SOCIETATEA DE ȘTIINȚE MATEMATICE DIN ROMÂNIA
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## LA GACETA

## DE LA REAL SOCIEDAD MATEMÁTICA ESPAÑOLA

**PROBLEMA 320.** ***Propuesto por D. M. Bătinețu – Giurgiu, "Matei Basarab" National College, Bucarest, Rumanía, y Neculai Stanciu, "George Emil Palade" Secondary School, Buzau, Rumanía***

***Sea $\{a_n\}_{n\geq 1}$ una sucesión de numerous reales positivos tal que $\lim_{n\to\infty}(a_n + 1 - a_n) = a$, siendo $a$ un cierto valor real positivo.***

***Consideramos la sucesión dada por $x_1 = 1$ y***

$x_n = \sqrt[n]{\sqrt{3!}\sqrt[3]{5!}\sqrt[4]{7!}\ldots\sqrt[n]{(2n-1)!}}$ ***. Evaluar los límites***

$$\lim_{n\to\infty}\left(\frac{a_{n+1}^3}{x_{n+1}} - \frac{a_n^3}{x_n}\right) \qquad y \qquad \lim_{n\to\infty}\left(\frac{x_{n+1}}{a_{n+1}} - \frac{x_n}{a_n}\right).$$

*Solution by Soumitra Mandal-Chandar Nagore-India*

***Let $\{a_n\}_{n\geq 1}$ be a positive real sequence and***

$$\lim_{n\to\infty}(a_{n+1} - a_n) = a$$

***Let $\{x_n\}_{n\geq 1}$ be another positive sequence where $x_1 = 1$ and***

$$x_n = \sqrt[n]{\sqrt{3!}\sqrt[3]{5!}\sqrt[4]{7!}\ldots\sqrt[n]{(2n-1)!}}$$

***Calculate***

$$\lim_{n\to\infty}\left(\frac{a_{n+1}^3}{x_{n+1}} - \frac{a_n^3}{x_n}\right)$$

$$\Omega = \lim_{n\to\infty}\left(\frac{a_{n+1}^3}{x_{n+1}} - \frac{a_n^3}{x_n}\right) = \lim_{n\to\infty}\left\{\left(\frac{a_n}{n}\right)^3 \frac{n^2}{x_n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n\right\}$$

***Where***

$$u_n = \frac{x_n}{x_{n+1}} \cdot \left(\frac{a_{n+1}}{a_n}\right)^3 \text{ for all } n \geq 2. \text{ Now}$$

$$\lim_{n\to\infty}\left(\frac{a_n}{n}\right)^3 = \lim_{n\to\infty}\left(\frac{a_{n+1}-a_n}{n+1-n}\right)^3 = a^3$$

$$\therefore \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n = \lim_{n\to\infty}\left(\left(1+\frac{a_{n+1}-a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right)^{\frac{a_n}{n}(a_{n+1}-a_n)} = e$$

***By Cauchy D-Alembert's Theorem:***

$$\lim_{n\to\infty}\frac{x_n}{n^2} = \lim_{n\to\infty}\sqrt[n]{\frac{\sqrt{3!}\sqrt[3]{5!}\ldots\sqrt[5]{(2n-1)!}}{n^{2n}}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{2n}\cdot\frac{1}{(n+1)^2}\cdot\sqrt[n+1]{(2n+1)!}$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{2n}}\cdot\lim_{n\to\infty}\sqrt[n+1]{\frac{(2n+1)!}{(n+1)^{2(n+1)}}} = \frac{1}{e^2}\lim_{n\to\infty}\frac{(2n+3)!}{(n+2)^{2(n+2)}}\cdot\frac{(n+1)^{2(n+1)}}{(2n+1)!}$$

$$= \frac{1}{e^2}\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n+1}\right)^{2n(n+1)}}\cdot\lim_{n\to\infty}\frac{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)^2} = \frac{4}{e^4}$$

$$\lim_{n\to\infty}u_n = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^3\cdot\lim_{n\to\infty}\left(\frac{(n+1)^2}{x_{n+1}}\right)\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2$$

$$= \lim_{n\to\infty}\left(\frac{a_{n+1}}{n+1}\right)^3\lim_{n\to\infty}\frac{1}{\left(\frac{a_n}{n}\right)^3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^3 = \lim_{n\to\infty}\left(\frac{a_{n+2}-a_{n+1}}{n+2-n-1}\right)^3\lim_{n\to\infty}\frac{1}{\left(\frac{a_{n+1}-a_n}{n+1-n}\right)^3}$$

$$= 1. \text{ So when } u_n \to 1 \text{ then } \frac{u_1-1}{\ln u_n} \to 1 \text{ as } n \to \infty$$

$$\therefore \lim_{n\to\infty}u_n^n = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^{3n}\cdot\lim_{n\to\infty}\frac{x_n^n}{x_{n+1}^{n+1}}\cdot\lim_{n\to\infty}x_{n+1}$$

$$e^3 = \lim_{n\to\infty}\frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!}}\cdot\lim_{n\to\infty}\frac{x_{n+1}}{(n+1)^2} = e^3\cdot\frac{e^2}{4}\cdot\frac{4}{e^4} = e$$

$$\therefore \lim_{n\to\infty}\ln u_n^n = 1 \therefore \Omega = \lim_{n\to\infty}\left(\frac{a_n}{n}\right)^3\cdot\frac{n^2}{x_n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n = a^3\cdot\frac{e^4}{4}\cdot\ln e = \frac{e^3e^4}{4}$$