ab^3) \geq$$

$$2\left(a^5bc^2 + b^5ca^2 + c^5ab^2 + a^3b^3c^2 + b^3c^3a^2 + c^3a^3b^2\right)$$

$$\Leftrightarrow ab^7 + bc^7 + ca^7 + a^4bc^3 + b^4ca^3 + c^4ab^3 \geq$$

$$\geq a^5bc^2 + b^5ca^2 + c^5ab^2 + a^3b^3c^2 + b^3c^3a^2 + c^3a^3b^2 \Rightarrow$$ ***(1) is true (Proved)***

SP.049. Prove that the following inequality holds for all positive real numbers x, y:

$$x^{y-x} \cdot y^{x-y} \leq 1 .$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by SK Rejuan-West Bengal-India

$x, y \in \mathbb{R}^+$ ***we have to prove,*** $x^{y-x}y^{x-y} \leq 1$

$$\Leftrightarrow x^y \cdot y^x \leq x^x y^y \qquad (1)$$

Now, let us take x, y with the associated weight x and y respectivelly.

By weighted GM $\geq$ AM we get,

$$x^x \cdot y^y \geq \left(\frac{x+y}{\frac{x}{x}+\frac{y}{y}}\right)^{x+y} \Rightarrow x^x y^y \geq \left(\frac{x+y}{2}\right)^{x+y} \qquad (2)$$

Now, let us take x, y with the associated weigth, y and x respectively by applying AM $\geq$ GM we get,

$$\left(\frac{x\cdot y + y\cdot x}{x+y}\right) \geq (x^y \cdot y^x)^{\frac{1}{x+y}} \Rightarrow \left(\frac{2xy}{x+y}\right)^{x+y} \geq x^y y^x \qquad (3)$$

We know that, by AM $\geq$ GM $(x+y)^2 \geq 4xy \Rightarrow \frac{x+y}{2} \geq \frac{2xy}{x+y} \quad [\because x, y \in \mathbb{R}^+]$

$$\Rightarrow \left(\frac{x+y}{2}\right)^{x+y} \geq \left(\frac{2xy}{x+y}\right)^2 \geq x^y \cdot y^x \text{ [from (3)]} \Rightarrow \left(\frac{x+y}{2}\right)^{x+y} \geq x^y y^x \qquad (4)$$

Combining (3) & (4) we get, $x^x y^y \geq \left(\frac{x+y}{2}\right)^{x+y} \geq x^y y^x \Rightarrow x^x y^y \geq x^y y^x$

$$\Rightarrow x^{x-y} y^{y-x} \geq 1 \Rightarrow x^{y-x} y^{x-y} \leq 1$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\text{Let } g(t)=t+\ln t-1, 0<t\le 1;\ g'(t)=1+\frac{1}{t}>0 \text{ for } 0<t<1$$

$$\therefore g(t)<g(1) \text{ for } 0<t<1 \Rightarrow g(t)<0 \text{ for } 0<t<1$$

$$\text{For } a>0, \text{ consider } f(x)=\left(\frac{x}{a}\right)^{a-x}, x>0;\ \ln f(x)=(a-x)\ln\left(\frac{x}{a}\right)$$

$$\frac{f'(x)}{f(x)}=-\ln\left(\frac{x}{a}\right)+\frac{a-x}{x}\Rightarrow f'(x)=\left(\frac{a}{x}+\ln\left(\frac{a}{x}\right)-1\right)f(x)$$

$$f'(x)>0 \text{ if } 0<x<a$$

$$=0 \text{ if } x=a$$

$$<0 \text{ if } x>a$$

$$[\because f'(x)=g\left(\frac{a}{x}\right)<0 \text{ for } x>a]$$

Thus, $f(x)$ *is strictly increasing on* $(0,a]$ *an decreasing on* $[0,\infty)$

$$\therefore \text{ if } 0<x\le a, f(x)\le f(a) \text{ and if } x\ge a, f(x)\le f(a)$$

$$\text{i.e. } f(x)\le f(a)\ \ \forall x>0\Rightarrow\left(\frac{x}{a}\right)^{a-x}\le 1\ \ \forall x>0 \therefore x,y>0, \left(\frac{x}{y}\right)^{y-x}\le 1 \text{ or } x^{y-x}y^{x-y}\le 1$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\text{Knonw, results } \frac{x+y}{2}\ge\frac{2xy}{x+y} \text{ and } x^xy^y\ge\left(\frac{x+y}{2}\right)^{x+y}$$

$$\text{Now, } x^yy^x \overset{WEIGHTED\ AM\ge GM}{\lessapprox} \left(\frac{xy+yx}{x+y}\right)^{x+y}=\left(\frac{2xy}{x+y}\right)^{x+y}\le\left(\frac{x+y}{2}\right)^{x+y}\le x^xy^y$$

$$\therefore x^{y-x}y^{x-y}\le 1 \text{ (proved) equality at } x=y$$

Solution 4 by Imad Zak-Saida-Lebanon

$$x+y>0 \text{ show that } x^{y-x}\cdot y^{x-y}\le 1\Leftrightarrow (y-x)\ln x+(x-y)\ln y\le 0\ ??$$

$$\text{WLOG suppose } y\ge x\Rightarrow y-x\ge 0\ \ \&\frac{x}{y}\le 1\Leftrightarrow \ln\left(\frac{x}{y}\right)\le 0\Rightarrow \underset{(+)}{(y-x)}\cdot\underset{(-)}{\ln\left(\frac{x}{y}\right)}\le 0$$

$$\Leftrightarrow (y-x)(\ln x-\ln y)\le 0\Leftrightarrow (y-x)\ln x+(x-y)\ln y\le 0$$

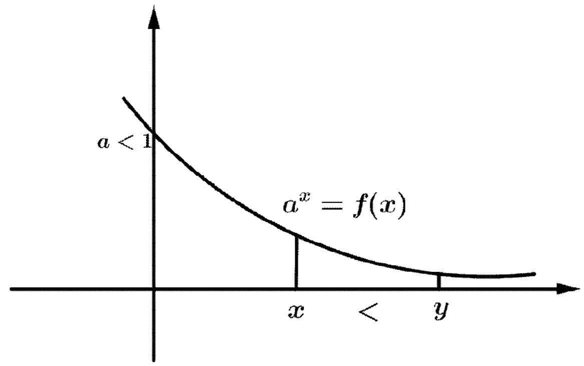
$$\ll=\gg \text{ for } x=y \text{ QED}$$

Solution 5 by Sladjan Stankovic-Macedonia

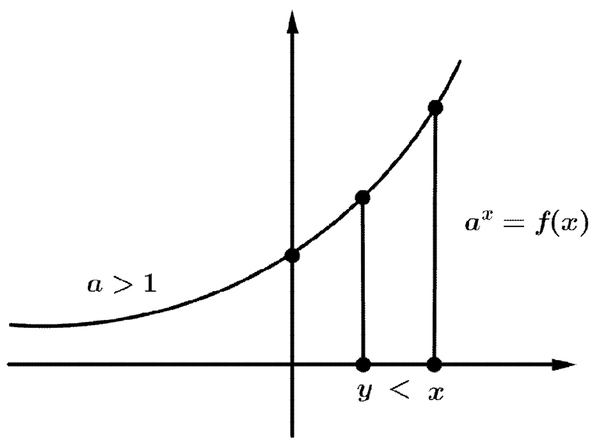
$$\frac{x^y}{x^x}\cdot\frac{y^x}{y^y}\le 1\Leftrightarrow\left(\frac{x}{y}\right)^y\le\left(\frac{x}{y}\right)^x$$

Let $a = \frac{x}{y}$

1) If $x < y \Rightarrow \frac{x}{y} < 1 \Rightarrow \left(\frac{x}{y}\right)^y \le \left(\frac{x}{y}\right)^x$



2) If $x > y \Rightarrow \frac{x}{y} > 1 \Rightarrow \left(\frac{x}{y}\right)^y \le \left(\frac{x}{y}\right)^x$



Solution 6 by Mirza Uzair Baig-Lahore-Pakistan

Assume $x \le y$ ***then apply*** $\ln(\cdot)$ ***to both sides.***

$$\Leftrightarrow (y - x)\ln x + (x - y)\ln y \le 0 \Leftrightarrow (y - x)\ln x \le (y - x)\ln y$$

$\Leftrightarrow \ln x \le y \Leftrightarrow x \le y$***. Similarly, we can prove for*** $y \ge x$***.***

SP.050. Let $a \ge b \ge c > 0$**. Prove that:**

$$a^{a-b}b^{b-c}c^{c-a} \ge 1\,.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by SK Rejuan -West Bengal-India

Given $a \ge b \ge c > 0$. We have to prove

$$a^{a-b}b^{b-c}c^{c-a} \ge 1 \Leftrightarrow a^a b^b c^c \ge a^b b^c c^a \quad (1)$$

Let us take $a, b, c \in \mathbb{R}^+$ with the associated weight a, b, c respectively, by applying

$$AM \ge HM \text{ we get, } (a^a b^b c^c)^{\frac{1}{a+b+c}} \ge \frac{a+b+c}{\frac{a}{a}+\frac{b}{b}+\frac{c}{c}} \quad [\because a, b, c \ne 0]$$

$$\Rightarrow a^a b^b c^c \ge \left(\frac{a+b+c}{3}\right)^{a+b+c} \quad (2)$$

Now let us take $a, b, c > 0$ with the associated weight b, c, a respectively by applying

AM $\ge$ GM we get,

$$\left(\frac{a\cdot b+b\cdot c+c\cdot a}{b+c+a}\right) \ge (a^b \cdot b^c \cdot c^a)^{\frac{1}{a+b+c}} \Rightarrow \left(\frac{ab+bc+ca}{a+b+c}\right)^{a+b+c} \ge a^b b^c c^a \quad (3)$$

$$\text{Now, } (a+b+c)^2 - 3(ab+bc+ca) = \sum a^2 - \sum ab$$

$$= \frac{1}{2}\left\{\sum (a-b)^2\right\} \ge 0 \Rightarrow (a+b+c)^2 \ge 3(ab+bc+ca)$$

$$\Rightarrow \left(\frac{a+b+c}{3}\right) \ge \left(\frac{ab+bc+ca}{a+b+c}\right) \Rightarrow \left(\frac{\sum a}{3}\right)^{a+b+c} \ge \left(\frac{\sum ab}{\sum a}\right)^{a+b+c} \quad (4)$$

Combining (3) & (4) we get,

$$\left(\frac{\sum a}{3}\right)^{a+b+c} \ge \left(\frac{\sum ab}{\sum a}\right)^{a+b+c} \ge a^b b^c c^a \Rightarrow \left(\frac{\sum a}{3}\right)^{a+b+c} \ge a^b b^c c^a \quad (5)$$

Combining (2) & (5) we get $a^a b^b c^c \ge \left(\frac{\sum a}{3}\right)^{a+b+c} \ge a^b b^c c^a \Rightarrow a^a b^b c^c \ge a^b b^c c^a$

$$\Rightarrow a^{a-b}b^{b-c}c^{c-a} \ge 1 \text{ [Proved]}$$

Solution 2 by Ravi Prakash-New Delhi-India

As $a \ge b \ge c$,

$$\frac{a}{c} \ge 1 \text{ and } a-b \ge 0 \Rightarrow \left(\frac{a}{c}\right)^{a-b} \ge 1 \quad (1)$$

$$\text{Also, } b \ge c \Rightarrow \frac{b}{c} \ge 1, b-c \ge 0 \Rightarrow \left(\frac{b}{c}\right)^{b-c} \ge 1 \quad (2)$$

From (1), (2)

$$\left(\frac{a}{c}\right)^{a-b}\left(\frac{b}{c}\right)^{b-c} \ge 1 \Rightarrow a^{a-b}b^{b-c}c^{b-a+c-b} \ge 1 \Rightarrow a^{a-b}b^{b-c}c^{c-a} \ge 1$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$a^b b^c c^a \overset{WEIGHTED\ AM\geq GM}{\leq} \left(\frac{ab+bc+ca}{a+b+c}\right)^{a+b+c}$$

$$\leq \left(\frac{a+b+c}{3}\right)^{a+b+c} \quad [\because (a+b+c)^2 \geq 3(ab+bc+ca)]$$

$$= \left(\frac{a+b+c}{\frac{b}{b}+\frac{a}{a}+\frac{c}{c}}\right)^{a+b+c} \overset{WEIGHTED\ GM\geq HM}{\leq} a^a b^b c^c$$ *(proved)* equality at $a=b=c$

SP.051. If $a,b,x,y \in (0,\infty)$ and $m \in [0,\infty)$ then:

$$\frac{x}{(ay+bz)^{m+1}} + \frac{y}{(az+bx)^{m+1}} + \frac{z}{(ax+by)^{m+1}} \geq \frac{3^{m+1}}{(a+b)^{m+1}(x+y+z)^m}$$

Proposed by D.M. Bătinețu-Giurgiu; Neculai Stanciu – Romania

Solution 1 by Marian Ursărescu-Romania

We must show:

$$\frac{x^{m+2}}{(axy+bxz)^{m+1}} + \frac{y^{m+2}}{(ayz+bxy)^{m+1}} + \frac{z^{m+2}}{(axz+byz)^{m+1}} \geq \frac{3^{m+1}}{(a+b)^{m+1}(x+y+z)^m} \quad (1)$$

From Hölder's inequality we have: $\frac{x^{m+2}}{(axy+bxz)^{m+1}} + \frac{y^{m+2}}{(ayz+bxy)^{m+1}} + \frac{z^{m+2}}{(axz+byz)^{m+1}} \geq$

$$\geq \frac{(x+y+z)^{m+2}}{(axy+bxz+ayz+bxy+axz+byz)^{m+1}} = \frac{(x+y+z)^{m+2}}{(a+b)^{m+1}(xy+xz+yz)^{m+1}} \quad (2)$$

From (1)+(2) we must show: $\frac{(x+y+z)^{m+2}}{(a+b)^{m+1}(xy+xz+yz)^{m+1}} \geq \frac{3^{m+1}}{(a+b)^{m+1}(x+y+z)^m} \Leftrightarrow$

$$\Leftrightarrow ((x+y+z)^2)^{m+1} \geq 3^{m+1}(xy+xz+yz)^{m+1} \Leftrightarrow (x+y+z)^2 \geq 3(xy+xz+yz) \Leftrightarrow$$

$$\Leftrightarrow x^2+y^2+z^2 \geq xy+xz+yz$$ *(true)*

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{x}{(ay+bz)^{m+1}} = \sum_{cyc}\frac{x^{m+2}}{(axy+bzx)^{m+1}} \overset{RADON'S\ INEQUALITY}{\geq} \frac{(x+y+z)^{m+2}}{(a+b)^{m+1}(xy+yz+zx)^{m+1}} \geq$$

$$\geq \frac{(x+y+z)^{m+2}}{(a+b)^{m+1}\frac{(x+y+z)^{2m+2}}{3^{m+1}}} = \frac{3^{m+1}}{(a+b)^{m+1}(x+y+z)^m}$$

(proved)

SP.052. In any triangle ABC the following relationship holds:

$$\frac{1}{(\cos A+\cos B)^2}+\frac{1}{(\cos B+\cos C)^2}+\frac{1}{(\cos C+\cos A)^2}\ge 3$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}=2\sin\frac{C}{2}\cos\frac{A-B}{2}$$

$$\because 0<A,B<\pi,\therefore -\frac{\pi}{2}<\frac{A-B}{2}<\frac{\pi}{2}\therefore 0<\cos\left(\frac{A-B}{2}\right)\le 1$$

$$\therefore 2\sin\frac{C}{2}\cos\frac{A-B}{2}\le 2\sin\frac{C}{2}\left(\because \sin\frac{C}{2}>0\right)$$

$$\therefore \cos A+\cos B\le 2\sin\frac{C}{2}\Rightarrow(\cos A+\cos B)^2\le 4\sin^2\frac{C}{2}$$

$$\left(\because \cos A+\cos B=2\sin\frac{C}{2}\cos\frac{A-B}{2}>0\right)$$

$$\Rightarrow\frac{1}{(\cos A+\cos B)^2}\underset{(1)}{\ge}\frac{1}{4}\csc^2\frac{C}{2}$$

Similarly, $\frac{1}{(\cos B+\cos C)^2}\underset{(2)}{\ge}\frac{1}{4}\csc^2\frac{A}{2}$ *and* $\frac{1}{(\cos C+\cos A)^2}\underset{(3)}{\ge}\frac{1}{4}\csc^2\frac{B}{2}$

$$(1)+(2)+(3)\Rightarrow LHS\ge\frac{1}{4}\sum\csc^2\frac{A}{2}\ge\frac{1}{4}\cdot\frac{1}{3}\left(\sum\csc\frac{A}{2}\right)^2$$

$$\left(\because \sum x^2\ge\frac{1}{3}\left(\sum x\right)^2\right)\therefore LHS\ge\frac{1}{12}\left(\sum\csc\frac{A}{2}\right)^2$$

$$\because f(x)=\csc\frac{x}{2}\ \forall x\in(0,\pi)\text{ is convex as } f''(x)=\frac{\csc^3\left(\frac{x}{2}\right)+\cot^2\left(\frac{x}{2}\right)\csc\left(\frac{x}{2}\right)}{4}>0$$

$$\therefore LHS\ge\frac{1}{12}\left(\sum\csc\frac{A}{2}\right)^2\overset{Jensen}{\ge}\left\{3\csc\left(\frac{A+B+C}{6}\right)\right\}^2=3$$

Solution 2 by George Apostolopoulos-Messolonghi-Greece

Using the AM-GM inequality, we have $\frac{1}{(\cos A+\cos B)^2}+\frac{1}{(\cos B+\cos C)^2}+\frac{1}{(\cos C+\cos A)^2}\ge$

$$\ge\frac{3}{\sqrt[3]{\left((\cos A+\cos B)(\cos B+\cos C)(\cos C+\cos A)\right)^2}}$$

It is well – known that in any triangle ABC holds:

$$(\cos A+\cos B)(\cos B+\cos C)(\cos C+\cos A)=\frac{2R^2r^2+r^3+rs^2}{4R^3}$$

Also, we know that $R\geq 2r$ (Euler) and $s=\frac{a+b+c}{2}\leq\frac{3\sqrt{3}}{2}R$. So

$$\frac{1}{(\cos A+\cos B)^2}+\frac{1}{(\cos B+\cos C)^2}+\frac{1}{(\cos C+\cos A)^2}\geq$$

$$\frac{3}{\sqrt[3]{\left(\frac{2Rr^2+r^3+rs^2}{4R^3}\right)^2}}=\frac{3\sqrt[3]{4R^3}^2}{\sqrt[3]{(2Rr^2+r^3+rs^2)^2}}=$$

$$\frac{3R^2\sqrt[3]{16}}{\sqrt[3]{(2Rr^2+r^3+rs^2)^2}}\geq\frac{3R^2\sqrt[3]{16}}{\left(\sqrt[3]{2R\left(\frac{R}{2}\right)^2+\left(\frac{R}{2}\right)^3+\frac{R}{2}\cdot\frac{27R^2}{4}}\right)^2}=$$

$$\frac{3R^2\sqrt[3]{16}}{\sqrt[3]{\left(\frac{R^3}{2}+\frac{R^3}{8}+\frac{27R^3}{8}\right)^2}}=\frac{3R^2\sqrt[3]{16}}{\sqrt[3]{(4R^3)^2}}=\frac{3R^2\sqrt[3]{16}}{\sqrt[3]{16}\cdot R^2}=3$$

Equality holds when the triangle ABC is equilateral.

Solution 3 by Eliezer Okeke-Nigeria

In a triangle ABC prove the relation holds

$$\sum_{cyc}\frac{1}{(\cos A+\cos B)^2}\geq 3$$

$$\sum_{cyc}\frac{1}{(\cos A+\cos B)^2}=\sum_{cyc}\frac{1^3}{(\cos A+\cos B)^2}\overset{RADON}{\geq}\frac{3^3}{\left(2\sum_{cyc}\cos A\right)^2}\overset{JENSEN}{\geq}$$

$$\geq\frac{3^3}{2^2\left(3\cos\left(\frac{\sum A}{3}\right)\right)^2}=\frac{3^3}{2^2(3^2)\left(\frac{1}{2}\right)^2}=3$$

Solution 4 by Marin Chirciu-Romania

Using the inequality:

$x^2+y^2+z^2\geq\frac{(x+y+z)^2}{3}$, *with* $x=\frac{1}{\cos A+\cos B}$, $y=\frac{1}{\cos B+\cos C}$, $z=\frac{1}{\cos C+\cos A}$, *it suffices to prove that* $x+y+z\geq 3$. *We have:*

$x+y+z=\sum\frac{1}{\cos B+\cos C}=\frac{\sum(\cos A+\cos B)(\cos A+\cos C)}{\prod(\cos B+\cos C)}=\frac{\sum\cos^2 A+3\sum\cos B\cos C}{\prod(\cos B+\cos C)}$, ***and using the known identities in triangle:*** $\sum\cos^2 A=\frac{6R^2+4Rr+r^2-s^2}{2R^2}$, $\sum\cos B\cos C=\frac{s^2+r^2-4R^2}{4R^2}$, ***and***

$\prod(\cos B+\cos C)=\sum\cos A\sum\cos B\cos C-\prod\cos A$ ***and***

$\sum\cos A=1+\frac{r}{R}$, $\prod\cos A=\frac{s^2-(2R+r)^2}{4R^2}$, ***we obtain*** $\prod(\cos B+\cos C)=\frac{r(s^2+r^2+2Rr)}{4R^3}$. ***It follows that*** $\sum\frac{1}{\cos B+\cos C}=\frac{R}{r}\cdot\frac{s^2+5r^2+8Rr}{s^2+r^2+2Rr}$. ***The inequality*** $x+y+z\geq 3$ ***can be written:*** $\frac{R}{r}\cdot\frac{s^2+5r^2+8Rr}{s^2+r^2+2Rr}\geq 3\Leftrightarrow(R-3r)s^2+r(8R^2-Rr-3r^2)\geq 0$. ***We distinguish the cases:***

1) If $R-3r\geq 0$, ***the inequality is obviously, because*** $8R^2-Rr-3r^2>0$, ***from Euler's inequality.***

2) If $R-3r<0$, ***we write the inequality*** $(3r-R)s^2\leq r(8R^2-Rr-3r^2)$, ***it follows from Gerretsen's inequality*** $s^2\leq 4R^2+4Rr+3r^2$. ***It remains to prove that:***

$$(3r-R)(4R^2+4Rr+3r^2)\leq r(8R^2-Rr-3r^2)\Leftrightarrow 2R^3-4Rr^2-6r^3\geq 0\Leftrightarrow$$

$(R-2r)(2R^2+4Rr+3r^2)\geq 0$, ***obviously, from Euler's inequality*** $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

SP.053. If $x,y,z\in(0,\infty)$ then:

$$\frac{x}{(y+z)^3}+\frac{y}{(z+x)^3}+\frac{z}{(x+y)^3}\geq\frac{27}{8(x+y+z)^2}$$

Proposed by D. M. Bătinețu – Giurgiu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Si: $x,y,z\in<0,\infty>$. ***Probar la siguiente desigualdad:***

$$\frac{x}{(y+z)^3}+\frac{y}{(z+x)^3}+\frac{z}{(x+y)^3}\geq\frac{27}{8(x+y+z)^2}$$

1) Siendo: $x,y,z>0$, ***se cumple la siguiente desigualdad:***

$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{3}{2}$... ***(Desigualdad de Nesbitt)***

2) Si: $x,y,z\in R\rightarrow(x+y+z)^2\geq 3(xy+yz+zx)$

La desigualdad propuesta es equivalente, aplicando la desigualdad de Cauchy:

$$\frac{\left(\frac{x}{y+z}\right)^2}{x(y+z)}+\frac{\left(\frac{y}{z+x}\right)^2}{y(z+x)}+\frac{\left(\frac{z}{x+y}\right)^2}{z(x+y)}\ge\frac{\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)^2}{2(xy+yz+zx)}\ge\frac{9\times 3}{4\times 2(x+y+z)^2}=\frac{27}{8(x+y+z)^2}\ \dots\ \text{(LQQD)}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $x,y,z\in(0,\infty)$ *then*

$$\sum_{cyc}\frac{x}{(y+z)^3}\ge\frac{27}{8(x+y+z)^2}$$

$$\sum_{cyc}\frac{x}{(y+z)^3}=\sum_{cyc}\frac{x^4}{(xy+xz)^3}\overset{RADON}{\geq}\frac{(x+y+z)^4}{8(xy+yz+zx)^3}\ge\frac{(x+y+z)^4}{8\left(\frac{(x+y+z)^2}{3}\right)^3}$$

$$\ge\frac{27}{8(x+y+z)^2}\quad\text{(proved)}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

WLOG, we may assume $x\ge y\ge z;\ \frac{x}{y+z}\ge\frac{y}{z+x}\Leftrightarrow zx+x^2\ge y^2+yz\to$ *true* $\because x\ge y$

Similarly, $\frac{y}{z+x}\ge\frac{z}{x+y}\ \therefore\ \frac{x}{y+z}\ge\frac{y}{z+x}\ge\frac{z}{x+y}$. *Also* $\frac{1}{(y+z)^2}\ge\frac{1}{(z+x)^2}\Leftrightarrow x\ge y\to$ *true*

Similarly, $\frac{1}{(z+x)^2}\ge\frac{1}{(x+y)^2}\ \therefore\ \frac{1}{(y+z)^2}\ge\frac{1}{(z+x)^2}\ge\frac{1}{(x+y)^2}$

$$\therefore LHS=\left(\frac{x}{y+z}\right)\cdot\frac{1}{(y+z)^2}+\left(\frac{y}{z+x}\right)\cdot\frac{1}{(z+x)^2}+\left(\frac{z}{x+y}\right)\cdot\frac{1}{(x+y)^2}$$

$$\overset{Chebyshev}{\geq}\frac{1}{3}\sum\frac{x}{y+z}\sum\frac{1}{(y+z)^2}\overset{Nesbitt}{\geq}\frac{1}{3}\cdot\frac{3}{2}\sum\frac{1}{(y+z)^2}$$

$$=\frac{1}{2}\sum\frac{1^3}{(y+z)^2}\overset{Radon}{\geq}\frac{1}{2}\cdot\frac{(1+1+1)^3}{(2\sum x)^2}=\frac{27}{8(x+y+z)^2}=RHS\ \text{(Proved)}$$

Solution 4 by Imad Zak-Saida-Lebanon

$x,y,z>0$. *Prove that* $\sum\frac{x}{(y+z)^3}\ge\frac{27}{8(x+y+z)^2}$. *Homogeneous, so let* $x+y+z=3$, *it*

becomes $\sum\frac{x}{(3-x)^3}\ge\frac{3}{8}??\quad 0<x<3$

We have: $\frac{x}{(3-x)^3}-\left(\frac{5x}{16}-\frac{3}{16}\right)=\frac{\overbrace{(x-1)^2}^{(+)}\overbrace{(5x^2-38x+81)}^{\Delta<0\Rightarrow(+)}}{16(3-x)^3}\ge 0$

$$\Rightarrow\sum\frac{x}{(3-x)^3}\ge\sum\left(\frac{5x}{16}-\frac{3}{16}\right)=\frac{5(x+y+z)}{16}-\frac{9}{16}=\frac{6}{16}=\frac{3}{8}$$

$\ll=\gg$ *at* $x=y=z$ *Q.E.D.*

Solution 5 by George Apostolopoulos-Messolonghi-Greece

Let $x+y+z=k>0$*. Consider the function* $f(t)=\frac{t}{(k-t)^3}, t>0$*. Then*

$f''(t)>0$*. So the function* f *is convex on* $(0,+\infty)$*. By Jensen's Inequality, we have*

$$f(x)+f(y)+f(z)\geq 3f\left(\frac{x+y+z}{3}\right) \text{ namely}$$

$$\frac{x}{(y+z)^3}+\frac{y}{(z+x)^3}+\frac{z}{(x+y)^3}\geq 3f\left(\frac{k}{3}\right)=3\cdot\frac{\frac{k}{3}}{\left(k-\frac{k}{3}\right)^3}=\frac{27}{8k^2}=\frac{27}{8(x+y+z)^2}$$

Equality holds when $x=y=z$*.*

SP.054. Let $a\in\left(0,\frac{\pi}{2}\right), b\in[1,\infty), m,n\in\mathbb{R}_+^*$ **and** $f,g,h,k:[-a,a]\to\mathbb{R}$ **be continuous functions such that:** $f(-x)=-f(x), g(-x)=-g(x)$,
$h(-x)=h(x), k(-x)=k(x)$**. Evaluate**

$$\int_{-a}^{a}\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n}x}dx.$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Let

$$I=\int_{-a}^{a}\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+(\sin^{2n}x)k(x)}dx$$

Let

$$F(x)=\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n}x}$$

$$-a\leq x\leq a$$

$$F(-x)=\frac{f(-x)+g(-x)}{(b-\cos(-x))^m h(-x)+k(-x)(\sin(-x))^{2n}}$$

$$=\frac{-(f(x)+g(x))}{(b-\cos x)^m h(x)+k(x)(\sin x)^{2n}}=-F(x)$$

i.e. F *is an odd function.*

Thus,

$$\int_{-a}^{a} F(x)\,dx = 0$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $a \in \left(0, \frac{\pi}{2}\right), b \in (1, \infty)$ ***and*** $m, n \in \mathbb{R}^{+}. f, g, h, k: [-a, a] \to \mathbb{R}$ ***be a continuous function such that***

$f(-x) = -f(x), g(-x) = -g(x), h(-x) = h(x)$ ***and*** $k(-x) = k(x)$***. Calculate***

$$\int_{-a}^{a} \frac{f(x) + g(x)}{(b - \cos x)^m h(x) + k(x) \sin^{2n} x} dx$$

Let

$$\Omega = \int_{-a}^{a} \frac{f(x) + g(x)}{(b - \cos x)^m h(x) + k(x) \sin^{2n} x} dx$$

$$= \int_{-a}^{a} \frac{f(a - a - x) + g(a - a - x)}{\{b - \cos(a - a - x)\}^m h(a - a - x) + k(a - a - x) \sin^{2n}(a - a - x)} dx$$

$$= \int_{-a}^{a} \frac{f(-x) + g(x)}{(b - \cos(-x))^m h(-x) + k(-x) \sin^{2n}(-x)} dx =$$

$$= -\int_{-a}^{a} \frac{f(x) + g(x)}{(b - \cos x)^m h(x) + k(x) \sin^{2n} x} dx$$

$$= -\Omega. \therefore \Omega = 0$$

Solution 3 by Shivam Sharma-New Delhi-India

As we know, the following lemma,

If $f(x)$ ***is a continuous function defined on*** $[-a, a]$***, then,***

$$\int_{-a}^{a} f(x) = \begin{cases} 2\int_{0}^{a} f(x)\,dx, if\ f(x) is\ an\ even\ function \\ 0, if\ f(x) is\ an\ odd\ function \end{cases}$$

Replace $x \to -x$***, we get,***

$$I=\int_{-a}^{a}\frac{f(-x)+g(-x)}{(b-\cos(-x))^m h(-x)+k(-x)\sin^{2n}(-x)}dx$$

Given $\Rightarrow f(-x)=-f(x), g(-x)=-g(x), h(-x)=h(x), k(-x)=k(x)$.

And as we know,

$\cos(-x)=\cos(x)$, $\sin^{2n}(-x)=\sin^{2n}(x)$ ***because as if we put*** $n=1,2,3,\dots$ ***gives an even function. Using given conditions and the above lemma, we get,***

$$I=-\int_{-a}^{a}\frac{f(x)+g(x)}{(b-\cos(x))^m h(x)+k(x)\sin^{2n}(x)}dx$$

$I=-I$; $2I=0$; ***Hence,*** $I=0$ ***(Q.E.D)***

Solution 4 by Yen Thung Chung-Taichung-Taiwan

$$\int_{-a}^{a}\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n}x}dx$$

$$=\int_{0}^{a}\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n}x}dx+\underbrace{\int_{-a}^{0}\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n}x}dx}_{let\ x=-t\Rightarrow dx=-dt}$$

$$=\int_{0}^{a}\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n}x}dx+\int_{a}^{0}\frac{f(-t)+g(-t)}{(b-\cos(-t))^m h(-t)+k(-t)\sin^{2n}(-t)}\cdot(-dt)$$

$$=\int_{0}^{a}\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n}x}dx+\int_{0}^{a}\frac{-f(t)-g(t)}{(b-\cos t)^m h(t)+k(t)\sin^{2n}t}dt$$

$$=\int_{0}^{a}\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n}x}dx-\int_{0}^{a}\frac{f(x)+g(x)}{(b-\cos x)^m h(x)+k(x)\sin^{2n}x}dx=0$$

SP.055. Let m_a, m_b, m_c be the lengths of medians of a triangle ABC with inradius r. Prove that

$$\frac{m_a+m_b+m_c}{\sin^2 A+\sin^2 B+\sin^2 C}\geq 4r.$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC: $\frac{m_a+m_b+m_c}{\sin^2 A+\sin^2 B+\sin^2 C} \geq 4r$

Desde que: $m_a \geq \frac{b^2+c^2}{4R}, m_b \geq \frac{c^2+a^2}{4R}, m_c \geq \frac{a^2+b^2}{4R}, R \geq 2R$

La desigualdad es equivalente:

$$\Rightarrow \frac{m_a + m_b + m_c}{\sin^2 A + \sin^2 B + \sin^2 C} \geq \frac{\frac{a^2 + b^2 + c^2}{2R}}{\frac{a^2 + b^2 + c^2}{4R^2}} = \frac{4R^2}{2R} \geq 4r \to 4R(R - 2r) \geq 0$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Using Tereshin's Inequality, $m_a \geq \frac{b^2+c^2}{4R}, m_b \geq \frac{c^2+a^2}{4R}, m_c \geq \frac{a^2+b^2}{4R}$

$\therefore \sum m_a \geq \frac{2\sum a^2}{4R}$ *Again,* $\sum \sin^2 A = \frac{\sum a^2}{4R^2}$

$\therefore \frac{\sum a}{\sum \sin^2 A} \geq \frac{2\sum a^2}{4R} \cdot \frac{4R^2}{\sum a^2} = 2R \geq 4r$ *(Euler) (Proved)*

Solution 3 by Soumitra Mandal-Kolkata-India

We know, $m_a = \frac{b^2+c^2}{4R}, m_b = \frac{a^2+c^2}{4R}$ *and* $m_c = \frac{a^2+b^2}{4R}$

$\sin A = \frac{a}{2R}, \sin B = \frac{b}{2R}$ *and* $\sin C = \frac{c}{2R}$*, hence*

$$\frac{m_a+m_b+m_c}{\sin^2 A+\sin^2 B+\sin^2 C} = \frac{\frac{1}{4R}\sum_{cyc}(a^2+b^2)}{\frac{a^2+b^2+c^2}{4R^2}} = 2R \geq 4r \text{ (proved)} \because R \geq 2r$$

SP.056. Let ABC be a triangle such that

$$\left(\frac{1}{\sin B} + \frac{1}{\sin C}\right)(-\sin A + \sin B + \sin C) = 2.$$

Prove that $\measuredangle A \leq \frac{\pi}{3}$.

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Ravi Prakash-New Delhi-India

$$(-\sin A + \sin B + \sin C)\left(\frac{1}{\sin B} + \frac{1}{\sin C}\right) = 2 \quad \textit{(1)}$$

$$\Rightarrow \frac{(\sin B+\sin C)^2}{\sin B \sin C} - \frac{\sin A(\sin B+\sin C)}{\sin B \sin C} = 2 \Rightarrow \sin A = \frac{\sin^2 B+\sin^2 C}{\sin B+\sin C} \quad \textit{(1)}$$

Also, from (2): $\sin A(\sin B+\sin C)=\sin^2 B+\sin^2 C-\sin^2 A+\sin^2 A$

$$=\sin^2 B+\sin(C-A)\sin(C+A)+\sin^2 A=\sin^2 B+\sin(C-A)\sin B+\sin^2 A$$

$$=\sin B[\sin(C+A)+\sin(C-A)]+\sin^2 A$$

$$\Rightarrow \sin A(-\sin A+\sin B+\sin C)=2\sin B\cos A\sin C$$

$$\Rightarrow \sin A\left(\frac{2\sin B\sin C}{\sin B+\sin C}\right)=2\sin B\sin C\cos A \quad \textit{[using (1)]}$$

$$\Rightarrow \frac{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}=\cos A\Rightarrow \cos A=\frac{\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{B-C}{2}\right)}$$

$$\Rightarrow \cos A\ge \sin\left(\frac{A}{2}\right)>0\Rightarrow \cos^2 A\ge \sin^2\left(\frac{A}{2}\right)=\frac{1-\cos A}{2}$$

$$\Rightarrow 2\cos^2 A+\cos A-1\ge 0\Rightarrow (2\cos A-1)(\cos A+1)\ge 0$$

$$\Rightarrow 2\cos A\ge 1\Rightarrow \cos A\ge \frac{1}{2}\Rightarrow A\le \frac{\pi}{3}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sin B+\sin C-\sin A=2\sin\frac{B+C}{2}\cos\frac{B-C}{2}-2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$=2\cos\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)=2\cos\frac{A}{2}\cdot 2\sin\frac{B}{2}\sin\frac{C}{2}\overset{(1)}{=}4\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\frac{1}{\sin B}+\frac{1}{\sin C}\overset{(2)}{=}\frac{2\cos\frac{A}{2}\cos\frac{B-C}{2}}{4\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}}$$

$$\textit{(1)}\times\textit{(2)}\Rightarrow LHS=\frac{2\cos^2\frac{A}{2}\cos\frac{B-C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}=2$$

$$\Rightarrow (1+\cos A)\cos\frac{B-C}{2}=\cos\frac{B+C}{2}+\cos\frac{B-C}{2}$$

$$\Rightarrow \cos\frac{B-C}{2}+\cos A\cos\frac{B-C}{2}=\sin\frac{A}{2}+\cos\frac{B-C}{2}\Rightarrow \cos A\cos\frac{B-C}{2}=\sin\frac{A}{2} \quad \textit{(3)}$$

$$\because 0<a<\pi, \therefore 0<\frac{A}{2}<\frac{\pi}{2}\Rightarrow \sin\frac{A}{2}>0 \therefore RHS\ \textit{of (3)}>0$$

$$\textit{Now, }-\frac{\pi}{2}<\frac{B-C}{2}<\frac{\pi}{2}\Rightarrow 0<\cos\frac{B-C}{2}\le 1 \quad \textit{(4)}$$

$$\because RHS\ \textit{of (3)}>0 \therefore LHS\ \textit{of (3) also}>0$$

$$\Rightarrow \cos A\cos\frac{B-C}{2}>0\Rightarrow \cos A>0\ \left(\because \cos\frac{B-C}{2}>0, \textit{from (4)}\right)\Rightarrow 0<A<\frac{\pi}{2} \quad \textit{(5)}$$

$$\textit{(4)} \Rightarrow \cos\frac{B-C}{2} \le 1 \Rightarrow \cos A\cos\frac{B-C}{2} \le \cos A$$

$$\left(\because \cos A, \cos\frac{B-C}{2} > 0\right)$$

$$\Rightarrow \sin\frac{A}{2} \le \cos A \textit{ (using (3))} = 1 - 2\sin^2\frac{A}{2}$$

$$\Rightarrow 2t^2 + t - 1 \le 0 \left(t = \sin\frac{A}{2}\right) \Rightarrow (t+1)(2t-1) \le 0$$

$$\Rightarrow 2t - 1 \le 0 \left(\because t + 1 = 1 + \sin\frac{A}{2} > 1 > 0\right) \Rightarrow t \le \frac{1}{2}$$

$$\Rightarrow \sin\frac{A}{2} \le \frac{1}{2} \Rightarrow \frac{A}{2} \le \frac{\pi}{6}\left(\because 0 < \frac{A}{2} < \frac{\pi}{2}\right) \Rightarrow A \le \frac{\pi}{3}, \textit{ which also satisfies (5)}$$

SP.057. If $a, b, c, d \in \mathbb{R}_+^*, a < b$ and $f: \mathbb{R} \to \mathbb{R}_+^*$ is a continuous function such that $f(a+b-x) = f(x), \forall x \in \mathbb{R}$, then evaluate

$$\int_a^b \frac{f(x-a)\big(c + df(b-x)\big)}{c\big(f(x-a) + f(b-x)\big) + 2df(x-a)f(b-x)}dx.$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\textit{Let } \Omega = \int_a^b \frac{f(x-a)\big(c+df(b-x)\big)}{c\big(f(x-a)+f(b-x)\big)+2df(x-a)f(b-x)}dx$$

$$= \int_a^b \frac{f(a+b-x-a)(c+df(b-a-b+x))}{c\big(f(a+b-x-a) + f(b-a-b+x)\big) + 2df(a+b-x-a)f(b-a-b+x)}dx$$

$$= \int_a^b \frac{f(b-x)\big(c+df(x-a)\big)}{c\big(f(x-a)+f(b-x)\big) + 2df(x-a)f(b-x)}dx$$

$$\therefore 2\Omega = \int_a^b \frac{f(x-a)\big(c+df(b-x)\big)}{c\big(f(x-a)+f(b-x)\big) + 2df(x-a)f(b-x)}dx$$

$$+ \int_a^b \frac{f(b-x)\big(c+df(x-a)\big)}{c\big(f(x-a)+f(b-x)\big) + 2df(x-a)f(b-x)}dx = \int_a^b dx = b - a \therefore \Omega = \frac{b-a}{2}$$

Solution 2 by Shivam Sharma-New Delhi-India

As we know the following lemma,

If $f(x)$ is a continuous function defined on $[a,b]$, then,

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Using the above lemma, we get,

$$I = \int_a^b \frac{f(a+b-x-a)\big(c+df(b-a-b+x)\big)}{c\big(f(a+b-x-a)+f(b-a-b+x)\big)+2df(a+b-x-a)f(b-a-b+x)}dx$$

$$\Rightarrow \int_a^b \frac{f(b-x)\big(c+df(x-a)\big)}{c\big(f(b-x)+f(x-a)\big)+2df(b-x)f(x-a)}dx$$

$$2I = \int_a^b \frac{c\big(f(b-x)+f(x-a)\big)+2df(b-x)f(x-a)}{c\big(f(b-x)+f(x-a)\big)+2df(b-x)f(x-a)}dx$$

$2I = \int_a^b (1)\,dx \Rightarrow [x]_a^b;\ 2I = b-a$. *Hence,* $I = \frac{b-a}{2}$ *(Q.E.D)*

SP.058. Find:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\tan\frac{1}{n+i}-\tan\frac{1}{n+i+1}\right)\left(\cos\frac{1}{n+1}+\cos\frac{1}{n+2}+\cdots+\cos\frac{1}{n+i}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Rovsen Pirguliyev-Sumgait-Azerbaijan

I. The aproximations for $n\to+\infty$, $\tan\frac{1}{n+k}\approx\frac{1}{n+k}$, $\cos\frac{1}{n+k}\approx 1$

$$S_n = \sum_{k=1}^{n}\left(\tan\frac{1}{n+k}-\tan\frac{1}{n+k+1}\right)\left(\cos\frac{1}{n+1}+\cos\frac{1}{n+2}+\cdots+\cos\frac{1}{n+k}\right)\approx$$

$$\approx\sum_{k=1}^{n}\left(\frac{1}{n+k}-\frac{1}{n+k+1}\right)\cdot k = \sum_{k=1}^{n}\frac{k}{n+k}-\sum_{k=2}^{n}\frac{k-1}{n+k} = \sum_{k=1}^{n}\frac{1}{n+k}-\frac{n}{2n+1} = E_n$$

Using the fact that the sequence $\sum_{k=1}^{n}\frac{1}{k}-\ln n$ *is convergent with the limit* γ *(the Euler-Mascheroni constant), results*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}=\lim_{n\to\infty}\left(\sum_{k=1}^{2n}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{k}\right)=\lim_{n\to\infty}[(\gamma+\ln 2n)-(\gamma+\ln n)]=\ln 2$$

$$\lim_{n\to\infty}S_n\approx\lim_{n\to+\infty}E_n=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{n+k}\right)-\lim_{n\to\infty}\frac{n}{2n+1}=\ln 2-\frac{1}{2}$$

II. The evaluation of the errors. For $0<x<\frac{\pi}{4}, x<\tan x<x+x^3$.

$$\frac{1}{n+k}-\frac{1}{n+k+1}-\frac{1}{(n+k+1)^3}<\tan\frac{1}{n+k}-\tan\frac{1}{n+k+1}<\frac{1}{n+k}-\frac{1}{n+k+1}+\frac{1}{(n+k)^3}.$$

$$k-\left(\cos\frac{1}{n+1}+\cos\frac{1}{n+2}+\dots+\cos\frac{1}{n+k}\right)=\sum_{i=1}^{n}\left(1-\cos\frac{1}{n+i}\right)=\sum_{i=1}^{n}2\sin^2\frac{1}{2(n+i)}$$

$$0<\sum_{i=1}^{n}2\sin^2\frac{1}{2(n+i)}<\sum_{i=1}^{n}\frac{1}{2(n+i)^2}<\frac{k}{2(n+1)^2}<\frac{1}{2n}\Rightarrow$$

$$\Rightarrow k-\frac{1}{2n}<\cos\frac{1}{n+1}+\cos\frac{1}{n+2}+\dots+\cos\frac{1}{n+k}<k$$

$$S_n>\sum_{i=1}^{n}\left(\frac{1}{n+k}-\frac{1}{n+k+1}-\frac{1}{(n+k+1)^3}\right)\left(k-\frac{1}{2n}\right)>$$

$$>E_n-\frac{n^2}{(n+2)^3}-\frac{1}{2n}\cdot\frac{n}{(n+1)(2n+1)}+\frac{1}{2}\cdot\frac{1}{(2n+1)^3}\Rightarrow\lim_{n\to\infty}S_n\geq\lim_{n\to\infty}E_n$$

$$S_n<\sum_{k=1}^{n}\left(\frac{1}{n+k}-\frac{1}{n+k+1}+\frac{1}{(n+k)^3}\right)\cdot k<E_n+\frac{n^2}{(n+1)^3}\Rightarrow\lim_{n\to\infty}S_n\leq\lim_{n\to\infty}E_n$$

Results: $\lim_{n\to\infty}S_n=\lim_{n\to\infty}E_n=\ln 2-\frac{1}{2}$

SP.059. Compute:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(1^p+2^p+\dots+k^p)^2}{n^{p+1}(1^p+2^p+\dots+n^p)};p\in\mathbb{N}$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

Let be $x_n=\frac{1}{n^{p+1}}(1^p+2^p+\dots+n^p)$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n^{p+1}} \sum_{k=1}^{n} k^p = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^p =$$

$$= \int_0^1 f(x)\,dx = \int_0^1 x^p\,dx = \frac{x^{p+1}}{p+1}\Big|_0^1 = \frac{1}{p+1}$$

$$a_{nk} = \frac{1^p + 2^p + \dots + k^p}{1^p + 2^p + \dots + n^p}; \lim_{n\to\infty} a_{nk} = 0$$

By Toeplitz's theorem:

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk} x_k = \lim_{n\to\infty} x_n = \frac{1}{p+1}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk} x_k = \lim_{n\to\infty} \frac{1^p + 2^p + \dots + k^p}{1^p + 2^p + \dots + n^p} \cdot \frac{1}{n^{p+1}} (1^p + 2^p + \dots + k^p) =$$

$$= \lim_{n\to\infty} \frac{(1^p + 2^p + \dots + k^p)^2}{n^{p+1}(1^p + 2^p + \dots + n^p)} = \frac{1}{p+1}$$

SP.060. Prove that if $a, b, c, d \in \mathbb{R}; a^2 + b^2 \neq 0; c^2 + d^2 \neq 0$ then:

$$\frac{(ad - bc)(3(a^2 + b^2)(c^2 + d^2) - 4(ad - bc)^2)}{\left((a^2 + b^2)(c^2 + d^2)\right)^{\frac{3}{2}}} \leq 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Let $z_1 = a + ib, z_2 = d + ic;\ z_1 z_2 = (ad - bc) + i(ac + bd)$

$$ad - bc = \frac{1}{2}(z_1 z_2 + \overline{z_1 z_2}); ac + bd = \frac{1}{2i}(z_1 z_2 - \overline{z_1 z_2})$$

Also, $|z_1|^2 = z_1 \overline{z_1} = a^2 + b^2;\ |z_2|^2 = z_2 \overline{z_2} = c^2 + d^2$*. Now,*

$$3(a^2 + b^2)(c^2 + d^2) - 4(ad - bc)^2 = 3 z_1 \overline{z_1} z_2 \overline{z_2} - \frac{4}{4}(z_1 z_2 + \overline{z_1 z_2})^2$$

$$= 3 z_1 \overline{z_1} z_2 \overline{z_2} - (z_1 z_2 + \overline{z_1 z_2})^2 = -\left[z_1^2 z_2^2 + \overline{z_1}^2 \overline{z_2}^2 + 2 z_1 z_2 \overline{z_1 z_2} - 3 z_1 z_2 \overline{z_1 z_2}\right]$$

$$= -\left(z_1^2 z_2^2 + \overline{z_1}^2 \overline{z_2}^2 - z_1 z_2 \overline{z_1 z_2}\right)$$

Num $= -\frac{1}{2}(z_1 z_2 + \overline{z_1 z_2})\left(z_1^2 z_2^2 + \overline{z_1}^2 \overline{z_2}^2 - z_1 z_2 \overline{z_1 z_2}\right) = -\frac{1}{2}[(z_1 z_2)^2 - (\overline{z_1 z_2})^3]$

$$|Num| \leq \frac{1}{2}\left|(z_1z_2)^3 - (\overline{z_1z_2})^3\right|$$

$$\leq \frac{1}{2}[|z_1z_2|^3 + |\overline{z_1z_2}|^3] = |z_1z_2|^3 = \left((a^2+b^2)(c^2+d^2)\right)^{\frac{3}{2}} = \textbf{\textit{DEN}} \Rightarrow \frac{|NUM|}{DEN} \leq 1$$

$$\therefore \frac{NUM}{DEN} \leq \frac{|NUM|}{|DEN|} \leq 1$$

Solution 2 by Ravi Prakash-New Delhi-India

Let $z_1 = a + ib = r_1(\cos\theta + i\sin\theta)$ ***and*** $z_2 = d + ic = r_2(\cos\theta + i\sin\theta)$

$$z_1z_2 = (ad - bc) + i(ac + bd) = r_1r_2[\cos(\theta+\phi) + i\sin(\theta+\phi)]$$

Also, $|z_1| = \sqrt{a^2+b^2} = r_1, |z_2| = \sqrt{d^2+c^2} = r_2$***. Now,***

$$\frac{(ad-bc)[3(a^2+b^2)(c^2+d^2) - 4(ad-bc)^2]}{(a^2+b^2)^{\frac{3}{2}}(c^2+d^2)^{\frac{3}{2}}}$$

$$= \frac{r_1r_2\cos(\theta+\phi)\left[3r_1^2r_2^2 - 4r_1^2r_2^2\cos^2(\theta+\phi)\right]}{r_1^3r_2^2}$$

$$= 3\cos(\theta+\phi) - 4\cos^3(\theta+\phi) = -\cos(3\theta+3\phi) \leq 1$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

Let $(a^2+b^2)(c^2+d^2) = x^2$, $\therefore$ ***we need to prove*** $\frac{(ad-bc)(3(a^2+b^2)(c^2+d^2)-4(ad-bc)^2)}{\left((a^2+b^2)(c^2+d^2)\right)^{\frac{3}{2}}} \leq 1$

$$\Leftrightarrow \frac{(ad-bc)(3x^2-4(ad-bc)^2)}{x^3} \leq 1 \Leftrightarrow x^3 - 3x^2(ad-bc) + 4(ad-bc)^3 \geq 0$$

$$\Leftrightarrow x^3 + (ad-bc)^3 - 3(ad-bc)\{x^2 - (ad-bc)^2\} \geq 0$$

$$\Leftrightarrow (x + ad - bc)(x^2 - x(ad-bc) + (ad-bc)^2) - 3(ad-bc)(x^2 - (ad-bc)^2) \geq 0$$

$$\Leftrightarrow (x + ad - bc)(x^2 - 4x(ad-bc) + 4(ad-bc)^2) \geq 0$$

$\Leftrightarrow (x + ad - bc)\left(x - 2(ad-bc)\right)^2 \geq 0$***, which is true and*** $x \geq bc - ad$

since, $(ac+bd)^2 \geq 0$ ***for all*** $a, b, c, d \in \mathbb{R}$

$\therefore \frac{(ad-bc)(3(a^2+b^2)(c^2+d^2)-4(ad-bc)^2)}{\left((a^2+b^2)(b^2+d^2)\right)^{\frac{3}{2}}} \leq 1$ ***(proved)***

Solution 4 by Soumava Chakraborty-Kolkata-India

$$(ac+bd)^2 + (ad-bc)^2 \overset{(a)}{=} (a^2+b^2)(c^2+d^2)$$

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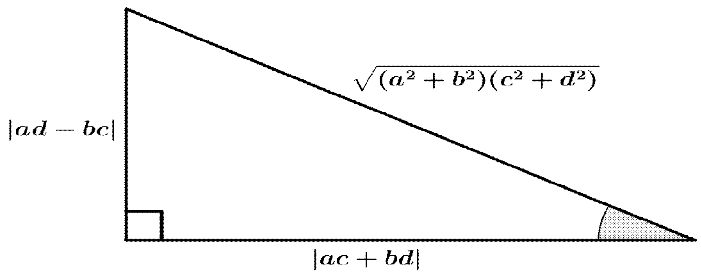
$$\therefore 3(a^2+b^2)(c^2+d^2)-4(ad-bc)^2$$

$$=3\{(ac+bd)^2+(ad-bc)^2\}-4(ad-bc)^2$$

$$\overset{(1)}{=}3(ac+bd)^2-(ad-bc)^2=3|ac+bd|^2-|ad-bc|^2$$

Case 1: $ac+bd\neq 0, ad-bc\neq 0$



$\therefore |ac+bd|=p\cos\theta$ ***(2)***

$|ad-bc|=p\sin\theta$ ***(3), where,*** $p=\sqrt{(a^2+b^2)(c^2+d^2)}$

$\therefore LHS=\frac{(ad-bc)(3p^2\cos^2\theta-p^2\sin^2\theta)}{p^3}$ ***(using (1), (2), (3))***

$$\overset{(4)}{=}\frac{(ad-bc)(3\cos^2\theta-\sin^2\theta)}{p}$$

Now, according as $ad-bc\geq 0$ ***or*** $ad-bc<0$

$ad-bc=\pm|ad-bc|$ ***(6) Again*** $\frac{|ad-bc|}{p}=\sin\theta$

$\therefore LHS=\pm\sin\theta(3\cos^2\theta-\sin^2\theta)$ ***(using (4), (5), (6))***

$$=\pm\sin\theta(3(1-\sin^2\theta)-\sin^2\theta)=\pm(3\sin\theta-4\sin^3\theta)=\pm\sin 3\theta$$

When $LHS=\sin 3\theta$***, then*** $LHS\leq 1, \because \sin 3\theta\leq 1$

$LHS=-\sin 3\theta$***, then, also,*** $LHS\leq 1, \because \sin 3\theta\geq -1$

$\therefore LHS=\pm\sin 3\theta\leq 1$ ***(proved under case (1))***

Case 2: $ad-bc=0$ ***(a)*** $\Rightarrow ac+bd\neq 0$ ***Then,*** $LHS=0\leq 1$

Case 3: $ac+bd=0$ ***(a)*** $\Rightarrow ad-bc\neq 0\ (\because (a^2+b^2)(c^2+d^2)\neq 0)$

$\therefore LHS=\frac{-(ad-bc)}{|ad-bc|}=\pm 1\leq 1$***. Hence, in all*** 3 ***cases,*** $LHS\leq 1$***. (Done)***

UP.046. Let a, b, c be positive real numbers such that $a + b + c = 1$.

Prove that

$$a^{a^2} b^{b^2} c^{c^2} \geq (a^2 + b^2 + c^2)^{a^2+b^2+c^2}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by SK Rejuan-West Bengal-India

Given $a, b, c \in \mathbb{R}^+$ such that $\sum a = 1$.

Let us take a, b, c with the associated weight a^2, b^2, c^2 respectively, hence by applying GM $\geq$ HM we get,

$$\left(a^{a^2} \cdot b^{b^2} \cdot c^{c^2}\right)^{\frac{1}{a^2+b^2+c^2}} \geq \frac{a^2 + b^2 + c^2}{\frac{a^2}{a} + \frac{b^2}{b} + \frac{c^2}{c}} \quad [\because a, b, c \neq 0]$$

$$\Rightarrow \left(a^{a^2} \cdot b^{b^2} \cdot c^{c^2}\right) \geq \left(\frac{a^2 + b^2 + c^2}{a + b + c}\right)^{a^2+b^2+c^2}$$

$$\Rightarrow a^{a^2} \cdot b^{b^2} \cdot c^{c^2} \geq (a^2 + b^2 + c^2)^{a^2+b^2+c^2}$$ *[as $\sum a = 1$] [Proved]*

Solution 2 by Shivam Sharma-New Delhi-India

As we know, G.M $\geq$ H.M. Using this, we get, $\left(a^{a^2} b^{b^2} c^{c^2}\right)^{\frac{1}{(a^2+b^2+c^2)}} \geq \frac{a^2+b^2+c^2}{\frac{a^2}{a}+\frac{b^2}{b}+\frac{c^2}{c}}$

$\left(a^{a^2} b^{b^2} c^{c^2}\right)^{\frac{1}{(a^2+b^2+c^2)}} \geq \frac{a^2+b^2+c^2}{a+b+c}$*. It is given that,* $a + b + c = 1$

Putting this, we get, $\left(a^{a^2} b^{b^2} c^{c^2}\right)^{\frac{1}{(a^2+b^2+c^2)}} \geq a^2 + b^2 + c^2$

Hence, $a^{a^2} b^{b^2} c^{c^2} \geq (a^2 + b^2 + c^2)^{(a^2+b^2+c^2)}$ *(Q.E.D.)*

UP.047. Let a, b, c be distinct rational numbers such that

$$\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b} = 0.$$

Prove that

$$\sqrt{\frac{(b-c)^4}{a^2} + \frac{(c-a)^4}{b^2} + \frac{(a-b)^4}{c^2}}$$

is a rational number.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Anas Adlany-Zemamra-Morroco

First of all we observe that if $xyz \neq 0$ and $x^2 + y^2 + z^2 = (x + y + z)^2$ then $\sum x^{-1} = 0$. Suppose that $(x + y + z)^2 = \sum x^2 \Leftrightarrow 2\sum xy = 0 \Leftrightarrow \sum x^{-1} = 0$.

Also, form the condition we have $\sum \frac{a}{b-c} = 0 \Rightarrow \frac{c}{(a-b)^2} = \frac{a+b+c}{(b-c)(c-a)}$, etc.

Now set $x = \frac{a+b+c}{(b-c)(c-a)}$, etc.

and this substitution verify the condition of the above lemma $\sum xy = 0$, therefore the number $\sqrt{\sum \left(\frac{(a-b)^4}{c^2}\right)^2}$ is a rational number, and we are done.

Solution 2 by Soumava Chakraborty-Kolkata-India

a, b, c are distinct rational numbers such that $\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b} = 0$

p.t $\underbrace{\sqrt{\frac{(b-c)^4}{a^2} + \frac{(c-a)^4}{b^2} + \frac{(a-b)^4}{c^2}}}_{e}$ is a rational no. $\because \frac{a}{b-c} + \frac{b}{c-a} = \frac{-c}{a-b}$

$$\therefore \frac{-c}{a-b} = \frac{ac - a^2 + b^2 - bc}{(b-c)(c-a)} = \frac{c(a-b) - (a+b)(a-b)}{(b-c)(c-a)}$$

$$\Rightarrow \frac{c}{a-b} = \frac{(a-b)(a+b-c)}{(b-c)(c-a)} \Rightarrow \frac{(a-b)^2}{c} = \frac{(b-c)(c-a)}{a+b-c}$$

(If $a + b - c = 0$, then $c = 0$. But for e to be defined $c \neq 0 \therefore a + b - c \neq 0$)

$$\therefore \frac{(a-b)^4}{c^2} = \frac{(b-c)^2(c-a)^2}{(a+b-c)^2} \quad (1)$$

Similarly, $\frac{a}{b-c} = \frac{(b-c)(b+c-a)}{(c-a)(a-b)}$

If $b + c - a = 0$, then $a = 0$. But for e to be defined, $a \neq 0 \therefore b + c - a \neq 0$

$$\therefore \frac{(b-c)^4}{a^2} = \frac{(c-a)^2(a-b)^2}{(b+c-a)^2} \quad (2)$$

Also, similarly, $\frac{b}{c-a} = \frac{(c-a)(c+a-b)}{(a-b)(b-c)}$

If $c + a - b = 0$, then $b = 0$. But for e to be defined, $b \neq 0$.

$$\therefore c + a - b \neq 0 \therefore \frac{(c-a)^4}{b^2} = \frac{(a-b)^2(b-c)^2}{(c+a-b)^2} \quad (3)$$

$$(1) + (2) + (3) \Rightarrow e^2 = \frac{(c-a)^2(a-b)^2}{(b+c-a)^2} + \frac{(a-b)^2(b-c)^2}{(c+a-b)^2} + \frac{(b-c)^2(c-a)^2}{(a+b-c)^2} = x^2 + y^2 + z^2$$

where $x = \frac{(c-a)(a-b)}{b+c-a}, y = \frac{(a-b)(b-c)}{(c+a-b)}, z = \frac{(b-c)(c-a)}{a+b-c}$

Now, $\sum xy = \{\prod(a-b)\}\left[\frac{a-b}{(b+c-a)(c+a-b)} + \frac{b-c}{(c+a-b)(a+b-c)} + \frac{c-a}{(a+b-c)(b+c-a)}\right]$

$$\Rightarrow \sum xy = \left\{\prod(a-b)\left[\frac{(a-b)(a+b-c)+(b-c)(b+c-a)+(c-a)(c+a-b)}{(a+b-c)(b+c-a)(c+a-b)}\right]\right\}$$

$$= \left\{\prod(a-b)\right\}\left[\frac{a^2-b^2-ca+bc+b^2-c^2-ab+ca+c^2-a^2-bc+ab}{(a+b-c)(b+c-a)(c+a-b)}\right] = 0$$

$$\therefore e^2 = \sum x^2 = \sum x^2 + 0 = \sum x^2 + 2\sum xy\left(\because \sum xy = 0\right)$$

$$= (x+y+z)^2 \Rightarrow e = |x+y+z|$$

$$= \left|\frac{(c-a)(a-b)}{(b+c-a)} + \frac{(a-b)(b-c)}{(c+a-b)} + \frac{(b-c)(c-a)}{a+b-c}\right|,$$

which is obviously a rational number as a, b, c are rational numbers (Proved).

UP.048. Let a, b, c be non-negative real numbers such that $a+b+c=1$.

Prove that: $a^4+b^4+c^4+26abc \le 1$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo: a, b, c ***números reales no negativos, de tal manera que:*** $a+b+c=1$

Probar que: $a^4+b^4+c^4+26abc \le 1$. ***Siendo:*** $a, b, c \ge 0$, ***se cumple:***

$$a^2+b^2+c^2 \ge ab+bc+ca \wedge (ab+bc+ca)^2 \ge 3abc(a+b+c) \ldots (A)$$

Es suficiente demostrar lo siguiente: $(a+b+c)^4 \ge a^4+b^4+c^4+26abc$

$$\left(a^2+b^2+c^2+2(ab+bc+ca)\right)^2 \ge a^4+b^4+c^4+26abc$$

$$(a^2+b^2+c^2)^2+4(ab+bc+ca)^2+4(a^2+b^2+c^2)(ab+bc+ca) \ge$$
$$\ge a^4+b^4+c^4+26abc$$

$$a^4+b^4+c^4+2(ab+bc+ca)^2-4abc(a+b+c)+4(ab+bc+ca)^2$$
$$+4(a^2+b^2+c^2)(ab+bc+ca) \ge a^4+b^4+c^4+26abc$$

$$6(ab+bc+ca)^2-4abc+4(a^2+b^2+c^2)(ab+bc+ca) \ge 26abc$$

Ahora bien de (A):

$$6(ab+bc+ca)^2+4(a^2+b^2+c^2)(ab+bc+ca)-4abc\geq$$

$$\geq 10(ab+bc+ca)^2-4abc\geq 26abc$$

Por lo tanto: $a^4+b^4+c^4+26abc\leq 1$ ***… (LQQD)***

Solution 2 by Imad Zak-Saida-Lebanon

$\sum a^4+26abc\leq 1$***. Let*** $\sum ab=\frac{p^2-q^2}{3}=\frac{1-q^2}{3};0\leq q\leq 1;p=1$

By VQBC inequality $r\leq\frac{1}{27}(p-q)^2(p+2q)=\frac{1}{27}(1-q)^2(1+2q)$

$$\sum a^4=\frac{-p^4+8p^2q^2+2q^4}{9}+4pr=\frac{-1+8q^2+2q^4}{9}+4r\Rightarrow$$

$$\sum a^4+26abc-1\leq\frac{-1+8q^2+2q^4}{9}+\frac{30}{27}(1-q)^2(1+2q)-1$$

$$=-\frac{2}{9}\underset{(+)}{(q+11)}\underset{(+)}{q^2}\underset{(+)}{(1-q)}\leq 0 \text{ Q.E.D.}$$

$\ll=\gg$ ***when:*** $*\,q=0\Rightarrow a=b=c=\frac{1}{3}$ ***or**** $q=1\Rightarrow\sum ab=0\ \&\ r=0\Rightarrow(a,b,c)=$

$(1,0,0)$ ***or permutations***

Solution 3 by Soumitra Mandal-Chandar Nagore-India

Let $a+b+c=p, ab+bc+ca=q$ ***and*** $r=abc$

$$\therefore\sum_{cyc}a^4=(p^2-2q)^2-2(q^2-2pr)=(1-2q)^2-2(q^2-2r),$$

since $p=1$

$=1-4q+2q^2+4r$ ***and*** $q\geq 9r\Rightarrow 30r\leq\frac{10q}{3}$

we need to prove, $1-4q+2q^2+4r+26r\leq 1$

$\Leftrightarrow 1-4q+2q^2+\frac{10q}{3}\leq 1\Leftrightarrow 2q^2-\frac{2q}{3}\leq 0\Leftrightarrow 2q\left(q-\frac{1}{3}\right)\leq 0$***, which is true:***

$\because\frac{1}{3}\geq q\therefore a^4+b^4+c^4+26abc\leq 1$ ***(proved) equality at*** $a=b=c=\frac{1}{3}$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$a^4+b^4+c^4+26abc\leq 1\Leftrightarrow a^4+b^4+c^4+26abc\leq(a+b+c)^4$$

$$\Leftrightarrow 2a^3b+2ab^3+2b^3c+2bc^3+2c^3a+2ca^3+3a^2b^2+3b^2c^2+3c^2a^2+$$

$$+6a^2bc+6b^2ca+6c^2ab\geq 13abc$$

$$\text{Now, } 3\sum a^2b^2 \underset{(a)}{\geq} 3abc(a+b+c)(\because \sum x^2 \geq \sum xy)$$

(true for $\forall\, a, b, c$)

$$= 3abc \quad \left(\because \sum a = 1\right)$$

$$\text{Also } 6a^2bc + 6b^2ca + 6c^2ab = 6abc\left(\sum a\right) \underset{(b)}{=} 6abc$$

(a), (b) ⇒ it remains to prove:

$$a^3b + ab^3 + b^3c + bc^3 + c^3a + ca^3 \geq 2abc \quad \textit{(1)}$$

$$\text{Now, } a^3b + ab^3 = ab(a^2+b^2) \geq ab \cdot 2ab\ (\because a^2+b^2 \geq 2ab \text{ and } a, b \geq 0)$$

$$\Rightarrow a^3b + ab^3 \underset{(c)}{\geq} 2a^2b^2. \text{ Similarly, } b^3c + bc^3 \underset{(d)}{\geq} 2b^2c^2 \text{ and } c^3a + ca^3 \underset{(e)}{\geq} 2c^2a^2$$

$$(c) + (d) + (e) \Rightarrow \sum(a^3b + ab^3) \geq 2\sum a^2b^2 \geq 2abc(a+b+c) = 2abc$$

⇒ (1) is true (Proved)

UP.049. Prove that the following inequality holds for any triangle ABC,

$$a^2(5m_a - m_b - m_c) + b^2(5m_b - m_c - m_a) + c^2(5m_c - m_a - m_b) \leq 12m_am_bm_c$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar la siguiente desigualdad en un triángulo ABC:

$$a^2(5m_a - m_b - m_c) + b^2(5m_b - m_c - m_a) + c^2(5m_c - m_a - m_b) \leq 12m_am_bm_c$$

Siendo ABC un triángulo ∧ si $x, y, z \in R$, se cumple la siguiente desigualdad:

$$x^2 + y^2 + z^2 \geq 2xy\cos A + 2yz\cos B + 2zx\cos C. \text{ En otras palabras:}$$

$$x^2 + y^2 + z^2 \geq xy\frac{b^2+c^2-a^2}{bc} + yz\frac{a^2+c^2-b^2}{ca} + zx\frac{a^2+b^2-c^2}{ab} \quad \textit{(A)}$$

Aplicando para un triángulo de longitudes m_a, m_b, m_c, tenemos:

$$x^2 + y^2 + z^2 \geq xy\frac{(m_b)^2 + (m_c)^2 - (m_a)^2}{m_bm_c} + \frac{(m_c)^2 + (m_a)^2 - (m_b)^2}{m_cm_a}yz +$$

$$+\frac{(m_a)^2 + (m_b)^2 - (m_c)^2}{m_am_b}zx$$

$$4(x^2+y^2+z^2)m_am_bm_c \geq$$

$$\geq xy(5a^2 - b^2 - c^2)m_a + yz(5b^2 - c^2 - a^2)m_b + zx(5c^2 - a^2 - b^2)m_c$$

Un caso particular, cuando $x = y = z = 1$

$$12m_am_bm_c \geq (5a^2 - b^2 - c^2)m_a + (5b^2 - c^2 - a^2)m_b + (5c^2 - a^2 - b^2)m_c$$

$$12m_am_bm_c \geq 5a^2m_a - b^2m_a - c^2m_a + 5b^2m_b - c^2m_b - a^2m_b - 5c^2m_c - a^2m_c - b^2m_c$$

$$a^2(5m_a - m_b - m_c) + b^2(5m_b - m_c - m_a) + c^2(5m_c - m_a - m_b) \leq 12m_am_bm_c$$

UP.050. Let a, b, c be positive real numbers such that $a^2b + b^2c + c^2a = 3$.

Prove that

$$\frac{1}{a(a+b)^2} + \frac{1}{b(b+c)^2} + \frac{1}{c(c+a)^2} \geq \frac{3}{4}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo: a, b, c números R^+, de tal manera que: $a^2b + b^2c + c^2a = 3$.

Probar que: $\frac{1}{a(a+b)^2} + \frac{1}{b(b+c)^2} + \frac{1}{c(c+a)^2} \geq \frac{3}{4}$

Si: $a, b, c > 0$, ***se cumple la siguiente desigualdad:*** $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$ ***... (Desigualdad de Nesbitt). La desigualdad propuesta es equivalente:*** $\frac{\left(\frac{c}{a+b}\right)^2}{c^2a} + \frac{\left(\frac{a}{b+c}\right)^2}{a^2b} + \frac{\left(\frac{b}{c+a}\right)^2}{b^2c} \geq \frac{3}{4}$

Aplicando la desigualdad de Cauchy:

$$\frac{\left(\frac{c}{a+b}\right)^2}{c^2a} + \frac{\left(\frac{a}{b+c}\right)^2}{a^2b} + \frac{\left(\frac{b}{c+a}\right)^2}{b^2c} \geq \frac{\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right)^2}{a^2b + b^2c + c^2a} \geq \frac{\frac{9}{4}}{3} = \frac{3}{4}$$ ***... (LQQD)***

UP.051. Let be $a \in [0, \infty)$; $f: (0, \infty) \to (0, \infty)$

$$f(x) = \left(\Gamma(x+1)\right)^{\frac{1}{x}}.$$ **Find:**

$$\Omega = \lim_{x\to\infty}\left(\left(f(x+1)\right)^a - \left(f(x)\right)^a\right) \cdot x^{1-a}$$

Proposed by D.M. Bătinețu-Giurgiu; Neculai Stanciu – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\Gamma(n+1) = n!, \lim_{n\to\infty} \frac{\sqrt[n]{\Gamma(n+1)}}{n} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} =$$

$$\overset{\substack{CAUCHY\\D'ALEMBERT}}{\Longleftrightarrow}\lim_{n\to\infty}\left(\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right)=\lim_{n\to\infty}\left(\frac{n}{n+1}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}\right)=\frac{1}{e}$$

Let $u_n=\left(\frac{\sqrt[n+1]{\Gamma(n+2)}}{\sqrt[n]{\Gamma(n+1)}}\right)^a$ *where* $n\in\mathbb{N}$, $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{\frac{\sqrt[n+1]{(n+1)!}}{n+1}}{\frac{\sqrt[n]{n!}}{n}}\cdot\left(1+\frac{1}{n}\right)\right)^a=1$

$$u_n\to 1\ \textit{then}\ \frac{u_n-1}{\ln u_n}\to 1\ \textit{as}\ n\to\infty$$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\left(\frac{(n+1)!}{n!}\cdot\frac{1}{n+1}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^a=e^a$$

$$\therefore\lim_{x\to\infty}\left(\left(f(x+1)\right)^a-\left(f(x)\right)^a\right)\cdot x^{1-a}=\lim_{\substack{n\to\infty\\n\to\mathbb{N}^*}}\left(\left(f(n+1)\right)^a-\left(f(n)\right)^a\right)n^{1-a}$$

$$=\lim_{n\to\infty}\left(\left(\frac{\sqrt[n]{n!}}{n}\right)^a\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right)=\frac{a}{e^a}$$

Solution 2 by Shafiqur Rahman-Bangladesh

$$\lim_{x\to\infty}\left(\left(f(x+1)\right)^a-\left(f(x)\right)^a\right)\cdot x^{1-a}=\lim_{x\to\infty}\left((x+1)^a\left(\frac{f(x+1)}{x+1}\right)^a-x^a\left(\frac{f(x)}{x}\right)^a\right)\cdot x^{1-a}=$$

$$=a\cdot\lim_{x\to\infty}\left(\frac{\left(\frac{f(x+1)}{x+1}\right)}{\left(\frac{f(x)}{x}\right)^x}\right)^a=a\cdot\lim_{x\to\infty}\left(\frac{\Gamma(x+2)}{(x+1)\Gamma(x+1)}\cdot\frac{1}{\left(1+\frac{1}{x}\right)^x}\right)^a$$

$$\therefore\lim_{x\to\infty}\left(\left(f(x+1)\right)^a-\left(f(x)\right)^a\right)\cdot x^{1-a}=\frac{a}{e^a}$$

UP.052. Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{a^6}{a^2+b}+\frac{b^2}{b^2+c}+\frac{c^6}{c^2+a}\ge\frac{3}{2}.$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by SK Rejuan-West Bengal-India

Given, $a,b,c\in\mathbb{R}^+$ ***and*** $a+b+c=3$

$LHS=\sum\frac{a^6}{a^2+b}=P$ ***(say)*** $\Rightarrow P=\sum\left(a^4-\frac{a^4b}{a^2+b}\right)=\sum a^4-\sum\frac{a^4b}{a^2+b}$ ***(1)***

By A.M. $\geq$ G.M. we get,

$$\left(\frac{a^2+b}{2}\right)^2 \geq a^2b \Rightarrow \frac{a^2+b}{4} \geq \frac{a^2b}{a^2+b} \Rightarrow -\frac{a^2b}{a^2+b} \geq -\frac{a^2+b}{4}$$

$$\Rightarrow -\frac{a^4b}{a^2+b} \geq -\frac{a^2}{4}(a^2+b) \Rightarrow \sum a^4 - \sum \frac{a^4b}{a^2+b} \geq \sum a^4 - \sum \frac{a^2}{4}(a^2+b)$$

$$\Rightarrow P \geq \frac{3}{4}\sum a^4 - \frac{1}{4}\sum a^2b \quad \textit{(2)}$$

Again by A.M. $\geq$ G.M. we get, $\frac{(a^2+b)^2}{4} \geq a^2b \Rightarrow \frac{2(a^4+b^2)}{4} \geq \frac{1}{4}(a^2+b^2)^2 \geq a^2b$

[by Cauchy inequality]

$$\Rightarrow -\frac{1}{4}\sum a^2b \geq -\frac{1}{8}\sum(a^4+b^2)$$

$$\Rightarrow \frac{3}{4}\sum a^4 - \frac{1}{4}\sum a^2b \geq \frac{3}{4}\sum a^4 - \frac{1}{8}\sum a^4 - \frac{1}{8}\sum b^2$$

$$\Rightarrow P \geq \frac{5}{8}\sum a^4 - \frac{1}{8}\sum b^2 \quad \textit{(3) [from (2)]}$$

$$\sum a^4 \geq 3\left(\frac{\sum a^2}{3}\right)^2 \Rightarrow \frac{5}{8}\sum a^4 \geq \frac{5}{24}\left(\sum a^2\right)^2$$

$$\Rightarrow \frac{5}{8}\sum a^4 - \frac{1}{8}\sum a^2 \geq \frac{5}{24}\left(\sum a^2\right)^2 - \frac{1}{8}\sum a^2 \Rightarrow P \geq \frac{1}{8}\sum a^2\left(\frac{5}{3}\sum a^2 - 1\right) \quad \textit{(4) [from (3)]}$$

$$\sum a^2 \geq 3\left(\frac{\sum a}{3}\right)^2 = 3 \ \textit{[as } \sum a = 3\textit{]} \Rightarrow \sum a^2 \geq 3 \quad \textit{(5)}$$

$$\therefore \left(\frac{5}{3}\sum a^2\right) - 1 \geq \frac{5}{3}\cdot 3 - 1 = 4 \ [\because a^2 \geq 3]$$

$$\Rightarrow \frac{1}{8}\sum a^2\left(\frac{5}{3}\sum a^2 - 1\right) \geq \frac{1}{8}\left(\sum a^2\right)\cdot 4 = \frac{1}{2}\sum a^2$$

$$\Rightarrow P \geq \frac{1}{2}\sum a^2 = \frac{3}{2} \ \textit{[from (5) we get } \sum a^2 \geq 3\textit{]} \Rightarrow P \geq \frac{3}{2}. \textit{ i.e. } \sum \frac{a^6}{a^2+b} \geq \frac{3}{2} \ \textit{[proved]}$$

Solution 2 by Kevin Soto Palacios – Huarmey – Peru

Siendo: a, b, c ***números*** R^+***: de ta manera que:*** $a+b+c=3$***. Probar que:***

$\frac{a^6}{a^2+b} + \frac{b^6}{b^2+c} + \frac{c^6}{c^2+a} \geq \frac{3}{2}$***. Cómo:*** $a, b, c > 0$***. Por la desigualdad de Cauchy:***

$$a^2+b^2+c^2 \geq \frac{1}{3}(a+b+c)^2 \geq 3$$

$$(a^3+b^3+c^3)(a+b+c) \geq (a^2+b^2+c^2)^2 \Leftrightarrow$$

$$\Leftrightarrow a^3+b^3+c^3 \geq \frac{(a^2+b^2+c^2)^2}{3} \geq a^2+b^2+c^2.$$ ***Por: MA ≥ MG***

$$\left(\frac{a^6}{a^2+b}+\frac{a^2+b}{4}\right)+\left(\frac{b^6}{b^2+c}+\frac{b^2+c}{4}\right)+\left(\frac{c^6}{c^2+a}+\frac{c^2+a}{4}\right) \geq a^3+b^3+c^3 \ldots (A)$$

Además, se demostro lo siguiente: $a^3+b^3+c^3 \geq a^2+b^2+c^2$***, a lo que es***

equivalente: $\frac{a^3}{4}+\frac{b^3}{4}+\frac{c^3}{4} \geq \frac{a^2}{4}+\frac{b^2}{4}+\frac{c^2}{4} \ldots (B)$

Luego, por la desigualdad de Holder: $\frac{3a^3}{4}+\frac{3b^3}{4}+\frac{3c^3}{4} \geq \frac{3}{4}\cdot\frac{(a+b+c)^3}{9}=\frac{9}{4} \ldots (C)$

Sumando: (A) + (B) + (C): $\frac{a^6}{a^2+b}+\frac{b^6}{b^2+c}+\frac{c^6}{c^2+a} \geq \frac{9-(a+b+c)}{4}=\frac{3}{2} \ldots (LQQD)$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{a^6}{a^2+b}+\frac{b^6}{b^2+c}+\frac{c^6}{c^2+a} \overset{Bergstrom}{\geq} \frac{(a^3+b^3+c^3)^2}{\sum a^2+\sum a}$$

$\overset{Chebyshev}{\geq} \frac{\left(\frac{1}{3}\sum a\sum a^2\right)^2}{\sum a^2+3}=\frac{t^2}{t+3}$***, where*** $t=\sum a^2 \therefore$ ***it suffices to prove:*** $\frac{t^2}{t+3} \geq \frac{3}{2}$

$$\Leftrightarrow 2t^2-3t-9 \geq 0 \Leftrightarrow (2t+3)(t-3) \geq 0 \Leftrightarrow t \geq 3 \quad \textbf{(1)}$$

$$(\because 2t+3>3>0)$$

Now, $t=\sum a^2 \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum a\right)^2=\frac{9}{3}=3 \Rightarrow t \geq 3 \Rightarrow$ ***(1) is true (Proved)***

Solution 4 by Sanong Hauyrerai – Nakhonpathom – Thailand

$\frac{a^6}{(a^2+b)}+\cdots \geq (a^2+\cdots)^{\frac{3}{3}}(a^2+\cdots+3)=\frac{x^3}{(3x+9)}$***, where*** $x=a^2+\cdots \geq 3$ ***is to be true***

Because $2x^3-9x-27=(x-3)(2x^2+6x+9) \geq 0, x \geq 3$

Therefore the expression is true.

UP.053. If $x,y,z \in \mathbb{C}^*; A,B,C \in M_n(\mathbb{C}); n \geq 2$ **are such that** $x^2A+B=xAB;$

$y^2B+C=yBC; z^2C+A=zCA$ **then:**

$$\left((y^2+1)x+\frac{x^2+1}{x}\right)A+\left((z^2+1)y+\frac{y^2+1}{x}\right)B+\left((x^2+1)z+\frac{z^2+1}{y}\right)C=$$

$$=(x+y+z)ABC$$

Proposed by D.M. Bătinețu-Giurgiu; Neculai Stanciu – Romania

Solution by Marian Ursărescu – Romania

We use the following theorem: If $M, N \in M_n(\mathbb{C})$ such that $MN = I_n \Rightarrow NM = I_n$, then

$$MN = NM.\ x^2A + B = xAB \Rightarrow xAB - x^2A - B = O_n \Rightarrow xA(B - xI_n) - B + xI_n = xI_n \Rightarrow$$

$$\Rightarrow (xA - I_n)(B - xI_n) = xI_n \Rightarrow (B - xI_n)(xA - I_n) = I_n \Rightarrow xBA - B - x^2A + xI_n = I_n \Rightarrow$$

$$\Rightarrow xBA = x^2A + B \Rightarrow AB = BA \text{ and similarly, } BC = CB \text{ and } AC = CA.$$

$$\left.\begin{array}{l} x^2A + B = xAB \Rightarrow x^2AC + BC = xABC \\ y^2B + C = yBC \Rightarrow y^2AB + AC = yABC \\ z^2C + A = zCA \Rightarrow z^2BC + AB = zABC \end{array}\right\} \Rightarrow$$

$$(x^2 + 1)AC + (y^2 + 1)AB + (z^2 + 1)BC = (x + y + z)ABC \quad (1)$$

$$x^2A + B = xAB \Rightarrow AB = xA + \frac{1}{x}B$$

$$y^2B + C = yBC \Rightarrow BC = yB + \frac{1}{y}C \quad (2)$$

$$z^2C + A = zCA \Rightarrow CA = zC + \frac{1}{z}A$$

From (1)+(2) we have:

$$(y^2 + 1)\left(xA + \frac{1}{x}B\right) + (z^2 + 1)\left(yB + \frac{1}{y}C\right) + (x^2 + 1)\left(zC + \frac{1}{z}A\right) = (x + y + z)ABC \Leftrightarrow$$

$$\Leftrightarrow \left(x(y^2 + 1) + \frac{x^2+1}{z}\right)A + \left((z^2 + 1)y + \frac{y^2+1}{x}\right)B + \left((x^2 + 1)z + \frac{z^2+1}{y}\right)C = (x + y + z)ABC$$

UP.054. If $x > 0$ then:

$$2\sqrt{x} \le \left(\frac{\Gamma(\sqrt{x} + 1)}{\Gamma\left(\sqrt{x} + \frac{1}{2}\right)}\right)^2 + \left(\frac{\Gamma(\sqrt[4]{x} + 1)}{\Gamma\left(\sqrt[4]{x} + \frac{1}{2}\right)}\right)^4 \le 2\sqrt{x} + \sqrt[4]{x} + \frac{1}{4}$$

where Γ denote the Euler's gamma function.

Proposed by Mihály Bencze – Romania

Solution by proposer

Applying Hölder's inequality:

$$\int_0^\infty f(t)\,g(t)dt \le \left(\int_0^\infty (f(t))^p dt\right)^{\frac{1}{p}} \left(\int_0^\infty (g(t))^q dt\right)^{\frac{1}{q}}$$

when $\frac{1}{p}+\frac{1}{q}=1$ *and* $p>1$. *Put* $p=\frac{1}{a},q=\frac{1}{1-a},f(t)=e^{-t}t^{x+a-1}$,

$g(t)=e^{-t}t^{x-1},\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}\,dt$ *we obtain* $\frac{\Gamma(x+a)}{x^a\Gamma(x)}\le 1$

$(\Gamma(x+1)=x\Gamma(x))$. *If in this case we take* $a\to 1-a,x\to x+a$ *we obtain*

$\left(\frac{x}{x+a}\right)^{1-a}\le\frac{\Gamma(x+a)}{x^a\Gamma(x)}$ *finally* $\left(\frac{x}{x+a}\right)^{1-a}\le\frac{\Gamma(x+a)}{x^a\Gamma(x)}\le 1$. *If* $a=\frac{1}{2}$

then we have $\sqrt{x}\le\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}\le\sqrt{x+\frac{1}{2}}$ *if* $x\to\sqrt{x}\Rightarrow$

$$\sqrt[4]{x}\le\frac{\Gamma(\sqrt{x}+1)}{\Gamma\left(\sqrt{x}+\frac{1}{2}\right)}\le\sqrt{\sqrt{x}+\frac{1}{2}}\Rightarrow\sqrt{x}\le\sqrt{x}\le\left(\frac{\Gamma(\sqrt{x}+1)}{\Gamma\left(\sqrt{x}+\frac{1}{2}\right)}\right)^2\le\sqrt{x}+\frac{1}{2}\quad (1)$$

$$x\to\sqrt[4]{x};\ \sqrt[8]{x}\le\frac{\Gamma\left(\sqrt[4]{x}+1\right)}{\Gamma\left(\sqrt[4]{x}+\frac{1}{2}\right)}\le\sqrt{\sqrt[4]{x}+\frac{1}{2}}\Rightarrow\sqrt{x}\le\left(\frac{\Gamma\left(\sqrt[4]{x}+1\right)}{\Gamma\left(\sqrt[4]{x}+\frac{1}{2}\right)}\right)^4\le\sqrt{x}+\sqrt[4]{x}+\frac{1}{2}\quad (2)$$

After addition (1)+(2) holds.

UP.055. Evaluate:

$$I=\int_0^1\frac{\ln^3 x}{2-x}dx$$

Proposed by Shivam Sharma – New Delhi – India

Solution 1 by Shivam Sharma-New Delhi-India

$$\Rightarrow\frac{1}{2}\int_0^1\ln^3(x)\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n dx\Rightarrow\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 x^n\ln^3(x)\,dx$$

$$\Rightarrow\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\cdot\frac{\partial^3}{\partial n^3}\left[\int_0^1 x^n\,dx\right]\Rightarrow\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\cdot\frac{\partial^3}{\partial n^3}\left[\frac{x^{n+1}}{n+1}\right]_0^1$$

$$\Rightarrow\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\left[\frac{x^{n+1}\ln^3(x)}{n+1}-\frac{3x^{n+1}\ln^2(x)}{(n+1)^2}+\frac{6x^{n+1}\ln(x)}{(n+1)^3}-\frac{6x^{n+1}}{(n+1)^4}\right]_0^1$$

$$\Rightarrow -6\sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}(n+1)^4}\right)\Rightarrow -6\sum_{n=0}^{\infty}\left(\frac{1}{2^n n^4}\right)$$

(OR)

$$I = -6Li_4\left(\frac{1}{2}\right) \text{ (Q.E.D.)}$$

Solution 2 by Shivam Sharma-New Delhi-India

$$\Rightarrow \frac{1}{2}\int_0^1 \ln^3(x)\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n dx \Rightarrow \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 x^n \ln^3(x)\, dx$$

$$\Rightarrow \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\cdot\frac{\partial^3}{\partial n^3}\left[\int_0^1 x^n dx\right] \Rightarrow \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\cdot\frac{\partial^3}{\partial n^3}\left[\frac{x^{n+1}}{n+1}\right]_0^1$$

$$\Rightarrow \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\left[\frac{x^{n+1}\ln^3(x)}{n+1} - \frac{3x^{n+1}\ln^2(x)}{(n+1)^2} + \frac{6x^{n+1}\ln(x)}{(n+1)^3} - \frac{6x^{n+1}}{(n+1)^4}\right]_0^1$$

$$\Rightarrow -6\sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}(n+1)^4}\right) \Rightarrow -6\sum_{n=0}^{\infty}\left(\frac{1}{2^n n^4}\right)$$

$$\text{(OR) } I = -6Li_4\left(\frac{1}{2}\right) \text{ (Q.E.D.)}$$

Solution 3 by Rovsen Pirguliev-Sumgait-Azerbaidian

$$\int_0^1 \frac{\ln^3 t}{2-t}dt = \int_0^1 \frac{\ln^3 t}{2\left(1-\frac{t}{2}\right)}dt = \frac{1}{2}\int_0^1 \ln^3 t\sum_{n=0}^{\infty}\left(\frac{t}{2}\right)^n dx = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\int_0^1 t^4 \ln^3 t\, dt =$$

$$= -6\sum_{n=0}^{\infty}\frac{1}{2^{n+1}(n+1)^4} = -6\cdot Li_4\left(\frac{1}{2}\right)$$

where Li_k is the polylogarithm of order k.

Solution 4 by Mirza Uzair Baig-Lahore-Pakistan

It should be easy to see that for $m \geq 0$ we have

$$\int_0^1 x^m \ln^3 x\, dx = -\frac{6}{(m+1)^4}$$

Now for $x \in (0,1)$ we have $\frac{1}{2-x} = \sum_{i=0}^{\infty}\frac{x^i}{2^{i+1}}$. This tells us that

$$I = -6\sum_{i=0}^{\infty}\frac{1}{2^i i^4} = -6Li_4\left(\frac{1}{2}\right).$$

where $Li_n(\cdot)$ is the Polylogarithm function.

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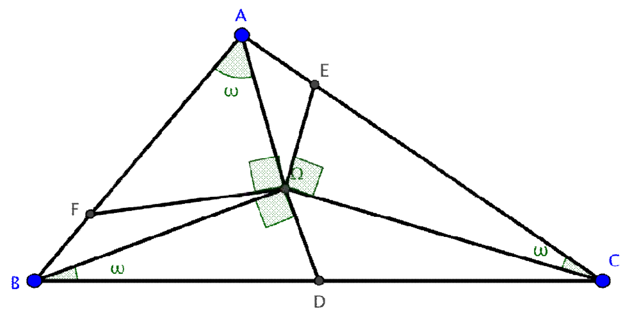
UP.056. Let ABC and be a triangle and Ω is first Brocard point of ABC. Let D, E, F be on the sides BC, CA, AB of ABC respectively. If $m(B\Omega D) = m(C\Omega E) = m(A\Omega F) = 90°$ then prove that $\frac{|BD|}{|BC|} + \frac{|CE|}{|CA|} + \frac{|AF|}{|AB|} = 2$.

Proposed by Mehmet Șahin – Ankara – Turkey

Solution by Marian Ursărescu – Romania



$$\Delta B\Omega D \Rightarrow \cos\omega = \frac{B\Omega}{BD} \Rightarrow BD = \frac{B\Omega}{\cos\omega} \quad (1)$$

But from Sines Law we have:

$$A\Omega = \frac{b}{a}2R\sin\omega,\ B\Omega = \frac{c}{b}2R\sin\omega \wedge C\Omega = \frac{a}{c}2R\sin\Omega \quad (2)$$

From (1)+(2)$\Rightarrow BD = \frac{c}{b}2R\tan\omega \Rightarrow \frac{BD}{BC} = \frac{c}{AB}2R\tan\omega \Rightarrow$

$$\Rightarrow \frac{BD}{BC} + \frac{CE}{CA} + \frac{AF}{AB} = 2R\tan\omega\left(\frac{c}{ab} + \frac{b}{ac} + \frac{a}{bc}\right) = 2R\tan\omega\frac{(a^2+b^2+c^2)}{abc} \quad (3)$$

But in any ΔABC *we have:* $\cot\omega = \frac{a^2+b^2+c^2}{4S}, S = [ABC] \Rightarrow \tan\omega = \frac{4S}{a^2+b^2+c^2}$ *(4)*

From (3)+(4)$\Rightarrow \frac{BD}{BC} + \frac{CE}{CA} + \frac{AF}{AB} = \frac{2R\cdot 4S}{a^2+b^2+c^2}\cdot\frac{(a^2+b^2+c^2)}{abc} = 2\cdot\frac{4RS}{abc} = 2$, *because* $abc = 4RS$

UP.057. Let $a, b \in \mathbb{R}$ such that $a + b > 0$ then

$$\left(\frac{a+b}{2}\right)^n \le \frac{1}{n+1}\sum_{k=0}^{n} a^k b^{n-k} \le \frac{a^n + b^n}{2}$$

Proposed by Abdallah El Farissi – Bechar – Algerie

Solution by Soumitra Mandal-Chandar Nagore-India

WLOG let us assume $b > a$

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Let us assume that the statement

$$\left(\frac{a+b}{2}\right)^n \le \frac{1}{n+1}\sum_{k=0}^{n} a^k b^{n-k} \le \frac{a^n+b^n}{2}$$

be $P(n)$***. Now,*** $P(1)$: $\frac{a+b}{2} \le \frac{a+b}{2} \le \frac{a+b}{2}$***, which is true***

so, $P(1)$ ***is true.*** $P(2)$: $\left(\frac{a+b}{2}\right)^2 \le \frac{1}{3}(a^2+ab+b^2) \le \frac{a^2+b^2}{2}$

now, $\frac{a^2+ab+b^2}{3} \ge \left(\frac{a+b}{2}\right)^2 \Leftrightarrow \frac{(a-b)^2}{12} \ge 0$***, which is true***

again, $\frac{a^2+b^2}{2} \ge \frac{a^2+ab+b^2}{3} \Leftrightarrow \frac{(a-b)^2}{6} \ge 0$***, which is true.***

$\therefore P(2)$ ***is established. Let us assume the statement is true for*** $n = m$***.***

$\therefore P(m)$: $\left(\frac{a+b}{2}\right)^m \le \frac{b^{m+1}-a^{m+1}}{(m+1)(b-a)} \le \frac{a^m+b^m}{2}$***. Similarly,*** $P(m-1)$ ***is also true.***

$$\therefore \left(\frac{a+b}{2}\right)^{m-1} \le \frac{b^m-a^m}{m(b-a)} \le \frac{a^{m-1}+b^{m-1}}{2}.$$

We need to prove, $n = m+1$. $\therefore \frac{a^{m+1}+b^{m+1}}{2} - \frac{b^{m+2}-a^{m+2}}{(m+2)(b-a)}$

$$= \frac{m(b-a)(a^{m+1}+b^{m+1})}{2(m+2)(b-a)} - \frac{ab}{m+2}\cdot\frac{b^m-a^m}{b-a} \ge$$

$$\ge \frac{m}{2(m+2)}(a^{m+1}+b^{m+1}) - \frac{abm}{2(m+2)}(a^{m-1}+b^{m-1})$$

$$= \frac{m}{2(m+2)}(b-a)(b^m-a^m) \ge \frac{m^2}{2(m+2)}(b-a)^2\left(\frac{a+b}{2}\right)^{m-1} \ge 0$$

$$\therefore \frac{a^{m+1}+b^{m+1}}{2} \ge \frac{1}{m+2}\cdot\frac{b^{m+2}-a^{m+2}}{b-a}$$

now, $\frac{b^{m+2}-a^{m+2}}{(m+2)(b-a)} - \left(\frac{a+b}{2}\right)^{m+1} = \frac{b^{m+2}-a^{m+2}}{(m+2)(b-a)} - \left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}\right)^m$

$$\ge \frac{b^{m+2}-a^{m+2}}{(m+2)(b-a)} - \frac{(a+b)(b^{m+1}-a^{m+1})}{2(m+1)(b-a)}$$

$$= \frac{2(m+1)(b^{m+2}-a^{m+2}) - (m+2)(a+b)(b^{m+1}-a^{m+1})}{2(b-a)(m+1)(m+2)}$$

$$= \frac{m(b^{m+1}+a^{m+1})}{2(m+1)(m+2)} - \frac{abm(b^{m-1}+a^{m-1})}{2(m+1)(m+2)} = \frac{m}{2(m+1)(m+2)}(b-a)(b^m-a^m)$$

$$\geq \frac{m^2}{2(m+1)(m+2)}(b-a)^2\left(\frac{a+b}{2}\right)^{m-1} \geq 0$$

$\therefore \frac{b^{m+2}-a^{m+2}}{(m+2)(b-a)} \geq \left(\frac{a+b}{2}\right)^{m+1}$. *Hence* $\left(\frac{a+b}{2}\right)^{m+1} \leq \frac{1}{m+2}\sum_{k=0}^{m+1} a^k b^{m-k} \leq \frac{a^{m+1}+b^{m+1}}{2}$

$\therefore P(m+1)$ ***is true. So, by theory of mathematical induction*** $P(n)$ ***is true.***

$$\left(\frac{a+b}{2}\right)^n \leq \frac{1}{n+1}\sum_{k=0}^{n} a^k b^{n-k} \leq \frac{a^n+b^n}{2}$$

UP.058. Let ABC be an arbitrary triangle and XYZ is the Kiepert triangle of ABC. If $K(\theta)$ is a Kiepert perspector and ω is first Brocard angle then prove that

a) $\frac{Area(XYZ)}{Area(ABC)} = \frac{1}{4}(3\tan^2\theta + 2\tan\theta \cdot \cot\omega + 1)$

b) **if $\theta = \omega$ then XYZ is Gallatly – Kiepert triangle takes the name.**

Prove that

$$\frac{Area\,(XYZ)}{Area\,(ABC)} = 3 \cdot \frac{a^2b^2 + b^2c^2 + c^2a^2}{(a^2+b^2+c^2)^2}$$

Proposed by Mehmet Şahin – Ankara – Turkey

Solution by Abdilkadir Altintas – Afyon – Turkey

Let $K(\theta)$ ***be Kiepert perspector.*** BXC, AYC ***and*** AZB ***triangles drawn outwardly to the sides of*** ABC. AX, BY ***and*** CZ ***are concurrent at point***

$$K(\theta) = \left(\frac{1}{S_A+S_\theta}, \frac{1}{S_B+S_\theta}, \frac{1}{S_c+S_\theta}\right)$$

where S_A, S_B ***and*** S_C ***are conway notations of*** ABC. ***If*** $\theta = w$ ***where*** w ***is Brocard angle than*** $S_w = \frac{a^2+b^2+c^2}{2}$; X ***has barycentric coordintes***

$$X = (-a^2 : S_C + S_w : S_B + S_w).$$

$$Y = (S_C + S_w : -b^2 : S_A + S_w),$$

$$Z = (S_B + S_w : S_A + S_w : -c^2).$$

Simplifying we get: $X = (-a^2 : a^2 + c^2 : a^2 + b^2)$; $Y = (a^2 + b^2 : -b^2 : b^2 + c^2)$

$Z = (a^2 + c^2 : b^2 + c^2 : -c^2)$. ***Using determinant to evaluate the area***

$$XYZ = \frac{3(a^2b^2 + a^2c^2 + b^2c^2)}{(a^2+b^2+c^2)^2}$$

UP.059. Let a, b, c be positive real numbers. Prove that

$$\frac{(a^2-ab+b^2)^2}{(a+b)^4}+\frac{(b^2-bc+c^2)^2}{(b+c)^4}+\frac{(c^2-ca+a^2)^2}{(c+a)^4}\geq\frac{3}{16}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo: a, b, c números R, probar que: $\frac{(a^2-ab+b^2)^2}{(a+b)^4}+\frac{(b^2-bc+c^2)^2}{(b+c)^4}+\frac{(c^2-ca+a^2)^2}{(c+a)^4}\geq\frac{3}{16}$

Es bien sabido que: $a^2-ab+b^2\geq\frac{1}{4}(a+b)^2\Leftrightarrow(a-b)^2\geq 0$

Por la tanto la desigualdad es equivalente: $\sum\frac{(a^2-ab+b^2)^2}{(a+b)^4}\geq\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{3}{16}$

Solution 2 by Ravi Prakash-New Delhi-India

For $a, b > 0$

$$\frac{a^3+b^3}{2}\geq\left(\frac{a+b}{2}\right)^3\Rightarrow\frac{a^3+b^3}{(a+b)^3}\geq\frac{1}{4}\Rightarrow\frac{a^2-ab+b^2}{(a+b)^2}\geq\frac{1}{4}\Rightarrow\frac{(a^2-ab+b^2)^2}{(a+b)^4}\geq\frac{1}{16}$$

Similarly for other two expressios. Thus,

$$\frac{(a^2-ab+b^2)^2}{(a+b)^4}+\frac{(b^2-bc+c^2)^2}{(b+c)^4}+\frac{(c^2-ca+a^2)^2}{(c+a)^4}\geq\frac{3}{16}$$

UP.060. Let a, b, c be positive real numbers with $a+b+c=1$. Prove that

$$\left(1+\frac{1}{2a+b}\right)^c\cdot\left(1+\frac{1}{2b+c}\right)^a\cdot\left(1+\frac{1}{2c+a}\right)^b\geq 2$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Imad Zak-Saida-Lebanon

$a, b, c > 0$ */* $a+b+c=1$. *Prove that:* $\left(1+\frac{1}{2a+b}\right)^c\cdot\left(1+\frac{1}{2b+c}\right)^a\cdot\left(1+\frac{1}{2c+a}\right)^b\geq 2$

Let $f(x)=\ln\left(1+\frac{1}{x}\right)$ *for* $x>0\Rightarrow$

$f'(x)=-\frac{1}{x+x^2}<0\Rightarrow f$ *is decreasing … (1)*

$f''(x)=\frac{2x+1}{x^2(x+1)^2}>0\Rightarrow f$ *is convex … (2)*

Weighted Jensen's on $f(x)\Rightarrow\sum c\ln\left(1+\frac{1}{2a+b}\right)=$

$$\sum c\cdot f(2a+b)\geq (c+a+b)f\left(\frac{c(2a+b)+a(2b+c)+b(2c+a)}{a+b+c}\right)$$

$$=1\cdot f\left(\frac{3\sum ab}{1}\right)=f\left(3\sum ab\right)\geq f(1)=\ln 2$$

because $\sum ab\leq \frac{p^2}{3}=\frac{1}{3}\Rightarrow 3\sum ab\leq 1$ f ***is*** $\searrow$

$$\therefore \sum c\cdot \ln\left(1+\frac{1}{2a+b}\right)\geq \ln 2\Leftrightarrow \sum \ln\left(1+\frac{1}{2a+b}\right)^c\geq \ln 2\Leftrightarrow$$

$\ln\left(\prod\left(1+\frac{1}{2a+b}\right)^c\right)\geq \ln 2$ ***ln is*** $\nearrow\Rightarrow$

$\prod\left(1+\frac{1}{2a+b}\right)^c\geq 2$ ***Q.E.D equality holds for*** $a=b=c=\frac{1}{3}$