Math Adventures on CutTheKnot Math 1-50

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MATH ADVENTURES
ON
CutTheKnotMath

1 - 50

By Alexander Bogomolny and Daniel Sitaru

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Special Thanks To Professor Alexander Bogomolny

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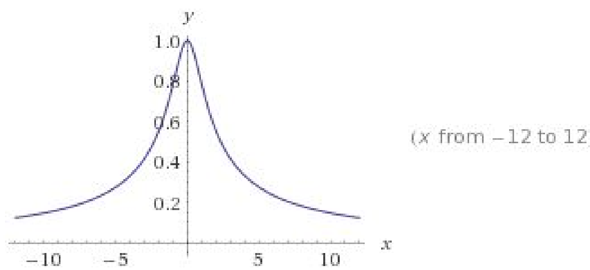
1. AN EASY INEQUALITY WITH THREE INTEGRALS

Let $a, b, c > 0$. Prove that

$$a^2 \int_0^b \frac{\arctan x}{x} dx + b^2 \int_0^c \frac{\arctan x}{x} dx + c^2 \int_0^a \frac{\arctan x}{x} dx < a^3 + b^3 + c^3$$

Proposed by Daniel Sitaru - Romania

Remark (by Alexander Bogomolny).
The starting point of the solutions below is the observation that $\frac{\arctan x}{x} < 1$, for all $x > 0$. The fractions has a limit of 1 as $x \to 0^+$ which allows for the definition (by continuity) $f(1) = 1$.



The inequality is equivalent to $\frac{\theta}{\tan \theta} < 1$, for $\theta \in \left(0, \frac{\pi}{2}\right)$.
Obviously, the inequality holds for any $f(x) \leq 1$ in leau of $\frac{\arctan x}{x}$.

Solution 1.

We have

$$\int_0^u \frac{\arctan x}{x} dx < 1.$$

It follows that

$$a^2 \int_0^b \frac{\arctan x}{x} dx + b^2 \int_0^c \frac{\arctan x}{x} dx + c^2 \int_0^a \frac{\arctan x}{x} dx < a^2 b + b^2 c + c^2 a$$

Suffice it to show that $a^2 b + b^2 c + c^2 a \leq a^3 + b^3 + c^3$. By the *AM-GM inequality*,

$$a^3 + a^2 + b^3 \geq 3\sqrt[3]{a^6 b^3} = 3a^2 b,$$
$$b^3 + b^3 + c^3 \geq 3b^2 c,$$
$$c^3 + c^3 + a^3 \geq 3c^2 a.$$

Summing up gives $3(a^2 + b^3 + c^3) \geq 3(a^2 b + b^2 c + c^2 a)$, as desired.

□

Solution 2.

This solution only differs from the above in treatment of $a^2 b + b^2 c + c^2 a \leq a^3 + b^3 + c^3$. This is simply true by the *rearrangement inequality*.

□

Acknowledgment (by Alexander Bogomolny).
The problem above has been kindly posted at the *CutTheKnotMath facebook page* by Daniel Sitaru, along with a solution (Solution 1) by Soumava Chakraborty. Leo Giugiuc and Ravi Prakash have commented with practically identical solutions (Solution 2). The inequality has been published in the *Romanian Mathematical Magazine*.

2. AN INEQUALITY FROM GAZETA MATEMATICA, MARCH 2016

Let a, b, c be positive numbers such that $a^2 + b^2 + c^2 = 3$.
Prove that $(a + c)(1 + b) \leq 4$.

Proposed by Daniel Sitaru, Leonard Giugiuc - Romania

Proof 1.

Define matrix $\begin{pmatrix} 1 & a \\ a & b \\ b & c \\ c & 1 \end{pmatrix}$. **We have**

$$A^t \cdot A = \begin{pmatrix} 1 & a & b & c \\ a & b & c & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a \\ a & b \\ b & c \\ c & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^2 + b^2 + c^2 + 1 & a + ab + bc + c \\ a + ab + bc + c & a^2 + b^2 + c^2 + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & (a + c)(1 + b) \\ (a + c)(1 + b) & 4 \end{pmatrix}$$

By Cauchy - Binet theorem, $\det(A^t \cdot A) \geq 0$. **Therefore,** $[(a + c)(1 + b)]^2 \leq 16$, **or** $(a + c)(1 + b) \leq 4$. □

Proof 2.
We use *spherical coordinates*. Let $b = \sqrt{3}\cos t, a = \sqrt{3}\sin t \cos u$, **and** $c = \sqrt{3}\sin t \sin u$, **where** $0 < t < \frac{\pi}{2}$.
We need to prove that $\sqrt{3}(\cos u + \sin u)(1 + \sqrt{3}\cos t)\sin t \leq 4$.
Observe that $1 < \sin u + \cos u \leq \sqrt{2}$.
Thus, suffice it to prove that $\sqrt{6}(1 + \sqrt{3}\cos t)\sin t \leq 4$.
Consider the function $f : \left(0, \frac{\pi}{2}\right) \to \mathbb{R}$, **defined by** $f(t) = (1 + \sqrt{3}\cos t)\sin t$.
We have $f'(t) = 2\sqrt{3}\cos^2 t + \cos t - \sqrt{3}\sin t$ **which implies** $\max f\left(\arccos\left(\frac{1}{\sqrt{3}}\right)\right) = 2\sqrt{\frac{2}{3}}$. **Therefore,**

$$\sqrt{6}(1 + \sqrt{3}\cos t)\sin t \leq \sqrt{6} \cdot 2\sqrt{\frac{2}{3}} = 4.$$

□

Proof 3.
$(a-b)^2+(b-c)^2+(a-1)^2+(c-1)^2 \geq 0$ which simplifies to $(a^2+b^2+c^2)+1-ab-bc-a-c \geq 0$.
This is exactly $(a+c)(1+b) \leq 4$ □

Proof 4.
By the AM-QM inequality, $\frac{a+b+c}{3} \leq \sqrt{\frac{a^2+b^2+c^2}{3}} = 1$. Further, by the $AM-GM$ inequality,

$$(a+c)(b+1) \leq \left(\frac{a+c+b+1}{2}\right) \leq \left(\frac{3+1}{2}\right)^2 = 4.$$

□

Proof 5.
From $a^2+b^2+c^2=3, 3\sum a^2 \geq \left(\sum a\right)^2$, implying $\left(\sum a\right)^2 \leq 9$, or $a+b+c \leq 3$. $a+b+c+1 \leq 4$. Therefore,

$$4 \geq (a+c)+(b+1) \geq 2\sqrt{(a+c)(b+1)}, \text{ i.e., } 2 \geq \sqrt{(a+c)(b+1)}, \text{ or}$$
$$4 \geq (a+c)(b+1)$$

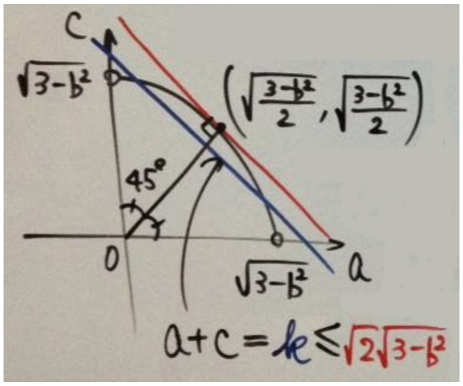
Equality is attained when $a+c=b+1$, which, with $a^2+b^2+c^2=3$ implies $a=b=c=1$. □

Proof 6.
From $(x-y)^2 \geq 0$ we have $2xy \leq x^2+y^2$. We use this with the couples $(a,b),(b,c),(1,a),(1,c)$:

$$2ab \leq a^2+b^2$$
$$2bc \leq b^2+c^2$$
$$2a \leq 1+a^2$$
$$2c \leq 1+c^2$$

adding which gives $2(a+c+ab+bc) \leq 2+2(a^2+b^2+c^2)=8$, and this is exactly $(a+c)(b+1) \leq 4$. □

Proof 7.



To continue:

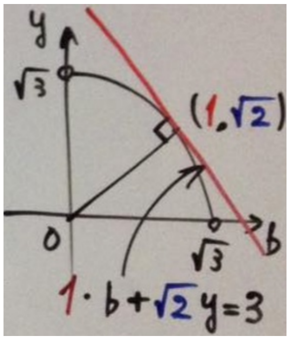
$$(a+c)(1+b) \leq \sqrt{2}\sqrt{3-b^2}(1+b)$$

$$\leq \sqrt{2}\sqrt{4-2b}(1+b) = \sqrt{2}\sqrt{(4-2b)(1+b)^2}$$
$$= \sqrt{2}\sqrt{(1+b)(1+b)(4-2b)} \leq \sqrt{2}\sqrt{\Big(\frac{1+b+1+b+4-2b}{3}\Big)^3} = 4$$

The equality is achieved for $1+b=1+b=4-2b$, making $b=1$ which, by the way, satisfies $3-b^2 \geq 0$. $a=c=\sqrt{\frac{3-b^2}{2}}$, i.e. $a=b=c=1$.

□

Proof 8.



To continue:

$$(a+c)(1+b) \leq \sqrt{2}\sqrt{3-b^2}(1+b)$$
$$\leq \sqrt{2}\frac{3-b}{\sqrt{2}}(1+b) \leq \Big(\frac{3-b+1+b}{2}\Big)^2 = 4$$

Equality is achieved for $3-b=1+b$ and $a=c=\sqrt{\frac{3-b^2}{2}}$, i.e., $a=b=c=1$.

□

Proof 9.
Observe that $(a+c)(1+b) = a\cdot 1 + a\cdot b + c\cdot 1 + b\cdot c$ such that by the ***Cauchy - Schwarz inequality***,

$$(a\cdot 1 + a\cdot b + c\cdot 1 + b\cdot b\cdot c)^2 \leq (a^2+b^2+c^2+b^2)(1^2+a^2+1^2+c^2)$$

which leads to a chain of inequalities

$$(a+ab+c+bc)^2 \leq (3+b^2)(2+a^2+c^2)$$
$$[(a+c)(1+b)]^2 \leq (3+b^2)(2+3-b^2)$$
$$= (3+b^2)(5-b^2) \leq \Big[\frac{3+b^2+5-b^2}{2}\Big]^2 = \Big[\frac{8}{2}\Big]^2,$$

and, therefore, $(a+c)(1+b) \leq 4$. □

Proof 10.
From $(a+c)^2 \leq 2(a^2+c^2)$ and $(1+b)^2 \leq 2(1+b^2)$ we obtain a sequence of inequalities:

$$(a+c)^2(1+b)^2 \leq 4(a^2+c^2)(1+b^2) \leq$$
$$\leq 4\Big(\frac{a^2+b^2+1+b^2}{2}\Big)^2 = 4\Big(\frac{4}{2}\Big)^2 = 4\cdot 4.$$

and, therefore, $(a+c)(1+b) \leq 4$. □

Proof 11.

We prove $\Big((a+b)(1+c)\Big)^2 \le 16$ when the point (a,b,c) lies on the sphere of radius $\sqrt{3}$ centred at the origin. At height c the sphere is a circle of radius $r = \sqrt{3-c^2}$ and the maximum of $a+b$ is $r\sqrt{2}$ (consider the line with slope -1 tangent to this circle in the first quadrant of the plane). We want thus the maximum of $2(1+c^2)(3-c^2)$ for $0 \le c \le \sqrt{3}$. The value 16 is attained for $c = 1$.
But $16 - 2(1+c)^2(3-c^2) = 2(c-1)^2(5+4c+c^2) \ge 0$, for all real c.

□

Proof 12.

Use Lagrange multipliers to prove that

$$\max_{a^2+b^2+c^2=3} (a+c)(1+b) = 4. \tag{1}$$

Let $J = (a+c)(1+b) + \lambda(a^2+b^2+c^2-3)$.
Taking $\frac{\partial J}{\partial a} = \frac{\partial J}{\partial b} = \frac{\partial J}{\partial c} = \frac{\partial J}{\partial \lambda} = 0$ yields

$$1 + b + 2a\lambda = 0, \tag{2}$$

$$1 + c + 2b\lambda = 0, \tag{3}$$

$$1 + b + 2c\lambda = 0, \tag{4}$$

$$a^2 + b^2 + c^2 = 3 \tag{5}$$

$$a = c \text{ (from 2 and 4)} \tag{6}$$

$$2a^2 + 2ab\lambda = 0 \text{ (from 3 and 6)} \tag{7}$$

$$b + b^2 + 2ab\lambda = 0 \text{ (from 2)} \tag{8}$$

$$b + b^2 = 2a^2 \text{ (from 7 and 8)} \tag{9}$$

$$2a^2 + b^2 = 3 \text{ (from 5 and 6)} \tag{10}$$

$$2b^2 + b - 3 = 0, b = 1, -\frac{3}{1} \text{ (from 9 and 10)} \tag{11}$$

Hence, $b = 1$ and from 10, $a = \pm 1$, implying $a = c = 1$ and $b = 1$, and 1 follows.

□

Remark (by Alexander Bogomolny).
It is clear that the equality is attained for $a = b = c = 1$ - a symmetric condition whereas the inequality itself is asymmetric. In analogy with the above derivation, we can show $(b+a)(1+c) \le 4$ and $(b+c)(1+a) \le 4$. The sum of the three gives $(a+b+c) + (ab+bc+ca) \le 6$ which is just more symmetric.

Acknowledgment (by Alexander Bogomolny).
Proofs 1 and 2 are by Leo Giugiuc and Daniel Sitaru; Proof 3 is by Nevena Sybeva; Proof 4 is by Augustini Moraru; Proof 5 is by Imad Zak

and independently by Rahim Shahbazov; Proof 6 is by Robert Kosova; Proof 7 and 8 are by Kunihiko Chikaya; Proof 9 and 10 are by Sk Rejuan; Proof 11 is by Grégoire Nicollier; Proof 12 is by Michalos Nikolau.

3. AN INEQUALITY IN CYCLIC QUADRILATERAL III

Prove that in a cyclic quadrilateral $ABCD$, with sides $AB = a, BC = b$, $CD = c, DA = d$, and the semiperimeter $s = \dfrac{a+b+c+d}{2}$, the following inequality holds

$$\sin A \sin B \leq \left(\frac{s}{a} - 1\right)\left(\frac{s}{b} - 1\right)\left(\frac{s}{c} - 1\right)\left(\frac{s}{d} - 1\right).$$

Proposed by Daniel Sitaru - Romania

Solution.
By *Brahmagupta's theorem*, the required inequality is equivalent to

$$abcd \sin A \sin B \leq [ABCD]^2$$

where $[ABCD]$ denotes the area of the quadrilateral. By the *AM-GM inequality*, this is equivalent to

$$abcd \sin A \sin B \leq \frac{(ad+bc)\sin A}{2} \cdot \frac{(ab+cd)\sin A}{2},$$

meaning $abcd \leq \frac{(ad+bc)}{2} \cdot \frac{(ab+cd)}{2}$.
But, by the AM-GM inequality, $\sqrt{abcd} \leq \frac{ad+bc}{2}$ and, similarly, $\sqrt{abcd} \leq \frac{ab+cd}{2}$.
The product of the two is the required $abcd \leq \frac{(ad+bc)}{2} \cdot \frac{(ab+cd)}{2}$.
Equality is achieved when $ad = bc$ and $ab = cd$. Talking the product: $a^2bd = c^2bd$, or $a^2 = c^2$, and, subsequently, $a = c$.
But then $b = d$, implying that $ABCD$ is a parallelogram, and, being cyclic, it is a square. □

Acknowledgment (by Alexander Bolgomolny).
The problem from his book *Math Accent* has been posted at the *CutTheKnotMath facebook page* by Daniel Sitaru, along with practically identical proofs by Leo Giugiuc, Adil Abdullayev, and Ravi Prakash.

4. A TRICKY INTEGRAL INEQUALITY

Let $a > 0$, and define

$$\Omega_1 = \int_0^a \left(\int_0^a \sqrt{x^2 + y^2 - 6x + 9}dx\right)dy$$

$$\Omega_2 = \int_0^a \left(\int_0^a \sqrt{x^2 + y^2 - 8y + 16}dx\right)dx$$

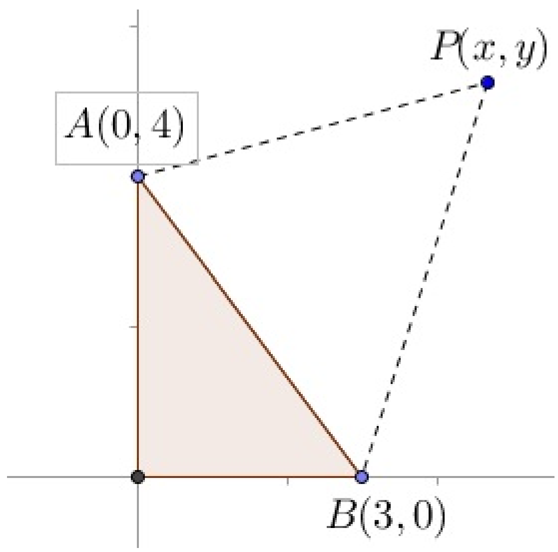
Prove that $\Omega_1 + \Omega_2 \geq 5a^2$.

Proposed by Daniel Sitaru - Romania

Solution (by Ravi Prakash - New Delhi - India).
First note, that due to *Fubini's theorem*, both repeated integrals can be treated as double. Next observe that the integrands are the (Euclidean) distance functions:

$\sqrt{x^2 + y^2 - 6x + 9} = dist(B, P)$ **and** $\sqrt{x^2 + y^2 - 8y + 16} = dist(A, P)$,

where A, B, P **are defined below:**



By the *triangle inequality* then

$$\Omega_1 + \Omega_2 = \int_0^a \int_0^a \left(\sqrt{x^2 + y^2 - 6x + 9} + \sqrt{x^2 + y^2 - 8y + 16}\right) dxdy$$

$$= \int_0^a \int_0^a \Big(dist(B, P) + dist(A, P)\Big) dxdy \geq \int_0^a \int_0^a dist(A, B) dxdy =$$

$$= \int_0^a \int_0^a 5 = 5a^2.$$

□

Extra (by Alexander Bogomolny).
The inequality just proved is always strict and can be improved for specific values of a**. For example, it is not hard to see that, for** $a \leq 0.5$, $dist(B, P) + dist(A, P) > 6$.
The beauty of the problem is in the implied generality. Indeed, any distance function ca be used in place of the Euclidean distance to make

the problem even more intriguing. For example, the *taxicab distance* leads to the following inequality:

$$\int_0^a \int_0^a (|x-3|+|y|dx)dy + \int_0^a \left(\int_0^a (|x|+|y-4|)dy \right) dx \geq 7a^2.$$

Using the *bounded distances* disguises the problem even further. For example, define

$$\Omega_1 = \int_0^a \left(\int_0^a \frac{\sqrt{x^2+y^2-6x+9}}{1+\sqrt{x^2+y^2-6x+9}} dx \right) dy$$

$$\Omega_2 = \int_0^a \left(\int_0^a \frac{\sqrt{x^2+y^2-8y+16}}{1+\sqrt{x^2+y^2-8y+16}} dy \right) dx.$$

Then $\Omega_1 + \Omega_2 \geq \frac{5}{6}a^2$. For another example, if

$$\Omega_1 = \int_0^a \left(\int_0^a \frac{|x-3|+|y|}{1+(|x-3|+|y|)} dx \right) dy$$

$$\Omega_2 = \int_0^a \left(\int_0^a \frac{|x|+|y-4|}{1+(|x|+|y-4|)} dy \right) dx,$$

then $\Omega_1 + \Omega_2 \geq \frac{7}{8}a^2$.

Acknowledgment (by Alexander Bogomolny).
The problem from the *Romanian Mathematical Magazine* has been posted at *CutTheKnotMath facebook page* by Daniel Sitaru, with a solution by Ravi Prakash.

5. COSPHERICAL POINTS

Find $x \in \mathbb{R}$ ***such that:***

$$A\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{4}, \frac{1}{4}\right), B\left(\frac{\sqrt{3}}{2}, \frac{1}{4}, \frac{\sqrt{3}}{4}\right), C\left(\frac{1}{2}, \frac{3}{4}, \frac{\sqrt{3}}{4}\right), D\left(0, \frac{1}{2}, x\right)$$

are cospherical points.

Proposed by Daniel Sitaru - Romania

Solution (by Leo Giugiuc), Comments (by Alexander Bogomolny).
In the Euclidean 2D space any three distinct points are *concylics*, unless they are *collinear*. In the Euclidean 3D space any four distinct points are *cospherical*, unless they are *coplanar* by not concyclic.
The three distinct points A, B, C define a plane, say α and a circle, say ω. It is immediately verifiable that $A, B, C \in S$, where S denotes the

unit sphere $x^2+y^2+z^2=1$. In particular, $\omega=\alpha\cap S$.
The condition for the four points be coplanar is given by

$$\begin{vmatrix} 2\sqrt{3} & \sqrt{3} & 1 & 1 \\ 2\sqrt{3} & 1 & \sqrt{3} & 1 \\ 2 & 3 & \sqrt{3} & 1 \\ 0 & 2 & 4x & 1 \end{vmatrix} = 0.$$

The equation implies $x=\frac{5}{4}$, i.e., $D=\left(0,\frac{1}{2},\frac{5}{4}\right)$, meaning $D\notin S$ and, therefore, $D\notin\omega$. So, if $x=\frac{5}{4}, A, B, C, D$ are coplanar, but not cospherical. If $x\notin\frac{5}{4}$, then A, B, C, D are not coplanar and, hence, are cospherical.
Thus the answer is $\mathbb{R}\setminus\left\{\frac{5}{4}\right\}$. □

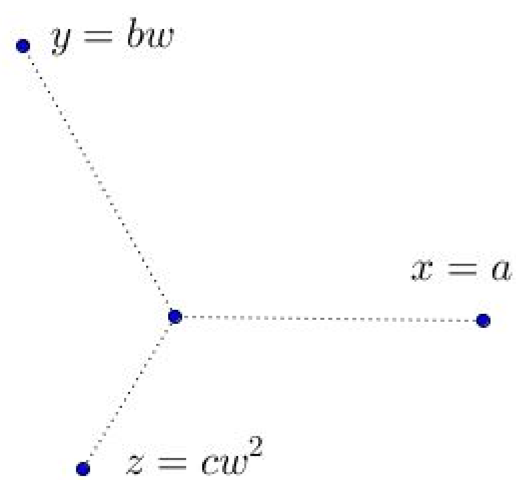
6. CYCLIC INEQUALITY

For $a,b,c>0$ the following inequality holds:

$$\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}$$
$$\leq a+b+c+\sqrt{a^2+b^2+c^2-ab-bc-ca}$$

Source: AOPS

Solution (by Claudia Nănuţi, Diana Trăilescu, Daniel Sitaru, Leo Giugiuc).
Set $w=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$, which is a rotations by $120°$ counterclockwise.
Define $x=a, y=bw, z=cw^2$.



First of all $|x|=|a|, |y|=|b|, |z|=|c|$. Then also, $|x+y|=\sqrt{a^2-ab+b^2}$, $|y+z|=\sqrt{b^2-bc+c^2}, |z+x|=\sqrt{c^2-ca+a^2}$, and
$|x+y+z|=\sqrt{a^2+b^2+c^2-ab-bc-ca}$. I'll verify the later identity:

$$x+y+z=\left(a-\frac{1}{2}b-\frac{1}{2}c\right)+i\left(\frac{\sqrt{3}}{2}b-\frac{\sqrt{3}}{2}c\right)$$

It follows that

$$|x+y+z|^2 = \left(a - \frac{1}{2}b - \frac{1}{2}c\right)^2 + \frac{3}{4}(b-c)^2$$

$$= a^2 + \frac{1}{4}b^2 + \frac{1}{4}c^2 - ab - ac + \frac{1}{2}bc + \frac{3}{4}b^2 + \frac{3}{4}c^2 - \frac{3}{2}bc$$

$$= a^2 + b^2 + c^2 - ab - bc - ca,$$

as required. It is now clearly seen that the problem is simply a reformulations of Hlawka's inequality:

$$|x+y| + |y+z| + |z+x| \leq |x| + |y| + |z| + |x+y+z|$$

true for any three complex numbers x, y, z. □

7. AN APPLICATION OF SCHUR'S INEQUALITY - II

Prove that for $x, y, z > 0$ such that $xyz = 1$, the following inequality holds:

$$\sum(x^4 + y^3 + z) \geq \sum\left(\frac{x^2+y^2}{z}\right) + 3$$

Proposed by Daniel Sitaru, Leonard Giugiuc - Romania

Proof 1 (by proposers).
We use Schur's inequality twice:
With $r = 1$ in the form $\sum x^3 + 3xyz \geq \sum xy(x+y)$ and,
with $r = 2$ in the form $\sum x^3 + xyz \sum x \geq \sum xy(x^2+y^2)$.
Since $xyz = 1$ the two can be rewritten as

$$\sum x^3 + 3 \geq \sum\left(\frac{x+y}{z}\right),$$

$$\sum x^4 + \sum x \geq \sum\left(\frac{x^2+y^2}{z}\right).$$

Adding up,

$$\sum(x^4 + x^3 + x) + 3 \geq \sum\left(\frac{x^2+y^2}{z}\right) + \frac{x}{z} + \frac{z}{x} + \frac{y}{x} + \frac{x}{y} + \frac{y}{z} + \frac{z}{y}.$$

But $\sum\left(\frac{x}{z} + \frac{z}{x}\right) \geq 2 + 2 + 2 = 6$. It follows that

$$\sum(x^4 + x^2 + x) \geq \sum\left(\frac{x^2+y^2}{z}\right) + 6 - 3$$

or

$$\sum(x^4 + y^2 + z) \geq \sum\left(\frac{x^2+y^2}{z}\right) + 3$$

□

Proof 2 (by Imad Zak - Saida - Lebanon).
We start with Schur's inequality:

$$\sum x^4 + xyz\sum x = \sum x^4 + \sum x = \sum(x^4 + x) \geq$$

$$\geq \sum xy(x^2 + y^2) = \sum \frac{x^2 + y^2}{z}.$$

Further, by the AM-GM inequality,

$$\sum x^3 \geq 3xyz = 3.$$

Adding this to the previous inequality yields the required result.
Equality holds for $x = y = z = 1$. □

8. A4 - VARIABLE INEQUALITY FROM ROMANIAN MATHEMATICAL MAGAZINE

Prove that, for $x, y, z, \eta \in \mathbb{R}$,

$$|(a-b)(b-c)(c-a)| \leq \sum_{cyc} |a-b||a+c+\eta||b+c+\eta|$$

Proposed by Daniel Sitaru - Romania

Proof (by Ravi Prakash - New Delhi - India).
Denote $x = b+c+\eta, y = c+a+\eta, z = a+b+\eta$. Then, e.g., $a-b = y-x$, and the inequality to prove becomes

$$|(x-y)(y-z)(z-x)| \leq \sum_{cycl} |(x-y)xy|$$

We have,

$$RHS = \sum_{cycl} |(x-y)xy| \geq |(x-y)xy + (y-z)yz + (z-x)zx|$$

$$= |(x-y)(y-z)(z-x)| = |(a-b)(b-c)(c-a)|,$$

as required.
Equality is achieved when $x = y = z$, i.e., when $a = b = c$. □

Acknowledgment (by Alexander Bogomolny).
Daniel Sitaru has kindly posted the above problem form the *Romanian Mathematical Magazine*, with a proof by Ravi Prakash, at the *CutTheKnotMath facebook page.*

9. A SYSTEM OF EQUATIONS IN DETERMINANTS

Statement (by Alexander Bogomolny).
Daniel Sitaru has kindly posted the following problem form the *Romanian Mathematical Magazine* and its solutions by Ravi Prakash at the

***CutTheKnotMath facebook page*:**
Find $A \in M_n(\mathbb{C})$ such that:

$$\begin{cases} \det(A+XY) = \det(A+YX), \forall X, Y \in M_n\mathbb{C} \\ \det A = 1 \end{cases}$$

Solution (by Ravi Prakash - New Delhi - India).
Let $E_{ij} = (\delta_{ij})$ be the matrix with all elements zero, except for the one in $i - s$ row and $j - s$ column which is 1.

$$E_{ij}E_{rs} = \begin{cases} E_{is}, \text{ if } j = r \\ 0, \text{ otherwise} \end{cases}$$

Consider $X = \alpha E_{ik}, \alpha \in \mathbb{C}$ and $Y = E_{ks}; XY = \alpha E_{is}, YX = 0$ unless $i = s$ and $X = E_{ii}$, otherwise. Thus, taking $i \neq s$,
$\det(A + \alpha E_{is}) = \det(A)$.
Now using the minor expansion of the determinant,

$$\det(A+\alpha E_{is}) = \sum_{j=1}^{n}(a_{ij} + \alpha\delta_{ij})M_{ij}(-1)^{i+j}$$

$$= \sum_{j=1}^{n} a_{ij}M_{ij}(-1)^{i+j} + \alpha M_{is}(-1)^{i+s}$$

$$= \det(A) + \alpha M_{is}(-1)^{i+s}$$

Since this equals $\det(A)$ and α is arbitrary, $M_{is} = 0, i \neq s$. Then from

$$\det(A) = \sum_{j=1}^{n} a_{ij}M_{ij}(-1)^{i+j} = a_{ii}M_{ii}$$

it follows that $a_{ii}M_{ii} = 1$, for all $i = 1, \ldots, n$.
In particular, for $i = 1, \ldots, n, a_{ii} \neq 0$ and $M_{ii} \neq 0$.
On the other hand, the choice $X = E_{rs}$ and $Y = E_{sr}$ leads to

$$\det(A) + \alpha M_{rr} = \det(A) + \alpha M_{ss}$$

so that all dialog minors of A are equal and, as consequence, so are all its diagonal elements: $a_{ii} = \lambda$, for $i = 1, \ldots, n$ and a fixed $\lambda \in \mathbb{C}$.
Next, for $i_1 \neq i_2$,

$$0 = \sum_{j=1}^{n} a_{i_1 j}M_{i_2 j}(-1)^{i_1+j} = a_{i_1 i_2}M_{i_2 i_2},$$

implying that all off - diagonal elements of A are zero: $a_{i_1 i_2} = 0$, for $i_1 \neq i_2$. Thus $A = \lambda E$, where E is the unit matrix.
Finally, $1 = \det(A) = \lambda^n$ implies $\lambda^n = 1$, i.e. λ is a root of unity. □

10. AN INEQUALITY IN INTEGERS

Statement (by Alexander Bogomolny).
The following inequality, due (1979) to professor Radu Gologan has posted at the *CutTheKnotMath facebook page* by Leo Giugiuc along wit a *solution* by Daniel Sitaru and Leo Giugiuc. Radu Gologan is the Romanian team leader for the IMO.

Let a and b be positive integers such that $\frac{a}{b} < \sqrt{7}$.

$$\textit{Prove that } \frac{a}{b} + \frac{1}{ab} < \sqrt{7}.$$

Proposed by Radu Gologan - Romania

Solution (by Leonard Giugiuc, Daniel Sitaru - Romania).
$a^2 < 7b^2$ so that $a^2 \le 7b^2 - 1$. In $\mathbb{Z}_7, a^2 \in \{0, 1, 2, 4\}$, making $a^2 = 7b^2 - 1$ impossible. Thus, necessarily, $a^2 \le 7b^2 - 2$. But then, again, $a^2 = 7b^2 - 2$ is also impossible such that, in fact $a^2 \le 7b^2 - 3$, or, $a \le \sqrt{7b^2 - 3}$. Introduce function $f(x) = x + \frac{1}{x}$ which is monotone increasing for $x \ge 1$. It follows that

$$\left(\sqrt{7b^2 - 3} + \frac{1}{\sqrt{7b^2 - 3}}\right)^2 \ge \left(a + \frac{1}{a}\right)^2$$

which is equivalent to

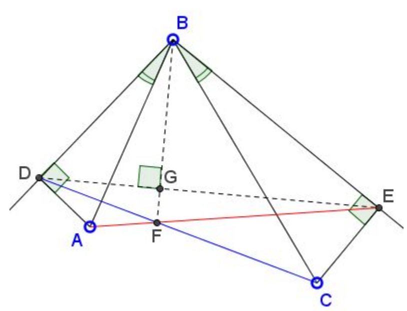
$$7b^2 - 1 + \frac{1}{7b^2 - 3} \ge \left(a + \frac{1}{a}\right)^2.$$

In addition, since b is a positive integer, $1 > \frac{1}{7b^2 - 3}$, such that $7b^2 > \left(a + \frac{1}{a}\right)^2$.

In other words, $7 > \left(\frac{a}{b} + \frac{1}{ab}\right)^2$, i.e., $\frac{a}{b} + \frac{1}{ab} < \sqrt{7}$, as required. □

11. ORTHOGONALITY IN ISOGONAL CONJUGACY

BD and BE are isogonal conjugate in ΔABC; $BE \perp CE$ and $BD \perp A$. F is the intersection of CD and AE.



Prove that $BF \perp DE$.

Proposed by Elberling Vargas Diaz - Peru

Solution (by Claudia Nănuţi, Diana Trăilescu, Daniel Sitaru, Leo Giugiuc).
Obviously, the angles BDC and BEA are acute. Consequently, the angles BDE and BED are also acute. So, without loss of generality, we may choose $D = -u, E = v$ and $B = i$, with $u, v > 0$. On the other hand, triangle BDA and BEC are inversely similar and right angled at D and E, respectively; hence there is $k > 0$ such that $\frac{A-D}{B-D} = \left(\frac{C-E}{B-E}\right) = -ki$. From here, $A = k - u - uki$ and $B = v - k - vki$. Let's write the equations of the straight lines AE and CD:

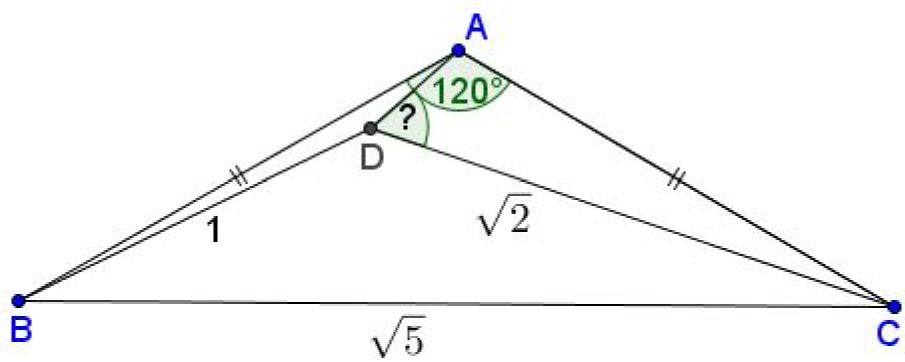
$$\begin{cases} AE : ukx - (u + v - k)y = uvk, \\ CD : vkx + (u + v - k)y = -uvk. \end{cases}$$

These give us $x = 0$, meaning that F lies on the y - axis. But DE was chosen to be the x - axis and the two meet at the right angle. □

Remark (by Alexander Bogomolny).
Francisco Javier García Capitán has observed that there is a second line, say, BE'. It is easy to see that $E' = CD \cap (BC)$ where (BC) is the circle with BC as a diameter. In this case, $F' = E'$. He posed this as an algebraic problem:
If we consider the vertices with coordinates $B = (0, 0), C = (a, 0)$ and $A = (u, v)$, the lines $y = mx$ and $y = nx$ are isogonal with respect to the angle B if $n = \frac{v-mu}{u+mv}$. For any m there exists another value of n, precisely $n = \frac{a+am^2-u-mv}{m(u+mv)}$ such that the lines $y = mx$ and $y = nx$ satisfy the conditions of the problem. Give some geometric description of this line.

12. ANGLES IN TRIANGLE: AN EXERCISE

In $\Delta ABC; AB = AC, \angle BAC = 120°$ and $BC = \sqrt{5}.D$ is a point inside ΔABC such that $BD = 1$ and $CD = \sqrt{2}$

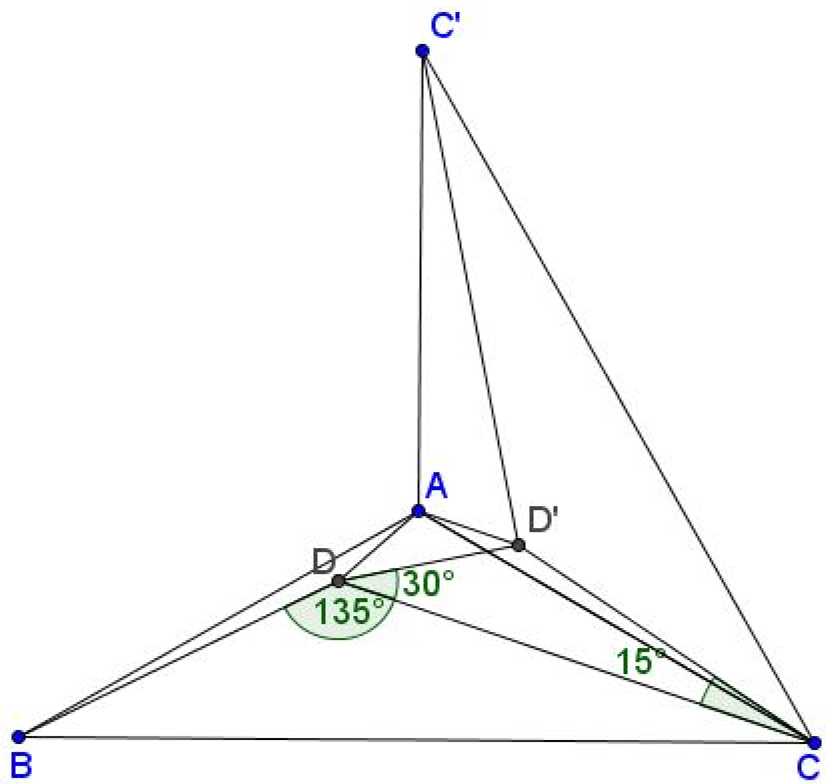


Prove that $\angle ADC = 60°$.

Proposed by Kadir Altintas - Afyon - Turkey

Solution 1 (by Alexander Bogomolny).

Rotate the diagram 120° counterclockwise around $A : B$ goes to C, C to C', and D to D'.



By the *Law of Cosines* in ΔBCD,

$$5 = 1 - 2\sqrt{2}\cos\angle BDC + 2,$$

such that $\cos\angle BDC = -\frac{\sqrt{2}}{2}$, giving $\angle BDC = 135^\circ$.
We thus have $\angle CBD + \angle BCD = 45^\circ, \angle C'CD' = \angle CBD, \angle BCC' = 60^\circ$, leading to $\angle DCD' = 15^\circ$.
***As we know,* $\sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}$ and $\cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$.**
Since $CD' = BD = 1$, we can employ the Law of Cosines in $\Delta DCD'$ to find DD':

$$(DD')^2 = 1 - 2\sqrt{2}\frac{\sqrt{6}+\sqrt{2}}{4} + 2 = 2 - \sqrt{3}$$

Further, by the *Law of Sines* in $\Delta DCD'$,

$$\frac{1}{\sin^2\angle CDD'} = \frac{2-\sqrt{3}}{(\sqrt{6}-\sqrt{2})/4} = 4,$$

implying $< CDD' = 30^\circ$.
Note that $\angle ADD' = 30^\circ$ also because, by the construction, $\angle DAD' = 120^\circ$ and $AD = AD'$. It follows that $\angle ADC = 60^\circ$. □

Solution 2 (by Claudia Nănuţi, Diana Trăilescu, Daniel Sitaru and Leonard Giugiuc).

Thinking of the points as complex numbers, choose $B = -\frac{2}{\sqrt{5}}, C = \frac{3}{\sqrt{5}}, D = \frac{i}{\sqrt{5}}$. With this choice, $A = \frac{1}{2\sqrt{5}} + i\frac{\sqrt{5}}{2\sqrt{3}}$. With $\phi = \angle ADC$, $\frac{A-D}{C-D} = \frac{AD}{CD}(\cos\phi + i\sin\phi)$ such that $\phi = \frac{Im(\frac{A-D}{C-D})}{Re(\frac{A-D}{C-D})}$. Thus, we get

$$\frac{A-D}{C-D} = \frac{1}{20\sqrt{3}}\left[\left(\sqrt{3} + i(5 - 2\sqrt{3})\right)(3 + i)\right] = \frac{\sqrt{3}-1}{4\sqrt{3}}(1 + i\sqrt{3}),$$

from which $\tan\phi = \sqrt{3}$, i.e., $\phi = 60°$. □

13. APPLICATION OF CAUCHY - SCHWARZ'S INEQUALITY

If $a, b, c \geq 1$, prove that:

$$\sqrt{a^2-1} + \sqrt{b^2-1} + \sqrt{c^2-1} \leq \frac{ab+bc+ca}{2}$$

Generalize!

Proposed by Dorin Marghidanu - Romania

Cauchy - Schwarz Inequality (by Alexander Bogomolny).

The two solutions below invoke the most important and useful mathematical tool - the *Cauchy - Schwarz inequality* that was covered almost in passing at the old an by now dysfunctional *Cut-The-Knot forum.* Below I state the inequality and give two proofs (out of a known great variety.)

For all real $x_i, y_i, i = 1, 2, \ldots, n$,

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2$$

***The equality is only attained when the two sequence (vectors) $\{x_1, \ldots, x_n\}$* and $\{y_1, \ldots, y_n\}$ are linearly dependent, i.e., when, say, there are u and v such that $ux_i + vy_i = 0$, for all $i, 1 \leq i \leq n$.**

Proof 1.

Consider

$$f(t) = \sum_{i=1}^{n}(tx_i + y_i)^2$$

$$= \sum_{i=1}^{n}(t^2x_i^2 + 2tx_iy_i + y_i^2)$$

$$= t^2\left(\sum_{i=1}^{n} x_i^2\right) + 2t\left(\sum_{i=1}^{n} x_i y_i\right) + \sum_{i=1}^{n} y_i^2$$

Since $f(t) \geq 0$, for all $t \in \mathbb{R}$, the discriminant
$D = (\sum_{i=1}^{n} x_i y_i)^2 - (\sum_{i=1}^{n} x_i^2)(\sum_{i=1}^{n} y_i^2)$ is not positive.
This is exactly the Cauchy - Schwarz inequality. □

Proof 2.
The Cauchy - Schwarz inequality is a direct consequence of a stronger result, *Lagrange's identity*:

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \left(\sum_{i=1}^{n} x_i y_i\right)^2 = \sum_{1 \leq i \leq j \leq n} (x_i y_i - x_i y_i)^2$$

□

Solution 1 (by Claudia Nănuţi, Diana Trăilescu, Daniel Sitaru and Leonard Giugiuc).

We denote $x = \sqrt{a^2 - 1}$, etc. Obviously, $x, y, z \geq 0$ and the task becomes to prove

$$\sum_{cycl} \sqrt{(x^2+1)(y^2+1)} \geq 2(x+y+z)$$

By the Cauchy - Schwarz inequality, $\sqrt{(x^2+1)(y^2+1)} \geq x + y$, with equality only when $xy = 1$. Applying this term-by-term yields the required inequality. Equality holds if $x = y = z = 1$, i.e., $a = b = c = \sqrt{2}$. For a generalization,

$$\sum_{i=1}^{n} \sqrt{a_i^2 - 1} \leq \frac{1}{2} \sum_{i=1}^{n} a_i a_{i+1}$$

where $a_{n+1} = a_1$. For n odd, the equality is only attained when all $a_i = \sqrt{2}$; for n even, whenever $x_1, x_2 = x_2 x_3 = \ldots = x_n x_1 = 1$. □

Solution 2 (by Alexander Bogomolny).
We'all go directly to a general case. By the Cauchy - Schwarz inequality,

$$\left(\sum_{i=1}^{n} \sqrt{a_i^2 - 1}\right)^2 \leq n \sum_{i=1}^{n} (a_i^2 - 1)$$

$$= n \sum_{i=1}^{n} a_i^2 - n$$

$$\leq n \sum_{i=1}^{n} a_i a_{i+1} - n$$

where $a_{n+1} = a_1$. Note that the first inequality becomes equality whenever all $\sqrt{a_i^2 - 1}$ are equal (i.e., whenever all a_i^2 are equal). The second inequality becomes equality whenever all a_i are equal. Thus the required inequality will be proved if we manage to prove

$$n \sum_{i=1}^{n} a_i a_{i+1} - n \leq \left(\frac{1}{2} \sum_{i=1}^{n} a_i a_{i+1}\right)^2$$

Denote $t = \sum_{i=1}^{n} a_i a_{i+1}$. We need to prove that

$$f(t) = \left(\frac{1}{2}t\right)^2 - nt + n \geq 0$$

This will be true for all $t \in \mathbb{R}$ provided the discriminant $D = (2n)^2 - 4n^2$ of the quadratic form $t^2 - 4nt + 4n^2$ is not positive. But as matter of fact, it is always zero, implying $f(t) \geq 0$, with $f(2n) = 0$. It follows that

$$n \sum_{i=1}^{n} a_i a_{i+1} - n \leq \left(\frac{1}{2} \sum_{i=1}^{n} a_i a_{i+1}\right)^2$$

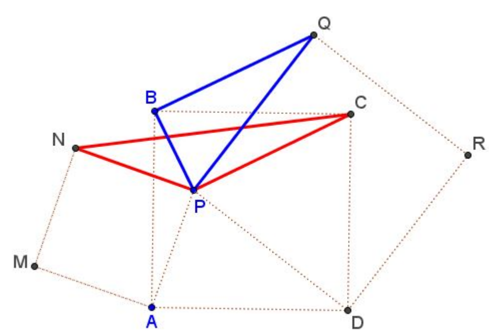
The equality is attained whenever $t = \sum_{i=1}^{n} a_i a_{i+1} = 2n$ and all a_i are equal, implying $a_i = \sqrt{2}$. □

Conclusion (by Alexander Bogomolny).

What do we learn from the above? Two solutions to the same problem, both using the Cauchy - Schwarz inequality, and, in the original problem (of three terms) producing the same results. However, the two methods lead to different generalisations for an increased number n of terms. The difference is only noticeable when n is even, and the second solution gives no clue that there might be a difference between the cases of odd and even number of terms. The only thing that comes to mind is that occasionally doing everything right may not necessarily yield a complete (not to use the term "right") answer. In a certain sense, the applications of the Cauchy - Schwarz inequality in the first solutions is more refined that its application in the second solution, but who could say that without first trying both ways?

14. AREAS IN THREE SQUARES

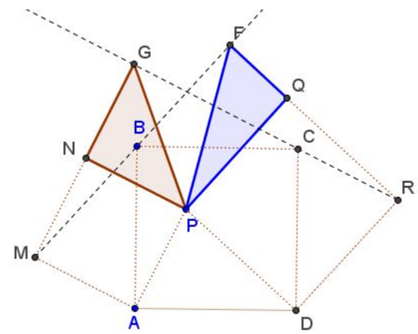
Given three squares $AMNP$, $ABCD$, and $DPQR$.



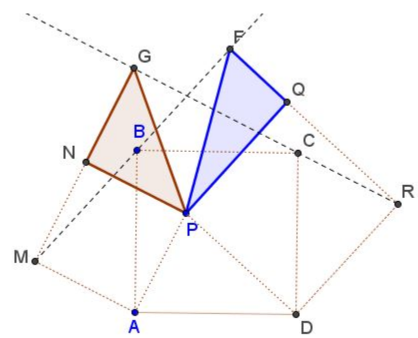
Prove that $[\Delta CPN] = [\Delta BPQ]$, where $[X]$ denotes the area of shape X.

Proposed by Miguel Ochoa Sanchez - Peru

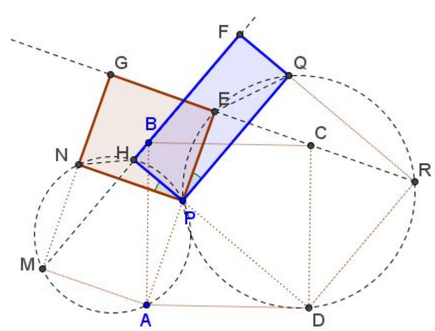
Solution 1 (by Alexander Bogomolny).
The solution is illustrated by a sequence of diagrams.
Draw $BF \parallel PQ$ and $CG \parallel NP$(F on QR or its extension; G on MN or its extension.
Then $[\Delta CPN] = [\Delta GPN]$ and $[\Delta BPQ] = [\Delta FPQ]$:



Complete the triangles GPN and FPQ to rectangles $GEPN$ and $FHPQ$.



We'd like to show that $[GEPN] = [FHPQ]$, i.e., that $EP \cdot NP = HP \cdot PQ$. Consider the circles (MP) and (PQ), with diameters MP and PR, respectively.



Obviously, $A, N \in (MP)$ but also $H \in (MP)$.
Similarly, $D, Q, E \in (PR)$.
Observe that

$$\angle HPN = \angle HMN = \angle GRF = \angle EPQ.$$

In addition, $\angle NHP = \angle PEQ = 135^\circ$ because both (inscribed) angles are subtended by the sides of squares inscribed into the circles (MP)

and (PR).
It follows that triangles HNP and EPQ are similar, from which we have the proposition $\frac{HP}{NP} = \frac{EP}{PQ}$ which is equivalent to $EP \cdot NP = HP \cdot PQ$. $\square$

Solution 2 (by Claudia Nănuţi, Diana Trăilescu, Daniel Sitaru, Leo Giugiuc).
We choose $A = -1 + -i, B = -1 + i, C = 1 + i, D = 1 - i$, and $P = a + bi$, where $-1 < a, b < 1$.
First of all, $\frac{N-P}{A-P} = -i$, implying $N = (a - b - 1) + i(a + b + 1)$. Similarly, $\frac{Q-P}{D-P} = i$, implying $Q = (a + b + 1) + i(-a + b + 1)$. Further,

$$2[\Delta BPQ] = \begin{vmatrix} a & b & 1 \\ a + b + 1 & -a + b + 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 2 - (a^2 + b^2)$$

Similarly, $2[\Delta PNC] = 2 - (a^2 + b^2)$. $\square$

15. AN INEQUALITY WITH JUST TWO VARIABLES II

Prove that, for positive a, b,

$$\left(\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2}\right)\left(\frac{a+b}{2ab} + \frac{1}{\sqrt{ab}} + \frac{2}{a+b}\right) \leq 5 + 2\left(\frac{a}{b} + \frac{b}{a}\right)$$

Proposed by Daniel Sitaru - Romania

Solution 1 (by Soumava Chakraborty - Kolkata - India).

$$LHS = 3 + \frac{4\sqrt{a}}{a+b} + \frac{a+b}{\sqrt{ab}} + \frac{4ab}{(a+b)^2} + \frac{(a+b)^2}{4ab}$$
$$\leq 3 + 2 + \frac{a+b}{\sqrt{ab}} + 1 + \frac{(a+b)^2}{4ab} = 6 + \frac{a+b}{\sqrt{ab}} + \frac{(a+b)^2}{4ab}.$$

Suffice it to show that

$$6 + \frac{a+b}{\sqrt{ab}} + \frac{(a+b)^2}{4b} \leq 5 + 2\left(\frac{a}{b} + \frac{b}{a}\right)$$

which is equivalent to

$$(1) \qquad 1 + \frac{a+b}{\sqrt{ab}} + \frac{(a+b)^2}{4ab} \leq \frac{2(a^2+b^2)}{ab}$$

Now, by the *GM-HM inequality*, $\sqrt{ab} \geq \frac{2ab}{a+b}$, hence $\frac{a+b}{\sqrt{ab}} \leq \frac{(a+b)^2}{2ab}$.

It follows that

$$(2) \qquad \frac{a+b}{\sqrt{ab}} + 1 + \frac{(a+b)^2}{4ab} \leq 1 + \frac{3(a+b)^2}{4ab}.$$

1 and 2 show that the problem will be solved if we manage to prove

$$\frac{3(a+b)^2 + 4ab}{4ab} \leq \frac{2(a^2+b^2)}{ab}$$

This is equivalent to $3a^2 + 3b^2 + 10ab \leq 8a^2 + 8b^2$, or $5a^2 + 5b^2 - 1ab \geq 0$, which is simply $5(a - b) \geq 0$.

$\square$

Solution 2 (by Kevin Soto Palacios - Huarmey - Peru).

We employ the obvious $\frac{2\sqrt{ab}}{a+b} \leq 1$ and $\frac{4ab}{(a+b)^2} \leq 1$ to obtain

$$LHS \leq 6 + \frac{3(a+b)^2}{4ab}.$$

Suffice it to prove that $6 + \frac{3(a+b)^2}{4ab} \leq 5 + 2\left(\frac{a}{b} + \frac{b}{a}\right)$.

Now note that

$$1 + \frac{3(a+b)^2}{4ab} \leq \frac{5}{2} + \frac{3}{4}\left(\frac{a}{b} + \frac{b}{a}\right),$$

Because $\frac{5}{2} \leq \frac{5}{4}\left(\frac{a}{b} + \frac{b}{a}\right)$.

□

Solution 3 (by Soumava Chakraborty - Kolkata - India).

Using the AM-GM inequality, $\frac{a+b}{2} \geq \sqrt{ab}$ and $\frac{2}{a+b} \leq \frac{1}{\sqrt{ab}}$,

$$LHS \leq \left(\sqrt{ab} + \sqrt{ab} + \frac{a+b}{2}\right)\left(\frac{a+b}{2ab} + \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{ab}}\right) =$$

$$= \left(2\sqrt{ab} + \frac{a+b}{2}\right)\left(\frac{a+b}{2ab} + \frac{2}{\sqrt{ab}}\right) = \frac{a+b}{\sqrt{ab}} + 4 + \frac{(a+b)^2}{4ab} + \frac{a+b}{\sqrt{ab}} =$$

$$= 4 + \frac{(a+b)^2}{4ab} + \frac{2(a+b)}{\sqrt{ab}}.$$

Thus, suffice it to prove that

$$4 + \frac{(a+b)^2}{4ab} + \frac{2(a+b)}{\sqrt{ab}} \leq 5 + 2\left(\frac{a}{b} + \frac{b}{a}\right)$$

which is equivalent to

$$\frac{(a+b)^2}{4ab} + \frac{2(a+b)}{\sqrt{ab}} \leq 1 + 2\frac{a^2+b^2}{ab} = \frac{2a^2+2b^2+ab}{ab}.$$

Now, from $\frac{1}{\sqrt{ab}} \leq \frac{a+b}{2ab}$ we get $\frac{2(a+b)}{\sqrt{ab}} \leq \frac{(a+b)^2}{ab}$, implying

$$\frac{(a+b)^2}{4ab} + \frac{2(a+b)}{\sqrt{ab}} \leq \frac{(a+b)^2}{4ab} + \frac{(a+b)^2}{a} = \frac{5(a+b)^2}{4ab}.$$

Thus, suffice it to prove $\frac{5}{4} \cdot \frac{(a+b)^2}{ab} \leq \frac{2a^2+2b^2+ab}{ab}$, i.e.,

$$5a^2 + 5b^2 + 10ab \leq 8a^2 + 8b^2 + 4ab$$

which reduces to $3(a-b)^2 \geq 0$.

□

Solution 4 (by Abdallah El Farissi - Bechar - Algerie).

Let

$$A=\Big(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\Big)\Big(\frac{a+b}{2ab}+\frac{1}{\sqrt{ab}}+\frac{2}{a+b}\Big)$$
$$=\Big(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\Big)\Big(\frac{2}{a+b}+\frac{1}{\sqrt{ab}}+\frac{a+b}{2ab}\Big)$$
$$=\frac{1}{ab}\Big(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\Big)^2\le\frac{1}{ab}(\sqrt{ab}+a+b)^2=$$
$$=\frac{1}{ab}\Big(ab+2\sqrt{ab}(a+b)+(a+b)^2\Big)\le\frac{1}{ab}\Big(ab+2(a+b)^2\Big)=$$
$$=\frac{1}{ab}\Big(5ab+2(a^2+b^2)\Big)=5+2\Big(\frac{a}{b}+\frac{b}{a}\Big).$$

□

Solution 5 (by Soumava Chakraborty - Kolkata - India).

Define $=\frac{2ab}{a+b}, y=\sqrt{ab}, z=\frac{a+b}{2}$. **We have** $x\le y\le z$ **and** $y^2=xz$.

$$LHS=\Big(\sum_{cycl}x\Big)\Big(\sum_{cycl}\frac{1}{x}\Big)=\frac{(\sum_{cycl}x)(xy+yz+zx)}{xyz}$$
$$=\frac{(\sum_{cycl}x(xy+yz+y^2)}{xyz}=\frac{(x+y+z)^2}{xz}=\frac{(x+y+z)^2}{y^2}$$
$$RHS=\frac{2a^2+2b^2+5ab}{ab}=\frac{2(a+b)^2+ab}{ab}=1+\frac{8z^2}{y^2}.$$

The required inequality is equivalent to $\frac{(x+y+z)^2}{y^2}\le 1+\frac{8z^2}{y^2}$ **which is**

$$(x+y+z)^2-y^2\le 8z^2, \text{ or, } (x+2y+z)(x+z)\le 8z^2.$$

Since $x\le y\le z, x+2y+z\le 4z$ **and** $x+z\le 2z$, **which shows that, indeed,**

$$(x+2y+z)(x+z)\le 8z^2.$$

□

Solution 6 (by Daniel Sitaru - Romania).

We know that, for positive a,b,c,

$$\frac{2ab}{a+b}\le\sqrt{ab}\le\frac{a+b}{2}$$

We'll use ***Schweitzer's inequality***:

$$\left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}\frac{1}{x_k}\right)\le\frac{(m+M)^2n^2}{4mM},$$

where $x_1,\dots,x_n\in[m,M], m>0$.

with $n=3, m=x_1=\frac{2ab}{a+b}, x_2=\sqrt{ab}$, **and** $x_3=\frac{a+b}{2}=M$, **we directly get**

$$A=\Big(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\Big)\Big(\frac{a+b}{2ab}+\frac{1}{\sqrt{ab}}+\frac{2}{a+b}\Big)$$

$$\leq 9\frac{\left(\frac{2ab}{a+b}+\frac{a+b}{2}\right)^2}{4ab} = \frac{9}{4ab}\left[\left(\frac{2ab}{a+b}\right)^2 + 2ab + \left(\frac{a+b}{2}\right)^2\right] =$$

$$\leq \frac{9}{4ab}\left[(\sqrt{ab})^2 + 2ab + \left(\frac{a+b}{2}\right)^2\right] = \frac{9}{4ab}\left[3ab + \left(\frac{a+b}{2}\right)^2\right] =$$

$$= \frac{9}{16ab}\left[14ab + (a^2+b^2)\right] = \frac{63}{8} + \frac{9}{16}\left(\frac{a}{b}+\frac{b}{a}\right).$$

Now, $1 \leq \frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right)$. It then follows that $\frac{23}{8} \leq \frac{23}{16}\left(\frac{a}{b}+\frac{b}{a}\right)$ and, subsequently,

$$\frac{63}{8} + \frac{9}{16}\left(\frac{a}{b}+\frac{b}{a}\right) \leq \frac{40}{8} + \frac{32}{16}\left(\frac{a}{b}+\frac{b}{a}\right) = 5 + 2\left(\frac{a}{b}+\frac{b}{a}\right).$$

□

16. AN INEQUALITY WITH CONSTRAINT VII

If $x, y, z \in \mathbb{R}, x + y - 5z = 0, x^2 + z^2 = 1,$ ***then:*** $|2x + 3y - 5z| \leq \sqrt{101}.$

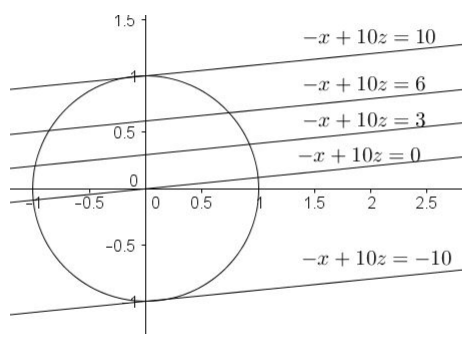
Proposed by Daniel Sitaru - Romania

Solution (by Alexander Bogomolny).

Since $x + y - 5z = 0$, the inequality at hand is equivalent to

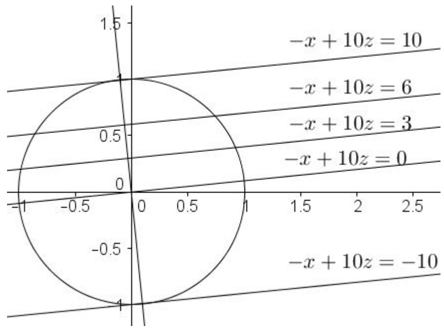
$$|(2x + 3y - 5z) - 3(x + y - 5z)| = |-x + 10z| \leq \sqrt{101}.$$

This is the one I shall prove under the restriction $x^2 + z^2 = 1$. After the fact, y may be found from $x + y - 5z = 0$.
The straight lines $-x + 10z = const$ may or may not meet the circle $x^2 + z^2 = 1$.



I shall employ geometric illustration. The value of $-x + 10z$ which is constant on each of the lines changing monotonically in the direction of their common normal: $(-10, 1)$. The extreme values are attained at the

intersection of $x^2 + z^2 = 1$ with $z = -10x$:



This happens when $x = \pm\frac{1}{\sqrt{101}}, z = \mp\frac{10}{\sqrt{101}}$ such that, at these points, $|-x + 10z| = \sqrt{101}$, which proves the required inequality. □

17. LIMIT OF A RECURSIVE SEQUENCE

Let $k \geq 2$ be a fixed integer; $x_0, x_1, \ldots, x_{k-1}$ complex numbers and

$$x_{n+1} = \frac{1}{k}\sum_{s=0}^{k=1} x_{n-s} \text{ for } n \geq k-1.$$

Find $\lim_{x\to\infty} x_n$.

Proposed by Arkady Alt - USA

Solution (by Leonard Giugiuc).
First, we introduce two lemmas:
Lemma 1
Let $\{a_n\}$ be a sequence of complex numbers such that $a_n \neq 0, n \geq 0$. If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1, \text{ then } \lim_{n\to\infty} a_n = 0.$$

For a proof, observe that Lemma holds for a real-valued sequence, and so does for $\{|a_n|\}$. It follows that $\lim_{n\to\infty}|a_n| = 0$ and, therefore, also $\lim_{n\to\infty} a_n = 0$.
Lemma 2
Let z be a complex number, with $|z| < 1$, and $m \geq 1$ an integer. Then

$$\lim_{n\to\infty} n^m z^n = 0$$

For a proof, assume $z \neq 0$, for, otherwise, there is nothing to prove.
Let $a_n = n^m z^n$. We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)^m \cdot |z|\right] = |z| < 1.$$

Hence, by Lemma 1, $\lim_{n\to\infty} a_n = 0$ i.e., $\lim_{n\to\infty} n^m z^n = 0$, as required. To continue, by the definition, sequence $\{a_n\}$ is a linear reccurence of

order k, with the characteristic polynomial

$$P(x) = kx^k - \sum_{s=0}^{k-1} x^s = (x-1)\sum_{s=0}^{k-1}(s+1)x^s.$$

$$\text{If } f(x) = \sum_{s=0}^{k-1}(s+1)x^s,$$

then $f(1) \neq 0$ so that the multiplicity of 1 as root of $P(x)$ is 1. Also $f(-1) \neq 0$.

Let z be a root of P, other than 1. We'll show that $|z| < 1$. Indeed, $P(z) = 0$ implies

$$kz^k = \sum_{s=0}^{k-1} z^s \text{ and, subsequently, } |kz^k| = \left|\sum_{s=0}^{k-1} z^s\right|$$

Let $|z| = r$. Assume, for a contradiction, that $r \geq 1$. Combined with the above, this would imply

$$kr^k \geq \sum_{s=0}^{k-1} r^s \text{ and, further, } kr^k = \sum_{s=0}^{k-1} r^s \text{ from which } r = 1.$$

Now, we have assumed that $z \neq 1$ and observed that, as root of $P, z \neq -1$. Hence —z— is not real. But the equality

$$\left|\sum_{s=0}^{k-1} z^s\right| = \sum_{s=0}^{k-1} r^s$$

only holds if z is real which is a desired contradiction. Thus, $|z| < 1$.

Let $z_1, z_2, \ldots, z_m$ be the roots of P (other than 1) with multiplicities $s_1, s_2, \ldots, s_m$. The the theory of linear recurrences informs us that there exists a complex number α and polynomials $P_t, t = 1, 2, \ldots, m$, such that $\deg P_t = s_t - 1$ and

$$x_n = \alpha + \sum_{t=1}^{m} P_t(n) \cdot z_t^n, \text{ for all } n \geq 0.$$

$$\text{By Lemma 2, } \lim_{n\to\infty}\left[\sum_{t=1}^{m} P_t(n) \cdot z_t^n\right] = 0, \text{ so that } \lim_{n\to\infty} x_n = \alpha.$$

On the other hand, we can easily verify that

$$\sum_{s=0}^{k-1}(k-s)x_{n-s} = \sum_{s=0}^{k-1}(s+1)x_s.$$

As we pass to the limit, we get

$$\frac{k(k+1)}{2} \cdot \alpha = \sum_{s=0}^{k-1}(s+1)x_s, \text{ such that } \alpha = \frac{2}{k(k+1)}\sum_{s=0}^{k-1}(s+1)x_s.$$

□

18. OPTIMAL QUADRILATERAL INSCRIBED IN A SQUARE

Let $x, y, z, t \in (0,1), \Omega = \sum_{cycl} \sqrt{x^2 + (1-y)^2}$.
Find $m = \inf \Omega$ ***and*** $M = \sum \Omega$.

Proposed by Daniel Sitaru - Romania

Solution (by Leonard Giugiuc - Romania).
We'll use complex numbers.
The function $f : [0,1]^4 \to \mathbb{R}$, **defined by** $f(x,y,z,t) = \sum_{cycl} \sqrt{x^2 + (1-y)^2}$, **is continuous and thus attains its extrema on** $[0,1]^4$. **The extremal values on** $[0,1]^4$ **will supply its supremum and infimum on** $(0,1)^4$.
Let's look for infimum first. $\sqrt{x^2 + (1-y)^2} = |x + i(1-y)|$, **etc. By the triangle inequality,**

$$\sum_{cycl} \sqrt{x^2 + (1-y)^2} = \sum_{cycl} |x + i(i-y)|$$
$$= |(x+y+z+t) + i[4 - (x+y+z+t)]|$$
$$= |k + i(4-k)|$$

where $k = x + y + z + t, 0 \le k \le 4$.
But $|k + i(4-k)| = \sqrt{k^2 + (4-k)^2} \ge 2\sqrt{2}$ **which is attained with** $k = 2$, **in particular when** $x = y = z = t = \frac{1}{2}$. **Thus,** $m = \inf \Omega = 2\sqrt{2}$.
To find the supremum, observe that, for real a **and** b, $\sqrt{a^2 + b^2} \le |a| + |b|$, **with equality only when** $ab = 0$. **In our case,**

$$\sum_{cycl} \sqrt{x^2 + (1-y)^2} \le \sum_{cycl} (x + 1 - y) = 4$$

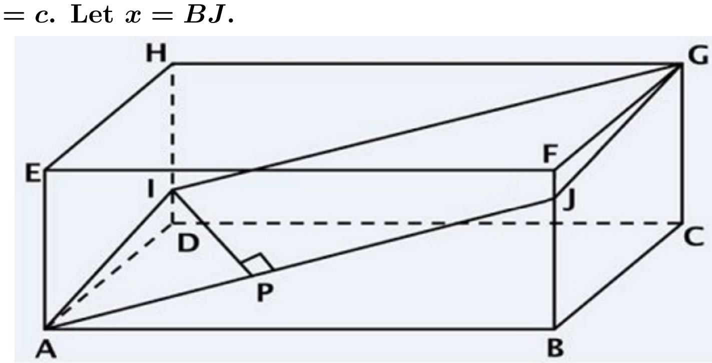
This is attained with $x = y = z = t = 0$. **Thus,** $m = \sup \Omega = 4$. $\square$

19. OPTIMAL IN PARALLELEPIPED

Given $a, b, c > 0$. ***Find minimum and maximum of***
$f(x) = \sqrt{a^2 + x^2} + \sqrt{b^2 + (c-x)^2}$ ***on interval*** $[0, c]$.

Proposed by Daniel Sitaru, Leonard Giugiuc - Romania

Solution 1 (by Daniel Sitaru).
Consider a parallelepiped (the diagram below), with $AB = a, BC = b$, **and** $BF = c$. **Let** $x = BJ$.

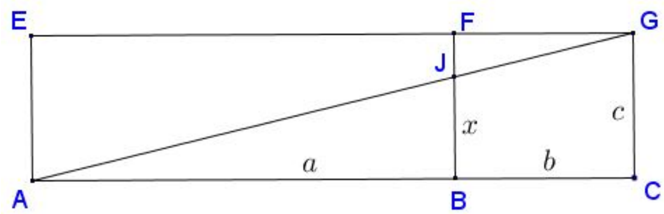


The diagram reveals several relationships between the elements:

$$AJ \leq AF, \text{ i.e., } \sqrt{a^2 + x^2} \leq \sqrt{a^2 + c^2}$$
$$GJ \leq GB = \sqrt{b^2 + c^2}, FJ = c - x,$$
$$GJ \leq GB, \text{ i.e., } = \sqrt{b^2 + (c-x)^2} \leq \sqrt{b^2 + c^2}.$$

If $AF + FG = \Omega_1$ and $AB + BG = \Omega_2$ then, $\max f(x) = \max\{\Omega_1, \Omega_2\}$

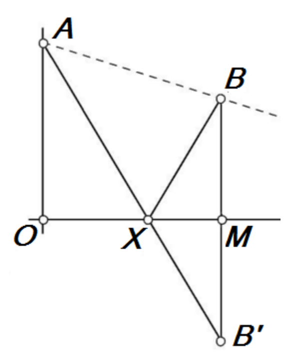
Consider just two faces of the parallelepiped:



$f(x)$ is exactly the path $AJ + JG$ which attains it minimum value only of AJG is a straight line. This happens exactly when $\frac{x}{c} = \frac{a}{a+b}$, so that $x = \frac{ac}{a+b}$.
Note that the algebraic formulation conceals an old problem that requires from a spider sitting in the vertex A to reach the fly in a vertex G in the shortest way possible. The second reveals another geometric interpretation. □

Solution 2 (by Francisco Javier García Capitán).
This solution is due to Francisco Javier García Capitán and it draws on the famous *Heron's problem.*

We observe that $f(x)$ is the distance $AX + BX$ where $A = (0, a)$, $B = (c, b)$, and $X = (0, x)$.



Since $AX + BX = AX + B'X$, where B' is the reflections of B in the x axis, $f(x)$ will be minimum when A, X and B' are collinear.

To find x we can use similar triangles AOX and $B'MX$ (with $M = (c, 0)$):$\frac{OX}{OA} = \frac{MX}{MB'}$, or $\frac{x}{a} = \frac{c-x}{b}$, from which $a = \frac{ac}{a+b}$. □

20. PROPORTIONS AND THE INCENTER

I ***is the incenter in*** ΔABC***. Prove that*** $\frac{BI}{CI}=\frac{AC}{AB}$ ***implies*** $\angle ABC=\angle ACB$***.***

Proposed by Miguel Ochoa Sanchez - Peru

Solution 1 (by Claudia Nănuţi, Diana Trăilescu, Daniel Sitaru, Leo Giugiuc).
We know that $BI=\frac{2ac}{a+b+c}\cdot\cos\frac{B}{2}$ **and** $CI=\frac{2ab}{a+b+c}\cdot\cos\frac{C}{2}$**, implying** $\frac{BI}{CI}=\frac{c}{b}\cdot\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}$**. It follows that** $\frac{c}{b}\cdot\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}=\frac{b}{c}$**, or** $b^2\cdot\cos\frac{c}{2}=c^2\cdot\cos\frac{B}{2}$**.**
By the *Law of Sines*,

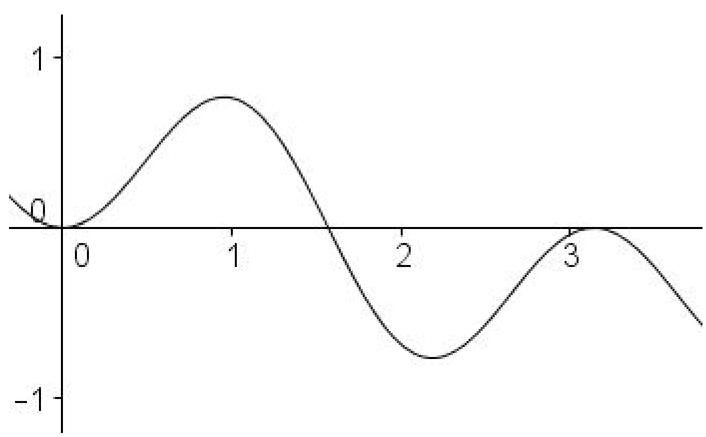
$$16R^2\cdot\sin^2\frac{B}{2}\cos^2\frac{B}{2}\cos\frac{C}{2}=16R^2\cdot\sin^2\frac{C}{2}\cos^2\frac{C}{2}\cos\frac{B}{2}$$

which simplifies to $16\sin^2\frac{B}{2}\cos\frac{C}{2}=\sin^2\frac{C}{2}\cos\frac{C}{2}$**.**
If $B\neq C$**, then** $\cos\frac{B}{2}\neq\cos\frac{C}{2}$**. Let** $\cos\frac{B}{2}=x$ **and** $\cos\frac{C}{2}=y$**. We have** $(1-x^2)x=(1-y^2)y$**, or,** $y^3-x^3=y-x$**. Since** $x\neq y$**, this simplifies to** $x^2+xy+y^2=1$**, or,** $xy=1-(x^2+y^2)$**.**
But $0<xy$ **implies** $x^2y^2<x^2y^2+xy$ **and subsequently,**

$$xy<\sqrt{x^2y^2+xy}=\sqrt{x^2y^2+1-(x^2+y^2)}=\sqrt{(x^2-1)(y^2-1)}$$

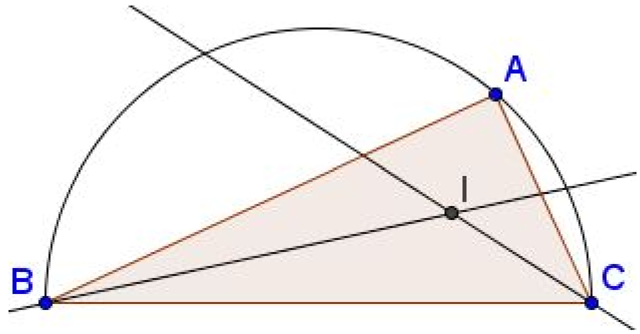
which says that $\cos\frac{B}{C}\cos\frac{C}{2}<\sin\frac{B}{2}\sin\frac{C}{2}$**, i.e.,** $\cos\frac{B+C}{2}<0$**, and, finally,** $\frac{B+C}{2}>\frac{\pi}{2}$**, which is absurd. Hence,** $\cos\frac{B}{2}=\cos\frac{C}{2}$ **and** $B=C$**.** □

Solution 2 (by Alexander Bogomolny).
Set $\angle ABC=2\beta,\angle ACB=2\theta$**.**
By the *Law of Sines* in ΔABC**,** $\frac{AC}{AB}=\frac{\sin 2\beta}{\sin 2\theta}$**. By the Law of Sines in** ΔIBC**,** $\frac{BI}{CI}=\frac{\sin\theta}{\sin\beta}$**. Thus we have (*)** $\sin 2\beta\cdot\sin\beta=\sin 2\theta\cdot\sin\theta$ **from which we should be able to conlude that** $\beta=\theta$**. It is clear that we may assume** $0<\beta,\theta<90^\circ$**. But on the interval** $[0,90^\circ]$ **function** $f(x)=\sin 2x\cdot\sin x$ **is experiencing a hump symmetric in** $x=45^\circ$**.**



It follows that (*) may only hold when, say, $\beta=45^\circ-\omega$ **and** $\theta=45^\circ+\omega$**, for** $-45^\circ\leq\omega\leq 45^\circ$**. Thus the only possibility to have** $f(\beta)=f(\theta)$ **is**

when $\beta + \theta = 90°$, making $\angle BAC = 90°$.



It is now intuitively clear that if $\frac{BI}{CI} > 1$, then $\frac{AC}{AB} < 1$ and vice versa. A rigorous justification makes use of *Leo's Lemma*.
Assume $0 < \beta < \theta < 90°$; then also $0 < 2\beta < 2\theta < 180°$. Then $\sin\beta < \sin\theta$ and, by Leo's Lemma also $\sin 2\beta < \sin 2\theta$ but then $\frac{BI}{CI} = \frac{AC}{AB}$ could not hold. It follows that $\beta = \theta$. $\square$

21. SCALAR PRODUCT OPTIMIZATION

Let x, y, z, b be real numbers such that $(x+1)^2 + y^2 = 1$ and $(a-2)^2 + b^2 = 4$. Find the extreme values of the expression $ax + by$.

Proposed by Leonard Giugiuc - Romania

Solution 1 (by Claudia Nănuţi, Diana Trăilescu, Daniel Sitaru, Leo Giugiuc).
The minimum is -8 because the two given circles are externally tangent at the origin and $ax + by$ is the scalar product of two vectors with the end points on the circles. Hence $2 * 4 * (-1) = -8$. To determine the maximum, consider the points $A(x, y), B(a, b)$, and $O(0, 0)$. Note that $ax + by = \overrightarrow{OA} \cdot \overrightarrow{OB} = OA \cdot OB \cdot \cos\angle AOB$. Hence, to achieve maximum, it is necessary that A and B lie on the corresponding circles. Observe also that $\angle AOB$ has to be acute which allows us to assume that the two points are in the upper half-plane. Thus, $A = (-1 + \cos u, \sin u)$ and $B = (2 + 2\cos t, 2\sin t)$, with $0 < t, u < \pi$. Setting $v = \angle AOB$, we get on one hand

$$\frac{A-O}{B-O} = \frac{OA}{OB}(\cos v + i \sin v)$$

while, on the other,

$$\frac{A-O}{B-O} = \frac{-1\cos u + i \sin u}{2 + 2\cos t + 2i \sin t} = \frac{i \sin\frac{u}{2}(\cos\frac{u}{2} + i \sin\frac{u}{2})}{2\cos\frac{t}{2}(\cot\frac{t}{2} + i \sin\frac{t}{2})} =$$

$$= \frac{\sin\frac{u}{2}}{2\cos\frac{t}{2}}\Big(\cos\frac{\pi + u - t}{2} + i \sin\frac{\pi + u - t}{2}\Big)$$

Since $\frac{\sin\frac{u}{2}}{2\cos\frac{t}{2}} > 0$, we may identify the angles: $v = \frac{\pi+u-t}{2}$ and so $\cos v = \sin\frac{t-u}{2}$, implying, in particular, that $t > u$. Substituting,

$$ax + by = OA \cdot OB \cdot \cos v = 8\sin\frac{u}{2}\cos\frac{t}{2}\sin\frac{t-u}{2}$$

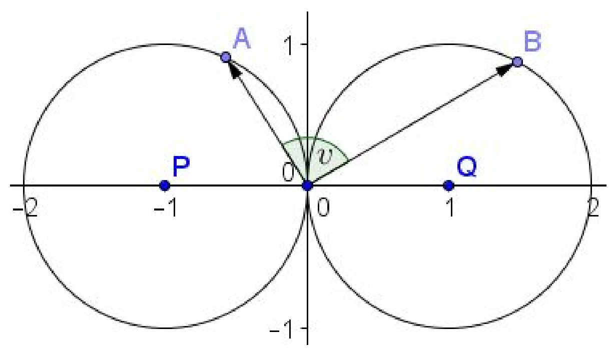
$$= 8\sin\frac{u}{2}\sin\frac{\pi-t}{2}\sin\frac{t-u}{2}.$$

But, $0 < \frac{u}{2}, \frac{\pi-t}{2}, \frac{t-u}{2}$ and $\frac{u}{2} + \frac{\pi-t}{2} + \frac{t-u}{2} = \frac{\pi}{2}$. By Jensen's inequality,

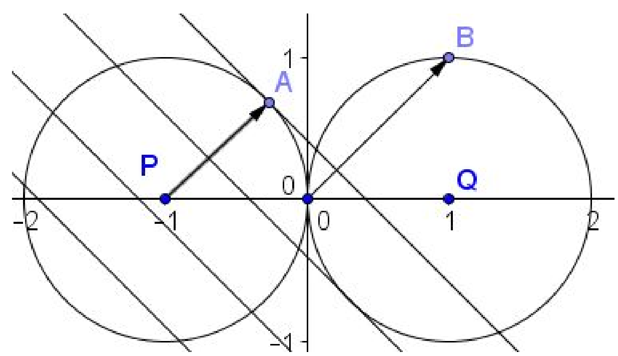
$$8\sin\frac{u}{2}\sin\frac{\pi-t}{2}\sin\frac{t-u}{2} \le 1$$

For equality we need $u = 60°$ and $t = 120°$. □

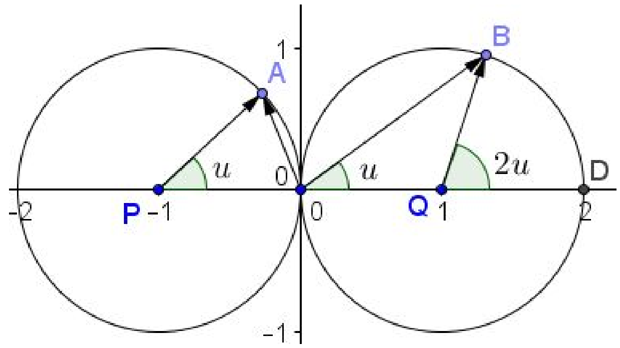
Solution 2 (by Alexander Bogomolny).
Note first that the problem could be generalized. Indeed, assuming that x, y, z, b satisfy $(x+r)^2 + y^2 = r^2$ and $(a-s)^2 + b^2 = s^2$, $(r, s > 0), ax + by = rs(\cos v + i\sin v)$, and, at the extremes of this expression, angle v is the same, independent of the radii r, s of the circles. So, below I shall assume $r = s = 1$ and use the same notations A, B, O as the first solution. Let $P(-0, 1)$ and $Q(1, 0)$ be the centers of the two circles



For a fixed point $B(a, b)$, the scalar product $ax + by$ is constant on straight lines perpendicular to $\overrightarrow{OB}$. The maximum value (for a given B) is attained for the line tangent to $(x+1)^2 + y^2 = 1$. If A is the point of tangency, PA is perpendicular to that line, and therefore, parallel to OB.



It follows that, if $A = (-1 + \cos u, \sin u)$ then $B = (1 + \cos 2u, \sin 2u)$:



It then follows that

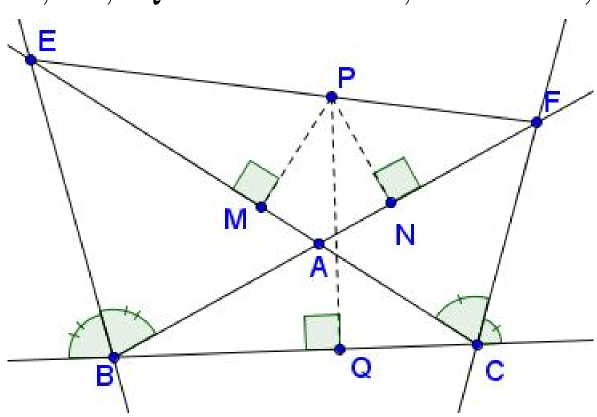
$$ax + by = (-1 + \cos u)(1 + \cos 2u) + \sin u \sin 2u$$
$$= -1 + \cos u - \cos 2u + (\cos 2u \cos u + \sin 2u \sin u)$$
$$= -1 + \cos u - (2\cos^2 u - 1) + \cos u$$
$$= -2\cos^2 u + 2\cos u = -2\cos u(\cos u - 1).$$

The parabola $f(t) = -2t(t-1)$ attains its maximum for $t = \frac{1}{2}$. Therefore, the maximum of $ax+by$ is attained when $\cos u = \frac{1}{2}$, i.e., at $u = 60°$, implying $2u = 120°$, and also $v = 60°$.

With these, the maximum of $ax + by$ equals $\frac{1}{2}$, or in the general case, $\frac{rs}{2}$. The minimum is rather obviously $(-2r)(2s) = -4rs$. □

22. THALES ON ANGLE BISECTORS

Let BE and CF be external angle bisectors in ΔABC; E is on AC or its extension, F is on AB or its extension. From point P on EF perpendiculars PM, PN, PQ are drawn to AC, AB and BC, respectively.



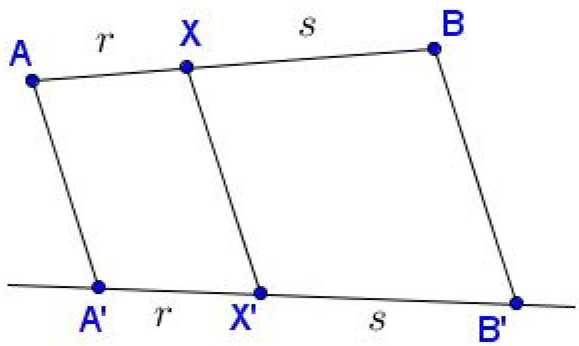
Prove that

$$PM + PN = PQ.$$

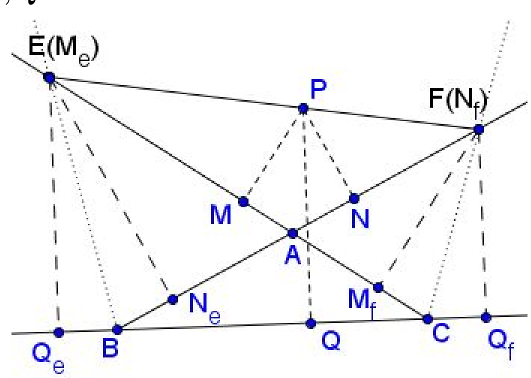
Proposed by Miguel Ochoa Sanchez - Peru

Solution 1 (by Alexander Bogomolny).
Let's first formulate simple *affine* lemma that is based on *Thales' Theorem*:

Lemma
Assume A, X, B are collinear as are A', X', B'.
Suppose $AA' \parallel XX' \parallel BB'$ and $AX : XB = r : S$.



Then $XX' = \frac{S}{r+s}AA' + \frac{r}{r+s}BB'$.
To continue with the solution assume $EP : PF = r : s, r + s = 1$, and consider two extreme positions of P: one when $P = E$, the other when $P = F$. Indices, $_e$ and $_f$ are added to distinguish between different endpoints M, N, Q.

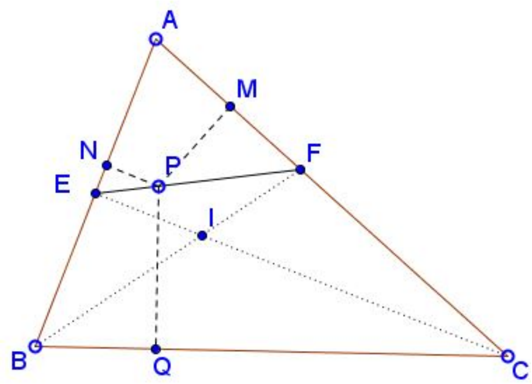


Note that $M_e = E$ and $N_f = F$, such that $EM_e = FN_f = 0$ and, since BE and CF are angle bisectors, $EN_e = EQ_e$ and $FM_f = FQ_f$. According to the lemma, $PM = sEM_e + rFM_f = rFM_f$,
$PN = sEN_e + rFN_f = sEN_e$,
$PQ = sEQ_e + rFQ_f$, so that
$PQ = sEQ_e + rFQ_f = sEN_e + rFM_f = PM + PN$. □

Solution 2 (by Claudia Nănuţi, Diana Trăilescu, Daniel Sitaru, Leo Giugiuc).
Since BC is the greatest side, we may choose $A(0,1), B(-b,0), C(c,0)$, where $b, c > 0$. We'll find the locus of point $X(n,m)$ such that $d(X, AB) + d(X, A) = d(X, B)$ and X is not in any open half - planes

$(B, AC), (C, AB)$ or (A, BC).
Under our assumptions, $-m + bn - b \geq 0, m + cn - c \geq 0$, and $n \geq 0$.
We have
$d(X, AB) = \frac{-m+bn-b}{\sqrt{b^2+1}}, d(X, AC) = \frac{m+cn-c}{\sqrt{c^2+1}}$, and $d(X, BC) = n$. We thus seek X for which $\frac{-m+bn-b}{\sqrt{b^2+1}} + \frac{m+bn-c}{\sqrt{c^2+1}} = n$, implying that the sought locus is a straight line. By the *external bisector theorem*,
$FA/FB = CA/CB$, such that F is defined by $\left(\frac{b\sqrt{c^2+1}}{b+c-\sqrt{c^2+1}}, \frac{b+c}{b+c-\sqrt{c^2+1}}\right)$.
By replacing we get immediately that $F \in l$. Similarly, $E \in l$. Hence, also $P \in EF$. We are done. □

Extra (by Alexander Bogomolny).
It was observed by Dao Thanh Oai that the same arguments applies in the case of internal bisectors:



$$PM + PN = PQ.$$

23. DIVIDE AND CONQUER IN CYCLIC SUMS

Let $a, b, c > 0$. *Prove that*
$$\sum_{cycl} c\left(\frac{4a}{b^2} + \frac{3b}{a^2}\right) \geq 12 + 3\sum_{cycl} \frac{a}{b}$$

Proposed by Daniel Sitaru - Romania

Solution (by Soumava Chakraborty - Kolkata - India).
First of all,
$$\sum_{cycl} c\left(\frac{4a}{b^2} + \frac{3b}{a^2}\right) = \sum_{cycl} \frac{4ac}{b^2} + \sum_{cycl} \frac{3bc}{a^2} =$$
$$= 4\sum_{cycl} \frac{ab}{c^2} + 3\sum_{cycl} \frac{ab}{c^2} = 7\sum_{cycl} \frac{ab}{c^2}.$$
By the *AM-GM inequality*,
$$4\sum_{cycl} \frac{ab}{c^2} \geq 4 \cdot 3\sqrt[3]{\frac{ab}{c^2} \cdot \frac{bc}{a^2} \cdot \frac{ca}{b^2}} = 12\sqrt[3]{1} = 12.$$

On the other hand,

$$\frac{ab}{c^2}+\frac{ab}{c^2}+\frac{ca}{b^2}\geq\sqrt[3]{\frac{a^3}{c^3}}=3\frac{a}{c},$$

$$\frac{bc}{a^2}+\frac{bc}{a^2}+\frac{ab}{c^2}\geq 3\sqrt[3]{\frac{b^3}{a^3}}=3\frac{b}{a},$$

$$\frac{ca}{b^2}+\frac{ca}{b^2}+\frac{bc}{a^2}\geq 3\sqrt[3]{\frac{c^3}{b^3}}=3\frac{c}{b},$$

Adding these to (1) we get

$$7\sum_{cycl}\frac{ab}{c^2}=4\sum_{cycl}\frac{ab}{c^2}+\sum_{cycl}\frac{ab}{c^2}\geq 12+3\Big(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\Big).$$

□

Acknowledgment (by Alexander Bogomolny).
The problem above has been kindly posted at *CutTheKnotMath facebook page* by Daniel Sitaru (from his book *Math Accent*), along with a solution by Soumva Chakraborty.

24. A CYCLIC INEQUALITY IN THREE VARIABLES

Let $a, b, c > 0$. Prove that

$$\frac{a^3}{b^2(5a+2b)}+\frac{b^3}{c^2(5b+2c)}+\frac{c^3}{a^2(5c+2a)}\geq\frac{3}{7}$$

Proposed by Daniel Sitaru - Romania

Solution 1 (by Imad Zak - Saida - Lebanon).
Consider $(x)=\frac{1}{x(5+2x)}+\frac{9}{49}\ln x-\frac{1}{7}$.

$$f'(x)=\frac{(x-1)(36x^2+216x+245)}{49x^2(2x+5)^2}$$

$f'(x)<0$, for $x<1$, and $f'(x)>0$, for $x>1$. Since $f(1)=0, f(x)\geq 0$, for $x>0$. Now, let $x=\frac{b}{a}, y=\frac{c}{b}, z=\frac{a}{c}.xyz=1$. We have to show that

$$\sum_{cycl}g(x)\geq\frac{3}{7}$$

where $(x)=\frac{1}{x(5+2x)}$.

$$\sum_{cycl}g(x)=\sum_{cycl}\Big(f(x)-\frac{9}{49}\ln x+\frac{1}{7}\Big)$$

$$=\sum_{cycl}\Big(f(x)\Big)-\frac{9}{49}\ln xyz+\frac{3}{7}=\sum_{cycl}\Big(f(x)\Big)+\frac{3}{7}\geq 0+\frac{3}{7}.$$

Equality is attained for $x=y=z=1$, i.e., $a=b=c$. □

Solution 2 (by Kevin Soto Palacios - Huarmey - Peru).
By the *Cauchy - Schwarz inequality*,

$$\sum_{cycl}\frac{a^3}{b^2(5a+2b)}\sum_{cycl}[ab^2(5a+2b)]\geq(a^2+b^2+c^2)^2.$$

Thus, suffice it so show that

$$\sum_{cycl}\frac{a^3}{b^2(5a+2b)}\geq\frac{(a^2+b^2+c^2)^2}{\sum_{cycl}[ab^2(5a+2b)]}\geq\frac{3}{7}$$

This is equivalent to

$$7(a^2+b^2+c^2)^2\geq 3\Big(5\sum_{cycl}(a^2b^2+2ab^3+2bc^3+2ca^3)\Big),$$

which can be written as

$$6\sum_{cycl}a^4+\sum_{cycl}a^4+14\sum_{cycl}a^2b^2\geq 15\sum_{cycl}(a^2b^2+6ab^3+6bc^3+6ca^3).$$

With the *AM-GM inequality*, we see that

$$a^4+a^4+a^4+c^4\geq 4a^3c,$$

$$b^4+b^4+b^4+a^4\geq 4b^3a,$$

$$c^4+c^4+c^4+b^4\geq 4c^3b.$$

And summing up these (times 6) and $a^4+b^4+c^4\geq a^2b^2+b^2c^2+c^2a^2$, *which we know is true*, we obtain

$$a^4+b^4+c^4\geq ab^3+bc^3+ca^3.$$

□

Solution 3 (by Soumitra Mandal - nickname Diego Alvariz - Chandar Nagore - India).

By *Radon's Inequality*, then the *Cauchy - Schwarz inequality*, and later, the *AM-GM inequality*,

$$\sum_{cycl}\frac{a^3}{b^2(5a+2b)}\geq\sum_{cycl}\frac{(a+b+c)^3}{(\sum_{cycl}\sqrt{b}c\det\sqrt{5ab+2b^2})^2}$$

$$\geq\frac{(a+b+c)^3}{\left(\sqrt{(a+b+c)\sum_{cycl}(2b^2+5ab)}\right)^2}$$

$$\geq\frac{(a+b+c)^3}{\left(\sqrt{(a+b+c)[2(a+b+c)^2+ab+bc+ca]}\right)^2}$$

$$\geq\frac{(a+b+c)^3}{\left(\sqrt{(a+b+c)[2(a+b+c)^2+\frac{1}{3}(a+b+c)^2]}\right)^2}=\frac{3}{7}$$

□

Solution 4 (by Daniel Sitaru - Romania).
Define $f:(0,\infty)\to\mathbb{R}$, by $f(x)=\frac{x^3}{2+5x}$. Then

$$f'(x)=\frac{3x^2(2+5x)-x^3\cdot 5}{(2+5x)^2}=\frac{6x^2+10x^3}{(2+5x)^2}>0.$$

Thus the function is strictly increasing. So, for any positive x_1,x_2,x_3,

$$f\Big(\sqrt[3]{x_1x_2x_3}\Big)\le f\Big(\frac{x_1+x_2+x_3}{3}\Big)$$

Further,

$$f''(x)=\frac{50x^3+10x^2+4x}{(2+5x)^4}>0,$$

making the function *convex*. By *Jensen's inequality* then,

$$f\Big(\sqrt[3]{x_1x_2x_3}\Big)\le f\Big(\frac{x_1+x_2+x_3}{3}\Big)\le\frac{1}{3}\Big[f(x_1)+f(x_2)+f(x_3)\Big].$$

Set $x_1=\frac{a}{b},x_2=\frac{b}{c},x_3=\frac{c}{a}$ to obtain

$$f\left(\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}}\right)\le\frac{1}{3}\left[f\Big(\frac{a}{b}\Big)+f\Big(\frac{b}{c}\Big)+f\Big(\frac{c}{a}\Big)\right].$$

Explicitly,

$$\frac{1}{7}=f(1)\le\frac{1}{3}\sum_{cycl}\frac{(\frac{a}{b})^3}{2+\frac{5a}{b}}=\frac{1}{3}\sum_{cycl}\frac{a^3}{2b^3+5ab^2}.$$

Hence, the required inequality. □

Solution 5 (by Hung Nguyen Viet - Hanoi - Vietnam).
By the *Cauchy - Schwarz inequality*,

$$\sum_{cycl}\frac{a^3}{b^2(2b+5a)}=\sum_{cycl}\frac{a^4}{ab^2(5a+2b)}$$

$$\ge\frac{(a^2+b^2+c^2)^2}{5(a^2b^2+b^2c^2+c^2a^2)+2(ab^3+bc^3+ca^3)}.$$

It remains to prove that

$$7(a^2+b^2+c^2)^2\ge 15(a^2b^2+b^2c^2+c^2a^2)+6(ab^3+bc^3+ca^3).$$

This is equivalent to

$$7(a^4+b^4+c^4)\ge(a^2b^2+b^2c^2+c^2a^2)+6(ab^3+bc^3+ca^3).$$

Since, $a^4+b^4+c^4\ge a^2b^2+b^2c^2+c^2a^2$, suffice it to show that

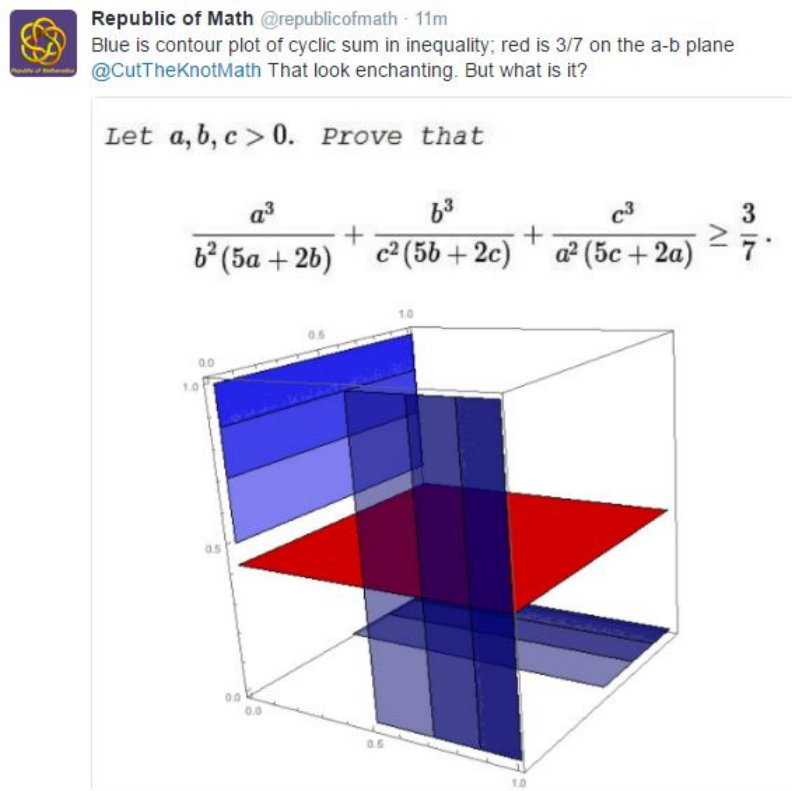
$$a^4+b^4+c^4\ge ab^3+bc^3+ca^3.$$

But this follows from summing up the inequalities below

$$a^4+b^4+b^4+b^4\ge 4ab^3,$$
$$b^4+c^4+c^4+c^4\ge 4bc^3,$$
$$c^4+a^4+a^4+a^4\ge 4ca^3.$$

The proof is complete. □

Illustration



Acknowledgment (by Alexander Bogomolny).
The problem above has been posted on the *CutTheKnotMath facebook page* by Daniel Sitaru. The problem came from his book *Math Storm.* Solution 1 is by Imad Zak (Lebanon); Solution 2 is by Kevin Soto Palacios (Peru); Solution 3 is by Diego Alvariz (India); Solution 4 is by Daniel Sitaru (Romania); Solution 5 is by Hung Nguyen Viet (Vietnam).

25. AN INEQUALITY WITH ABSOLUTE VALUES

Prove that, for $a,b,c\in(-1,1)$,

$$\sum_{cycl}\frac{|a|+|b|}{1-c^2}\geq\sum_{cycl}\frac{2|a|}{1-bc}$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Daniel Sitaru).
Since $a\in(-1,1), a^2<1, 1-a^2>0$, $\frac{1}{1-a^2}>0$. Similarly, $\frac{1}{1-b^2}>0$
By the *AM-GM inequality*,

$$\frac{1}{1-a^2}+\frac{1}{1-b^2}\geq 2\sqrt{\frac{1}{1-a^2}\cdot\frac{1}{1-b^2}}. \tag{1}$$

However,

$$(1-ab)^2 = 1-2ab+a^2b^2 \ge 1-(a^2+b^2)-a^2b^2 = (1-a^2)(1-b^2),$$

so that $\dfrac{1}{(1-a^2)(1-b^2)} \ge \dfrac{1}{(1-ab)^2}$**. This, together with 1, yields**

$$\frac{1}{1-a^2}+\frac{1}{1-b^2} \ge 2\sqrt{\frac{1}{(1-ab)^2}} = \frac{2}{1-ab}$$

So too,

$$\frac{|c|}{1-a^2}+\frac{|c|}{1-b^2} \ge \frac{2|c|}{1-ab}$$

Similarly,

$$\frac{|a|}{1-b^2}+\frac{|a|}{1-c^2} \ge \frac{2|a|}{1-bc}$$

$$\frac{|b|}{1-c^2}+\frac{|b|}{1-a^2} \ge \frac{2|b|}{1-ca}.$$

Adding the three gives the required inequality. The equality is achieved for $a=b=c$**.**

□

Proof 2 (by Kevin Soto Palacios - Huarmey - Peru).

Using *Bergström inquality* and, subsequently, the obvious $b^2+c^2 \ge 2bc$**,**

$$\frac{1}{1-b^2}+\frac{1}{1-c^2} \ge \frac{(1+1)^2}{2-b^2-c^2} \ge \frac{4}{2-2bc} = \frac{2}{1-bc}$$

so that

$$\frac{|a|}{1-b^2}+\frac{|a|}{1-c^2} \ge \frac{2|a|}{1-bc}.$$

Similarly,

$$\frac{|a|}{1-c^2}+\frac{|b|}{1-a^2} \ge \frac{2|b|}{1-ca},$$

$$\frac{|c|}{1-a^2}+\frac{|c|}{1-b^2} \ge \frac{2|c|}{1-ab}.$$

Adding the three gives the required inequality. The equality is achieved for $a=b=c$**.**

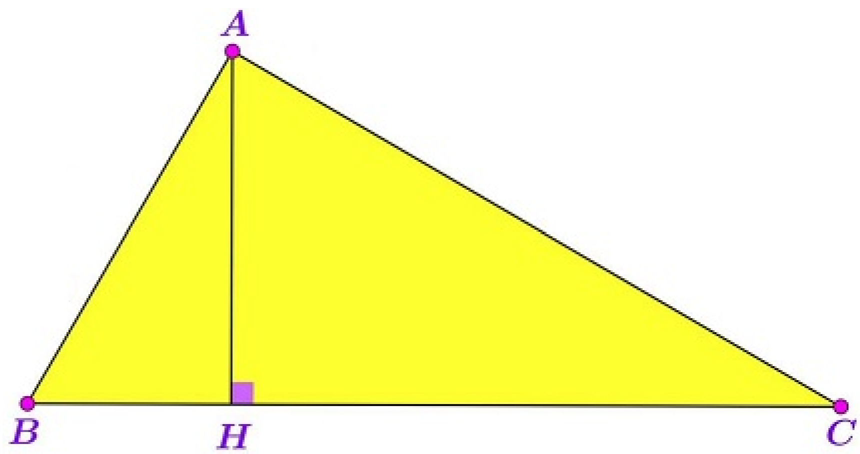
□

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly posted the above problem (from his book *Math Accent*), with a solution (Proof 1), at the *CutTheKnotMathfacebook page*. He later added another solution (Proof 2) by Kevin Soto Palacios.

26. PYTHAGOREAN PERIMETER THEOREM

Point H is the foot of the altitude in ΔABC to the side BC.



Let $p(\Delta)$ be the perimeter of triangle Δ. Prove that

$$\Big(p(ABH)\Big)^2 + \Big(p(ACH)\Big)^2 = \Big(p(ABC)\Big)^2$$

if $\angle BAC = 90^\circ$

Proposed by Miguel Ochoa Sanchez - Peru

Proof (by Claudia Nănuţi, Diana Trăilescu, Daniel Sitaru, Leo Giugiuc).
We choose $H = (0,0), A = (0,1), B = (-b,0), C = (c,0)$, with $b, c > 0$. We have $BH = b, CH = c, AH = 1, BC = b + c, AB = \sqrt{b^2+1}$, $AC = \sqrt{c^2+1}$. We need to prove that the conditions $AC \perp AB$ is equivalent to

$$\Big[1 + (b + \sqrt{b^2+1})\Big]^2 + \Big[1 + (c + \sqrt{c^2+1})\Big]^2 =$$

$$= \Big[(b + \sqrt{b^2+1}) + (c + \sqrt{c^2+1}\Big]^2$$

The latter is transformed into

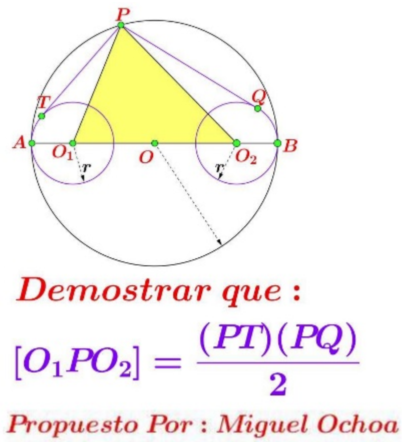
$$1 + 2(b + \sqrt{b^2+1}) + 1 + 2(c + \sqrt{c^2+1}) = 2(b + \sqrt{b^2+1})(c + \sqrt{c^2+1}),$$

$$\text{or, } 1 + (b + \sqrt{b^2+1}) + (c + \sqrt{c^2+1}) = (b + \sqrt{b^2+1})(c + \sqrt{c^2+1}).$$

Introduce $x, y > 1$ such that $b = \frac{x^2-1}{2x}$ and $c = \frac{y^2-1}{2y}$. Then $b + \sqrt{b^2+1} = x$ and $c + \sqrt{c^2+1} = y$. So that the identity at hand becomes $1 + x + y = xy$.
To make it clear, the problem has been reduced to showing that the condition $AC \perp AB$ is euqivalent to $bc = 1$ which, in turn, is equivalent to $(x^2-1)(y^2-1) = 4xy$, and this is algebraically manipulated into $(xy-1)^2 = (x+y)^2$. Since $x, y > 1, xy - 1 = x + y$, as required. $\square$

27. THREE CIRCLES AND AREA

Points O_1 and O_2 lie on the diameter AB of circle (O) . Circle $O_1(r)$ is tangent to (O) at $A, O_2(r)$ is tangent to (O) at B , for some $r > 0$.P is on (O); PT* is tangent to *$O_1(r), PQ$ is tangent to $O_2(r)$.
Prove that $\frac{PT \cdot PQ}{2} = [\Delta O_1PO_2]$, where $[F]$ denotes the area of shape F.

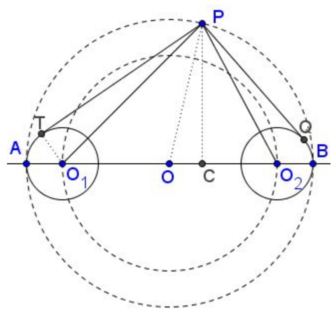


Proposed by Miguel Ochoa Sanchez - Peru

Solution by Claudia Nănuţi, Diana Trăilescu, Daniel Sitaru, Leo Giugiuc.
Assume (O) is described by the equation $x^2 + y^2 = 1; A = (-1, 0), B = (1, 0)$,
$P = (\cos t, \sin t)$, with $t \in (0, \pi)$. Obviously, $O_1 = (-1 + r, 0)$ and $O_2 = (1 - r, 0)$. From here

$$[\Delta O_1PO_2] = \frac{1}{2}O_1O_2 \cdot \sin t = \frac{2 - 2r}{2} \sin t = (1 - r) \sin t$$

On the other hand, by the *Pythagorean theorem* in triangles PO_1T and PCO_1,



or the *Power of a Point* theorem,

$$O_1P^2 - r^2 = PT^2 = 2(1 - r)(1 + \cos t), \text{ or } PT = \sqrt{2(1 - r)(1 + \cos t)}.$$

Similarly, $PQ = \sqrt{2(1 - r)(1 - \cos t)}$.
It thus follows that $PR \cdot PQ = 2(1 - r) \sin t$, as expected. □

28. A ONE-SIDED INEQUALITY IN TRIANGLE

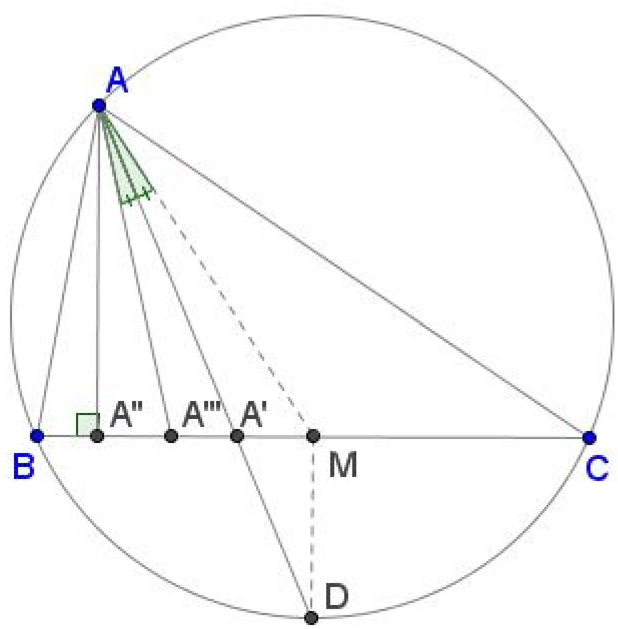
Prove that in any acute ΔABC the following inequality holds:

$$BA' \cdot CB' \cdot AC' + BA'' \cdot CB'' \cdot AC'' + BA''' \cdot CB''' \cdot AC''' < \frac{3abc}{8},$$

where **AA', BB', CC' are the angle bisectors, AA'', BB'', CC'' are the altitudes, and AA''', BB''', CC''' the symmedians; a, b, c the side lengths of triangle.**

Proposed by Daniel Sitaru - Romania

Proof by (Daniel Sitaru - Romania).
In an acute triangle, the angle bisector, the altitude, and the symmedian from the same vertex, all fall on the same side from the midpoint of the opposite side.



For this reason, we have

$$BA' < \frac{a}{2}, CB' < \frac{b}{2}, AC' < \frac{c}{2},$$

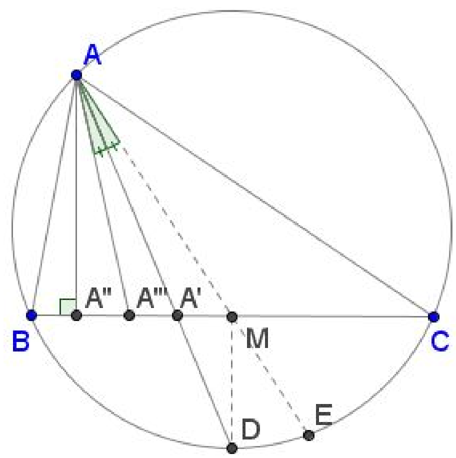
$$BA'' < \frac{a}{2}, CB'' < \frac{b}{2}, AC'' < \frac{c}{2},$$

$$BA''' < \frac{a}{2}, CB''' < \frac{b}{2}, AC''' < \frac{c}{2}$$

Multiplying the rows and adding the results yields the required inequality. □

Observations (by Alexander Bogomolny).

To clarify the inequalities, we make a couple of observations.



First, if M is the midpoint of AC, then A' lies between A'' and M. Second, if D is the midpoint of the arc BC, opposite A and E is the second intersection of AM with the circumcircle (ABC), then $\angle A'AA''' = \angle DAE < \angle CAD = \angle BAD$, implying that A''' is between B and M.

Acknowledgment (by Alexander Bogomolny).

The problem has been kindly posted by Daniel Sitaru at the *CutTheKnotMath facebook page.* The problem came from his book *Math Storm.*

29. TWO CONDITIONS FOR A TRIANGLE TO BE EQUILATERAL

Consider two statements in ΔABC:

$$P : \exists n \in \mathbb{Z}^*, \begin{cases} a^2 + (2n+1)a + n^2 = b \\ b^2 + (2n+1)b + n^2 = c \\ c^2 + (2n+1)c + n^2 = a \end{cases}$$

and

$$Q : r_a + r_b + r_c = l_a + l_b + l_c$$

***Prove that $P \Leftrightarrow Q$*, i.e., that the two statements are equivalent.**

Proposed by Daniel Sitaru - Romania

(The problem uses notations *common in triangle geometry.*)

Note (by Alexander Bogomolny).

Note that the proofs below in fact show more, viz., that both conditions only hold for equilateral triangle, which makes them automatically equivalent.

Solution 1 (by Soumava Chakraborty - Kolkata - India).
Adding up the three equation in P,

$$\sum_{cycl} a^2 + (2n+1)\sum_{cycl} a + 3n^2 = \sum_{cycl} a,$$

i.e.,

$$\sum_{cycl} a^2 + 2n\sum_{cylc} a + 3n^2 = 0,$$

which we rewrite as

$$3n^2 + \left(2\sum_{cycl} a\right)n + \sum_{cycl} a^2 = 0.$$

Since, it is given the equation has a solution in integers, hence in reals, the *disciminant* Δ of this qudratic (in n) equation is not negative

$$\Delta = 4\left(\sum_{cycl} a\right)^2 - 12\sum_{cycl} a^2 \geq 0$$

This is equivalent to

$$\sum_{cycl} a^2 + 2\sum_{cycl} ab - 3\sum_{cycl} a^2 \geq 0,$$

which reduce to

$$\sum_{cycl} ab \geq \sum_{cycl} a^2,$$

but, by say, the *Rearrangement inequality*, we have

$$\sum_{cycl} ab \leq \sum_{cycl} a^2,$$

implying that

$$\sum_{cycl} ab = \sum_{cycl} a^2.$$

From here

$$\sum_{cycl} (a-b)^2 = 0,$$

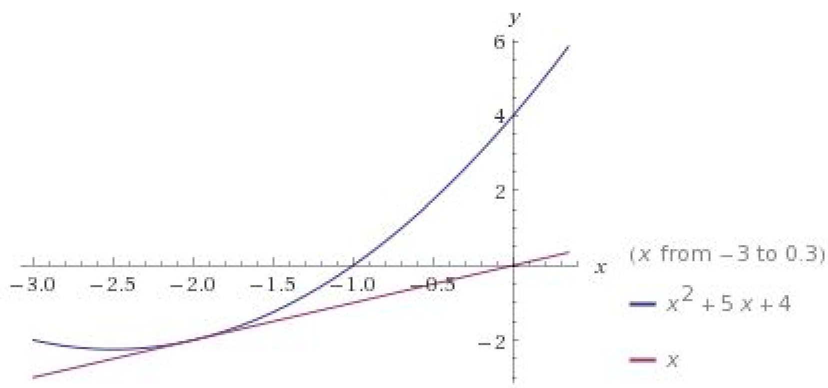
so that $a = b = c$.
For Q, we have

$$\sum_{cycl} r_a = 4R + r \geq \sum_{cycl} m_a \geq \sum_{cycl} l_a,$$

with equality only when $a = b = c$. □

Solution 2 (by Alexander Bogomolny).
For any $n \in \mathbb{Z}^*$, the graph of the function $y = f(x) = x^2 + (2n+1)x + n^2$

is tangent to the diagonal $y = x$ for $x = -n$, otherwise being above the diagonal.



Iterations $x_{k+1} = f(x_k)$ converge to the point of tangency, never forming a 3 - loop. It follows, therefore, from the definition of P that it may only hold when $a = b = c$.
Concerning Q, with $2p = a + b + c$, *we have*, say, by the *AM-GM inequality*,

$$l_a = \sqrt{p(p-a)}\frac{2\sqrt{bc}}{b+c} \leq \sqrt{p(p-a)} = \frac{S}{\sqrt{(p-a)(p-c)}},$$

by *Heron's formula*. (The equality is only for $b = c$.) *On the other hand*, say $r_a = \frac{S}{p-a}$. Thus,

$$l_a \leq \frac{S}{\sqrt{(p-b)(p-c)}} = \sqrt{r_b r_c}.$$

But then, by the *Cauchy - Schwarz inequality*, we get

$$\sum_{cycl} l_a \leq \sum_{cycl} \sqrt{r_b r_c} \leq r_a + r_b + r_c,$$

where the first inequality becomes equality only for $a = b = c$ while the second is the equality for $r_a = r_b = r_c$, which is the same. □

Acknowledgment (by Alexander Bogomolny).
The above problem from the *Romanian Mathematical Magazine* and a solution (Solution 1 by Soumava Chakraborty, Kolkata, India) has been kindly posted at the *CutTheKnotMath facebook page* by Daniel Sitaru.

30. AN ALL - INCLUSIVE INEQUALITY II

Prove that in an acute ΔABC the following relationship holds

$$\left(\sum_{cycl}\sqrt{\frac{m_a}{l_a}}\right)\left(\sum_{cycl}\sqrt{\frac{m_a}{h_a}}\right) \leq \frac{9R}{2r},$$

where m_a, m_b, m_c are the medians; h_a, h_b, h_c the altitudes; l_a, l_b, l_c the angle bisectors in triangle ABC, R and r its circumradius and inradius, respectively.

Proposed by Daniel Sitaru - Romania

Solution 1 (by Soumava Chakraborty - Kolkata - India).

$$LHS \leq \left(\sqrt{\sum_{cycl} m_a}\sqrt{\sum_{cycl}\frac{1}{h_a}}\right) \leq \left(\sum_{cycl} m_a\right)\sqrt{\sum_{cycl}\frac{1}{h_a}}\sqrt{\sum_{cycl}\frac{1}{h_a}},$$

for $l_a \geq h_a$, etc, implying $\sum \frac{1}{l_a} \leq \sum \frac{1}{h_a}$. Thus,

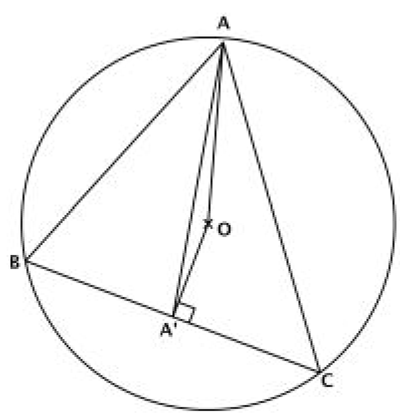
$$LHS \leq \left(\sum_{cycl} m_a\right)\left(\sum_{cycl}\frac{1}{h_a}\right) = \left(\sum_{cycl} m_a\right)\left(\frac{s}{S}\right),$$

where $s = \frac{a+b+c}{2}$ is the semiperimeter, S the area of ΔABC.
Using *Bottema's inequality*, $m_a + m_b + m_c \leq 4R + r$,

$$LHS = \left(\sum_{cycl} m_a\right)\left(\frac{s}{S}\right) \leq (4R+r)\left(\frac{s}{rs}\right) = \frac{4R+r}{r}.$$

Thus, suffice it to show that $\frac{4R+r}{r} \leq \frac{9R}{2r}$, but this is equivalent to $8R + 2r \leq 9R$, i.e., *Euler's inequality* $R \geq 2r$. □

Solution 2 (by Daniel Sitaru - Romania).
Let $AA' = m_a; OA' \perp BC; O$ - center of the circumcircle (ABC).



In ΔABC,

$$AA' \leq A'O + OA$$

It follows that

$$m_a \leq R\cos A + R = R(\cos A + 1) = R\left(2\cos^2\frac{A}{2} - 1 + 1\right)$$

$$= 2R\cos^2\frac{A}{2}$$

Further,

$$\frac{m_a}{h_a} \le \frac{2R\cos^2\frac{A}{2}}{h_a} = \frac{2Rp(p-a)}{bc\cdot\frac{2S}{a}} = \frac{Rap(p-a)}{bcS} =$$

$$= \frac{Ra^2p(p-a)}{abcS} = \frac{Ra^2p(p-a)}{4RS^2} = \frac{a^p(p-a)}{4S^2} = \frac{a^2p(p-a)}{4rpS} =$$

$$= \frac{a^2(p-a)}{4r^2p}.$$

Adding to that two analogous inequalities,

$$\sum_{cycl}\frac{m_a}{h_a} \le \sum_{cycl}\frac{a^2(p-a)}{4r^2p} = \frac{1}{4r^2p}\Big(p\sum a^2 - \sum a^3).$$

We are going to use the following two identities:

$$\sum_{cycl} a^2 = 2s^2 - 8Rr - 2r^2, \sum_{cycl} a^3 = s(2s^2 - 12Rr - 6r^2).$$

With these,

$$\sum\frac{m_a}{h_a} \le \frac{1}{4r^2p}\Big(p(2p^2 - 8Rr - 2r^2) - p(2p^2 - 12Rr - 6r^2)\Big)$$

$$= \frac{1}{4R^2}(2p^2 - 8Rr - 2r^2 - 2p^2 + 12Rr + 6r^2)$$

$$= \frac{1}{4R^2}(4Rr + 4r^2) = \frac{R}{r} + 1.$$

For the record,

(1) $$\sum_{cycl}\frac{m_a}{h_a} \le \frac{R}{r} + 1.$$

From that and $l_a \ge h_a$, etc., also

(2) $$\sum_{cycl}\frac{m_a}{l_a} \le \frac{R}{r} + 1.$$

By the *Cauchy - Schwarz inequality*, using 1,

(3) $$\left(\sum\sqrt{\frac{m_a}{h_a}}\cdot 1\right)^2 \le \Big(\sum\frac{m_a}{h_a}\Big)(1^2 + 1^2 + 1^2) = 3\sum\frac{m_a}{h_a} \le 3\Big(\frac{R}{r} + 1\Big).$$

Using 2, we similarly get

(4) $$\left(\sum\sqrt{\frac{m_a}{l_a}}\cdot 1\right)^2 \le 3\Big(\frac{R}{r} + 1\Big)$$

The product of 3 and 4, along with the Euler inequality, $R \ge 2r$ yields the required inequality:

$$\left(\sum\sqrt{\frac{m_a}{l_a}}\right)^2\left(\sum\sqrt{\frac{m_a}{h_a}}\right)^2 \le 9\Big(\frac{R+r}{r}\Big)^2 \le 9\left(\frac{R+\frac{R}{2}}{r}\right)^2$$

and, finally,

$$\left(\sum\sqrt{\frac{m_a}{l_a}}\right)\left(\sum\sqrt{\frac{m_a}{h_a}}\right)\leq 3\frac{\frac{3R}{2}}{r}=\frac{9R}{2r}.$$

□

Solution 3 (by Kevin Soto Palacios - Huarmey - Peru).
We shall use the following facts:

(1) $$l_a\geq h_a, l_b\geq h_b, l_c\geq h_c$$

and

(2) $$\frac{R}{2r}\geq\frac{m_a}{h_a},\frac{R}{2r}\geq\frac{m_b}{h_b},\frac{R}{2r}\geq\frac{m_c}{h_c}$$

So we have

(A) $$\sqrt{\frac{m_a}{h_a}}+\sqrt{\frac{m_b}{h_b}}+\sqrt{\frac{m_c}{h_c}}\leq 3\sqrt{\frac{R}{2r}}$$

and, therefore, also

(B) $$\sqrt{\frac{m_a}{l_a}}+\sqrt{\frac{m_a}{l_b}}+\sqrt{\frac{m_a}{l_c}}\leq 3\sqrt{\frac{R}{2r}}$$

The product of A and B is exactly the required inequality. □

Solution 4 (by Kevin Soto Palacios - Huarmey - Peru).
We know that

$$l_a\geq h_a, l_b\geq h_b, l_c\geq h_c,$$

$$m_a+m_b+m_c\leq 4R+r\leq\frac{9R}{2},$$

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}.$$

By the *Cauchy - Schwarz inequality*,

$$\left(\sqrt{\frac{m_a}{h_a}}+\sqrt{\frac{m_b}{h_b}}+\sqrt{\frac{m_c}{h_c}}\right)^2\leq(m_a+m_b+m_c)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right).$$

It follows that

$$\sqrt{\frac{m_a}{h_a}}+\sqrt{\frac{m_b}{h_b}}+\sqrt{\frac{m_c}{h_c}}\leq\sqrt{\left(\frac{9R}{2}\right)}\sqrt{\left(\frac{1}{r}\right)}.$$

It so, also

$$\sqrt{\frac{m_a}{l_a}}+\sqrt{\frac{m_b}{l_b}}+\sqrt{\frac{m_c}{l_c}}\leq\sqrt{\left(\frac{9R}{2}\right)}\sqrt{\left(\frac{1}{r}\right)}$$

The product of the two gives the required inequality. □

Acknowledgment (by Alexander Bogomolny).
The problem has been offered by Daniel Sitaru at the *CutTheKnotMath facebook page*; solutions added via comments and private communication. Solution 1 is by Soumava Chakraborty; Solution 2 is by Daniel Sitaru; Solution 3 and 4 are by Kevin Soto Palacios. The problem came

from Dan's book *Math Power* and has been published at the *Romanian Mathematical Magazine.*

31. A CYCLING INEQUALITY WITH INTEGRALS

$$\textit{Prove that, for } a,b,c>2,$$
$$2bc\Omega(a)+2ca\Omega(b)+2ab\Omega(c)<a^2+b^2+c^2$$
$$\textit{where } \Omega(t)=\int_0^1 \frac{1-x^2}{1+tx^2+x^4}dx$$

Proposed by Daniel Sitaru - Romania

Proof (by Ravi Prakash - New Delhi - India).
Note that

$$\Omega'(t)=\int_0^1 \frac{(1-x^2)(-1)x^2}{(1+tx^2+x^4)^2}dx<0,$$

for $t\geq 2$, making $\Omega(t)$ strictly decreasing on $[2,\infty)$. Further

$$\Omega(2)=\int_0^1 \frac{1-x^2}{1+2x^2+x^4}dx=\int_0^1\Big[\frac{2}{(1+x^2)^2}-\frac{1}{1+x^2}\Big]dx$$

But

$$\frac{\pi}{4}=\int_0^1\frac{dx}{1+x^2}$$
$$=\frac{x}{1+x^2}\Big|_0^1+\int_0^1\frac{x(2x)}{(1+x^2)^2}dx=\frac{1}{2}+\int_0^1\frac{2(x^2+1)-2}{(1+x^2)^2}dx$$
$$=\frac{1}{2}+\frac{\pi}{4}-\Omega(2),$$

implying $\Omega(2)=\frac{1}{2}$ such that, for $t>2, 0<\Omega(t)<\frac{1}{2}$. It follows that

$$2bc\Omega(a)+2ca\Omega(b)+2ab\Omega(c)<bc+ca+ab\leq$$
$$\leq\frac{b^2+c^2}{2}+\frac{c^2+a^2}{2}+\frac{a^2+b^2}{2}=a^2+b^2+c^2.$$

□

Acknowledgment (by Alexander Bogomolny).
Daniel Sitaru has kindly posted the above problem form the *Romanian Mathematical Magazine* (and his book *Math Accent*), with a proof by Ravi Prakash (India), at the *CutTheKnotMath facebook page.*
Note that the penultimate step in the proof could be shortened by noticing that

$$\int\frac{1-x^2}{(1+x^2)^2}dx=\frac{x}{1+x^2}+C$$

32. A CYCLIC INEQUALITY IN TRIANGLE

Prove that in any ΔABC,

$$\sum_{cycl}\frac{a^3(2s-a)}{b(2s-b)} \geq \frac{27a^2b^2c^2}{s^2},$$

$$\textit{where } s=\frac{a+b+c}{2}, \textit{ the semiperimeter of } \Delta ABC.$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Kevin Soto Palacios - Huarmey - Peru).

By the *AM-GM inequality,*

$$\frac{(a+b+c)^2}{4}\sum_{cycl}\frac{a^3(2s-a)}{b(2s-b)} \geq \frac{9\sqrt[3]{(abc)^2}}{4}\cdot\sqrt[3]{\prod_{cycl}a^2(2s-a)^2}.$$

Thus, suffice it to show that

(A) $$\frac{9\sqrt[3]{(abc)^2}}{4}\cdot 3\sqrt[3]{\prod_{cycl}a^2(2s-a)^2} \geq 27a^2b^2c^2.$$

Note that

(B) $$(2s-a)(2s-b)(2s-c) = (a+b)(b+c)(c+a) \geq 8abc.$$

Combining A and B we get

$$\frac{9\sqrt[3]{(abc)^2}}{4}\cdot\sqrt[3]{\prod_{cycl}a^2(2s-a)^2} \geq \frac{9\sqrt[3]{(abc)^2}}{4}\cdot 3\sqrt[3]{(abc)^2\cdot 64(abc)^2}$$

$$= 27a^2b^2c^2.$$

□

Proof 2 (by Soumava Chakraborty - Kolkata - India).

By the *AM-GM inequality,*

$$LHS \geq 3\sqrt[3]{a^2b^2c^2(a+b)^2(b+c)^2(c+a)^2}.$$

Suffice it to show that

$$\prod_{cycl}a^2(a+b)^2 \geq \frac{729a^6b^6c^6}{s^6}$$

We have a sequence of equivalent statements:

$$\prod_{cycl}(a+b) \geq \frac{27(a+b+c)^2}{s^3} = \frac{432R^2r^2}{s},$$

$$2abc+\prod_{cycl}ab(2s-c) \geq \frac{432R^2r^2}{s}, 2s^2\sum_{cycl}ab - 4Rr^2 \geq 432R^2r^2,$$

$$s^2(s^2+4Rr+r^2) - 2Rrs^2 - 216R^2r^2 \geq 0$$

To this we'll apply *Gerretsen's inequality* $s^2 \geq 16Rr - 5r^2$:

$$s^2(s^2 + 4Rr + r^2) - 2Rrs^2 - 216R^2r^2 \geq$$

$$\geq (16Rr - 5r^2)^2 + (16Rr - 5r^2)(2Rr + r^2) - 216R^2r^2.$$

Suffice it to show that:

$$(16Rr - 5r^2)^2 + (16Rr - 5r^2)(2Rr + r^2) - 216R^2r^2 \geq 0.$$

**But this is equivalent to $36R^2 - 77Rr + 10r^2 \geq 0$, or,
$(R-2r)(36R-5r) \geq 0$, which is true due to *Euler's inequality* $R \geq 2r$.** □

Proof 3 (by Alexander Bogomolny).

**By the *AM-GM inequality*, $LHS \geq 3\sqrt[3]{a^2b^2c^2(a+b)^2(b+c)^2(c+a)^2}$.
Suffice it to show that**

$$\prod_{cycl} a^2(a+b)^2 \geq \frac{3^3a^6b^6c^6}{s^6}.$$

This is equivalent to

$$(a+b+c)^3\prod_{cycl}(a+b) \geq 6^3a^2b^2c^2. \quad (*)$$

**In this form the inequality holds for $a, b, c \geq 0$, not necessarily the sides of a triangle. Now, $(a+b+c)^3 \geq 3^3abc$, whereas
$a+b \geq 2\sqrt{ab}, b+c \geq 2\sqrt{bc}, c+a \geq 2\sqrt{ca}$. Multiplying the four gives *** □

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly posted the above problem from his book *Math Accent*, with two proofs - one (Proof 1) by Kevin Soto Palacios (Peru), the other (Proof 2) by Soumava Chakraborty (India), at the *CutTheKnotMath facebook page.*

33. A CYCLCIC INEQUALITY IN TRIANGLE II

Prove that in any ΔABC,

$$\sqrt{abc}\Big(\frac{a^2}{\sqrt{b}} + \frac{b^2}{\sqrt{c}} + \frac{c^2}{\sqrt{a}}\Big)^2 \geq 16(\sqrt{a} + \sqrt{b} + \sqrt{c})S^2,$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Soumava Pal - Kolkata - India).

***WLOG*, assume $a \geq b \geq c$. Then $a^2 \geq b^2 \geq c^2$ but $\frac{1}{\sqrt{a}} \leq \frac{1}{\sqrt{b}} \leq \frac{1}{\sqrt{c}}$
. First employing the *Rearrangement inequality* and then the *AM-GM inequality*,**

$$\sum_{cycl} \frac{a^2}{\sqrt{b^2}} \geq \sum_{cycl} \frac{a^2}{\sqrt{a}} = a^{\frac{3}{2}} + b^{\frac{3}{2}} + c^{\frac{3}{2}} \geq 3\sqrt{abc}$$

Thus,

$$\sum_{cycl} \frac{a^2}{\sqrt{b}} \geq 3\sqrt{abc}. \tag{1}$$

Using the Rearrangement inequality the second time and then the *Chebyshev's inequality*,

$$\sum_{cycl} \frac{a^2}{\sqrt{b}} \geq \sum_{cycl} \frac{a^2}{\sqrt{a}} = a\sqrt{a} + b\sqrt{b} + c\sqrt{c} > \frac{1}{3}(a+b+c)(\sqrt{a}+\sqrt{b}+\sqrt{c})$$

So that

$$\sum_{cycl} \frac{a^2}{\sqrt{b}} \geq \frac{1}{3}\Big(\sum_{cycl} a\Big)\Big(\sum_{cycl} \sqrt{a}\Big) \tag{2}$$

Multiplying 1 and 2, we get

$$\sqrt{abc}\Big(\sum_{cycl} \frac{a^2}{\sqrt{b}}\Big) \geq (abc)(a+b+c)(\sqrt{a}+\sqrt{b}+\sqrt{c}). \tag{3}$$

Define $x = a+b-c, y = b+c-a, z = c+a-b$. Then $x+y \geq 2\sqrt{xy}$ and $b \geq \sqrt{xy}$. Similarly, $a \geq \sqrt{xz}$ and $c \geq \sqrt{yz}$. It follows that $abc \geq xyz$. Thus we may continue 3:

$$\sqrt{abc}\left(\sum_{cycl} \frac{a^2}{\sqrt{b}}\right) \geq (abc)(a+b+c)(\sqrt{a}+\sqrt{b}+\sqrt{c}) \geq$$

$$xyz(a+b+c)(\sqrt{a}+\sqrt{b}+\sqrt{c}) = 16S^2(\sqrt{a}+\sqrt{b}+\sqrt{c},$$

as required. □

Proof 2 (by Soumitra Mandal - nickname Diego Alvariz - Chandar Nagore - India).

Since $(x+y)(y+z)(z+x) \geq 8xyz$,

$$\sqrt{abc}\left(\sum_{cycl} \frac{a^2}{\sqrt{b}}\right)^2 \geq 3\sqrt[3]{\prod_{cycl} \frac{a^2}{\sqrt{b}}}\left(\sum_{cycl} \frac{a^2}{\sqrt{b}}\right)\sqrt{abc} = 3abc\left(\sum_{cycl} \frac{a^2}{\sqrt{b}}\right)$$

$$\geq \left(\prod_{cycl}(a+b+c)\right)\frac{3(a+b+c)^2}{\sum_{cycl}\sqrt{a}}$$

$$\left(\sum_{cycl}\sqrt{a}\right)(a+b+c)\left(\prod_{cycl}(a+b-c)\right) = (\sqrt{a}+\sqrt{b}+\sqrt{c})16S^2.$$

□

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly posted the above problem form his book *Math Accent*, with two proofs - one (Proof 1) by Soumava Pal, the other (Proof 2) by Diego Alvariz, at the *CutTheKnotMath facebook page.*

34. DAN SITARU'S INEQUALITY WITH TANGENTS

Prove that in any ΔABC,

$$\sum_{cycl} \sqrt[3]{\tan A}\sqrt[3]{\tan B}(\sqrt[3]{\tan A}+\sqrt[3]{\tan B}) \le 2\tan A\tan B\tan C$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Daniel Sitaru - Romania).

The starting point is *Heinz's inequality*:

$$\textit{For } x,y>0, \alpha\in[0,1], x^{1-\alpha}y^{\alpha}+x^{\alpha}y^{1-\alpha}\le x+y.$$

We apply Heingz's inequality with $\alpha=\frac{1}{3}$

$$x^{\frac{2}{3}}y^{\frac{1}{3}}+x^{\frac{1}{3}}y^{\frac{2}{3}}\le x+y$$

and set $x=\tan A$ and $y=\tan B$ to obtain

$$\sqrt[3]{\tan A}\sqrt[3]{\tan B}(\sqrt[3]{\tan A}+\sqrt[3]{\tan B})\le \tan A+\tan B$$

Similarly we get

$$\sqrt[3]{\tan A}\sqrt[3]{\tan B}(\sqrt[3]{\tan A}+\sqrt[3]{\tan B})\le \tan A+\tan B$$

$$\sqrt[3]{\tan C}\sqrt[3]{\tan A}(\sqrt[3]{\tan C}+\sqrt[3]{\tan A})\le \tan C+\tan Q.$$

yields the required inequality. □

Proof 2 (by Ritesh Dutta - India).

Let $\tan^{\frac{1}{3}}(A)=A; \tan^{\frac{1}{3}}(B)=b$, and $\tan^{\frac{1}{3}}(C)=c$. In any triangle $A+B+C=a^3+b^3+c^3=a^3b^3c^3$.
After the substitution we have to prove that

$$\sum cyclab(a+b)\le 2a^3b^3c^3$$

Now,

$$\sum_{cycl} ab(a+b)\le \sum \frac{a^2+b^2}{2}(a+b)=\sum_{cycl} a^3+\sum_{cycl}\frac{ab(a+b)}{2}$$

It follows that

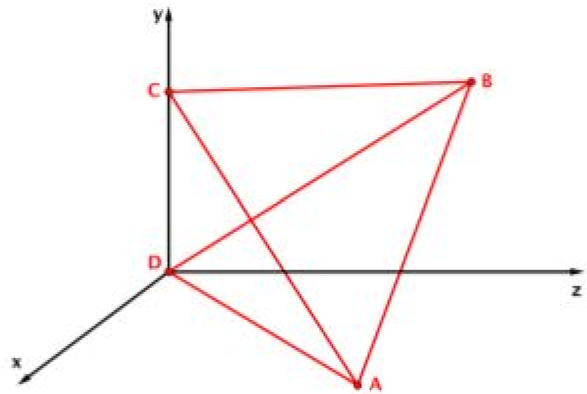
$$\sum_{cycl} ab(a+b)\le 2\sum_{cycl} a^3=2a^3b^3c^3$$

□

Acknowledgment (by Alexander Bogomolny)
Dan Sitaru has kindly posted the problem, with a solution (Proof 1), at the *CutTheKnotMath facebook page.* Proof 2 is by Ritesh Dutta (India).

35. VECTOR ALGEBRA IN TETRAHEDRON

Let $ABCD$be a tetrahedron where: $AC = \sqrt{11}; CD = 3; AD = \sqrt{14}$;
$AB = \sqrt{3}; BC = 2; BD = \sqrt{13}$.



Prove that: $m(\angle(AB, CD)) \geq 90^\circ$.

Proposed by Daniel Sitaru - Romania

Proof (by Daniel Sitaru - Romania).

We place the origin at D and $(yOz) \equiv (BCD)$. It follows that $D = (0,0,0)$ and $C = (0,3,0)$ because $CD = 3$.
Take $B = (0,a,b); B \in (yOz)$, and $(c,d,e), a,b,c,d,e \in \mathbb{R}$. Then, via the Pythagorean theorem,

$$\begin{cases} BC = 2 \\ AC = \sqrt{11} \\ BD = \sqrt{13} \\ AD = \sqrt{14} \\ AB = \sqrt{3} \end{cases} \Rightarrow \begin{cases} \sqrt{(a-3)^2 + b^2 = 2} \\ \sqrt{c^2 + d^2 + e^2} = \sqrt{14} \\ \sqrt{a^2 + b^2} = \sqrt{13} \\ \sqrt{c^2 + (d-3)^2 + e^2} = \sqrt{11} \\ \sqrt{c^2 + (d-a)^2 + (b-e)^2} = \sqrt{3} \end{cases}$$

$$\begin{cases} \sqrt{(a-3)^2 + b^2} = 2 \\ \sqrt{c^2 + d^2 + e^2} = \sqrt{14} \\ \sqrt{a^2 + b^2} = \sqrt{13} \\ \sqrt{c^2 + (d-3)^2 + e^2} = \sqrt{11} \\ \sqrt{c^2 + (d-a)^2 + (b-e)^2} = \sqrt{11} \end{cases} \Rightarrow \begin{cases} a^2 - 6a + 9 + b^2 = 4 \\ c^2 + d^2 + e^2 = 14 \\ a^2 + b^2 = 13 \\ c^2 + (d-3)^2 + e^2 = 11 \\ c^2 + (d-a)^2 + (b-e)^3 = e \end{cases}$$

□

Further,

$$\begin{cases} 13 + 9 - 6a = 4 \Rightarrow a = 3 \\ b^2 = 4 \Rightarrow b = 2 \\ c^2 + d^2 - 6d + 9 + e^2 = 11 \Rightarrow d = 2 \\ c^2 + 1 + (2-e)^2 = 3 \Rightarrow e = 3; c = 1 \end{cases}$$

Which is to say that $B = (0, 3, 2)$ and $A = (1, 2, 3)$. It follows (in customary notations) that $\overrightarrow{AB} = -i + j - k$ and $\overrightarrow{CD} = -3j$ so that $\overrightarrow{AB} \cdot \overrightarrow{CD} = -3$, implying $\cos\angle(\overrightarrow{AB}, \overrightarrow{CD}) < 0$, so that $\angle(\overrightarrow{AB}, \overrightarrow{CD}) \geq 90°$.

Remark (by Grego̍ire Nicollier)
A proof without any computation! Let $C*$ and $D*$ be the orthogonal projections of C and D on the line AB. The angle between the vectors AB and $C * D*$ have opposite directions. This is here clearly the case (without any sketch!) as ΔABC is obtuse at B whereas ΔABD is almost isosceles.

36. AN ELEMENTARY INEQUALITY BY NON-ELEMTARY MEANS

Prove that, for $a > 0$,

$$12(a\sin a + \cos a - 1)^2 \leq 2a^4 + a^3 \sin 2a.$$

Proposed by Daniel Sitaru - Romania

Proof (by Daniel Sitaru - Romania).

By the *Cauchy - Schwarz inequality*,

$$\left(\int_0^a x\cos x dx\right)^2 \leq \left(\int_0^a x^2 dx\right)\left(\int_0^a \cos^2 x dx\right).$$

We continue by evaluating integrals:

$$\left(x\sin x\Big|_0^a - \int_0^a \sin x\right)^2 \leq \frac{a^3}{3}\int_0^a \frac{1+\cos 2x}{2}dx$$

$$(a\sin a + \cos a - \cos 0)^2 \leq \frac{a^3}{6}\left(x\Big|_0^a + \frac{1}{2}\sin 2x\Big|_0^a\right).$$

$$6(a\sin a + \cos a - 1)^2 \leq a^3\left(a + \frac{1}{2}\sin 2a\right).$$

$$12(a\sin a + \cos a - 1)^2 \leq 2a^4 + a^3\sin 2a.$$

□

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly communicated in private message the above problem and its solutions. There is little doubt that the expression $a\sin a + \cos a$ betrays the integral origins of the problem. However, the inequality itself is quite elementary looking which makes one curious whether it has a more elementary solution that does not invoke calculus.

37. INTEGRAL OF A PIECE - WISE FUNCTION

For $a, b, c \in (0, \infty), a < b < c; f : [0, a] \to [0, b]$ ***and*** $g : [0, b] \to [0, c]$ **both continuous, surjective and strictly increasing functions.**

Prove that

$$\frac{1}{2}\int_0^a (g \circ g)^2(x)dx + \frac{1}{a}\int_0^c (f^{-1} \circ g^{-1})^2(x)dx \le ac.$$

Proposed by Daniel Sitaru - Romania

Solution.

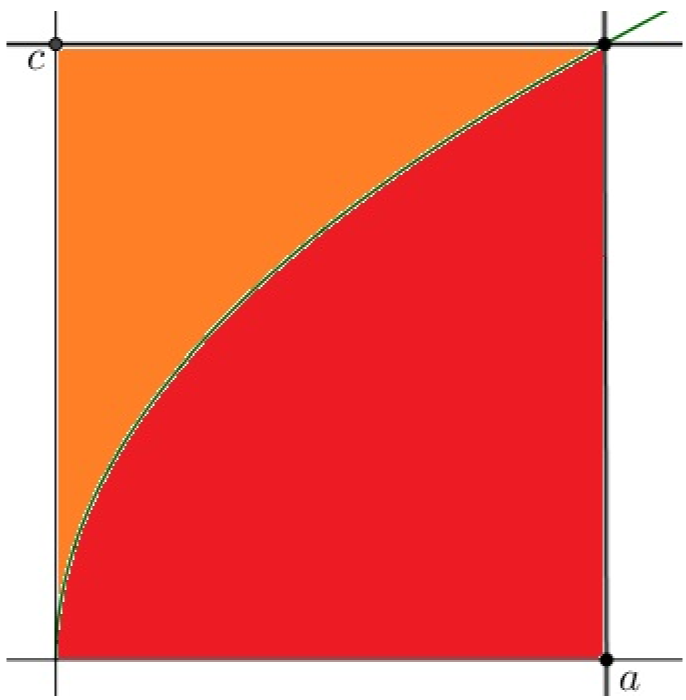
First of all, note that $h = g \circ f : [0, a] \to [0, c]$ **is a continuos, surjective and strictly increasing function. In particular,** $h(a) = c$ **and** $h^{-1} = f^{-1} \circ g^{-1}$**. We thus have to prove that**

$$\frac{1}{c}\int_0^a h^2(x)dx + \frac{1}{a}\int_0^c (h^{-1})^2(x)dx \le ac.$$

Next we observe that $h(x) \le c$ **and** $h^{-1}(x) \le a$ **and deduce that**

$$\frac{1}{c}\int_0^a h^2(x)dx + \frac{1}{a}\int_0^c (h^{-1})^2dx \le \int_0^a h(x)dx + \int_0^c h^{-1}(x)dx.$$

To this we apply the extreme case of ***Young's inequality*** **that may be called** ***Young's identity.*** **The latter is illustrated by the diagram below:**



The red represents $\int_0^a h(x)dx$ **and the orange are represents** $\int_0^c h^{-1}(x)dx$**. It follows that** $\int_0^a h(x)dx + \int_0^c h^{-1}(x)dx = ac$**.**

Acknowledgment (by Alexander Bogomolny)

The problem from the Romanian Mathematical Magazine (Problem 44) has been posted by Daniel Sitaru at the *CutTheKnotMath facebook page.* Leo Giugiuc and Daniel Sitaru commented with practically identical solutions. □

38. TWO - TRIANGLE INEQUALITY II

Given two triangles: ABC and $A'B'C'$, prove that

$$\frac{a+b+c}{3\sqrt{3}R} \le \frac{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}{\cos\frac{A'}{2}+\cos\frac{B'}{2}+\cos\frac{C'}{2}} \le \frac{3\sqrt{3}R'}{a'+b'+c'}$$

Proposed by Daniel Sitaru - Romania

Proof (by Kevin Soto Palacios - Huarmey - Peru).

We shall prove only the left inequality, as the right one is obtained from that by swapping A with A', a with a', etc.
Observe that $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \le 2\cos\frac{C}{2}$, implying that

$$\sin A + \sin B + \sin C \le \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}$$

It follows by the *Law of Sines* that

$$\frac{a+b+c}{2R} = \sin A + \sin B + \sin C \le \cos\frac{A}{2} + \cos\frac{C}{2} + \cos\frac{C}{2}$$

In addition, *we know that*

$$\frac{3\sqrt{3}}{2} \ge \cos\frac{A'}{2} + \cos\frac{B'}{2} + \cos\frac{C'}{2}$$

Dividing one by another we obtain

$$\frac{a+b+c}{3\sqrt{3}R} \le \frac{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}{\cos\frac{A'}{2}+\cos\frac{B'}{2}+\cos\frac{C'}{2}}$$

□

Acknowledgment (by Alexander Bogomolny)
Dan Sitaru has kindly posted the above problem, with two proofs by Kevin Soto Palacios (Peru), at *CutTheKnotMath facebook page.*

39. INEQUALITY WITH POWERS AND RADICALS

Prove that, for positive reala, b, cwe have:

$$\sum_{cycl} \sqrt[6]{ab^2c^3} \ge \sum_{cycl} \sqrt[30]{a^9b^{10}c^{11}}$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Dang Thanh Tung - Vietnam).

The inequality is equivalent to

$$\frac{1}{\sqrt[3]{abc}}\sum_{cycl}\sqrt[6]{ab^2c^3} \geq \frac{1}{\sqrt[3]{abc}}\sum_{cycl}\sqrt[30]{a^9b^{10}c^{11}}$$

which translates into

$$\sum_{cycl}\sqrt[6]{\frac{a}{b}} \geq \sum_{cycl}\sqrt[30]{\frac{a}{b}}$$

Defining $x = \sqrt[30]{\frac{a}{b}}$**, etc., the problem reduces to showing that**

$$\sum_{cycl} x^5 \geq \sum_{cycl} x$$

provided $x, y, z > 0$ **and** $xyz = 1$**. The AM-GM inequality, yields**

$$x^5 + 1 + 1 + 1 + 1 > 5x,$$
$$y^5 + 1 + 1 + 1 + 1 \geq 5y,$$
$$z^5 + 1 + 1 + 1 + 1 \geq 5z.$$

Summing up shows that

$$\sum_{cycl} x^5 \geq 5\sum_{cycl} x - 12 = \sum_{cycl} x + 4\left(\sum_{cycl} x - 3\right)$$
$$= \sum_{cycl} x + 4\left(\sum_{cycl} x - 3\sqrt[3]{\prod_{cycl} x}\right) \geq \sum_{cycl} x$$

because, by AM-GM inequality, $x + y + z \geq 3\sqrt[3]{xyz}$**. The equality is attained when** $x = y = z = 1$**, i.e., when** $a = b = c$**.** □

Proof 2 (by Ravi Prakash - New Delhi - India).

By the AM-GM inequality,

$$7(ab^2c^3)^{\frac{1}{6}} + 4(a^3bc^2)^{\frac{1}{6}} + 4(a^2b^3c)^{\frac{1}{6}} \geq (a^7b^{14}c^{21}a^{12}b^4c^8a^8b^{12}c^4)^{\frac{1}{6\times 15}}$$
$$\geq 15(a^{27}b^{30}c^{33})^{\frac{1}{90}}$$
$$= 15(a^9b^{10}c^{11})^{\frac{1}{30}}.$$

Similarly,

$$4(ab^2c^3)^{\frac{1}{6}} + 7(a^3bc^2)^{\frac{1}{6}} + 4(a^2b^3c)^{\frac{1}{6}} \geq 15(a^{11}b^9c^{10})^{\frac{1}{30}},$$
$$4(ab^2c^3)^{\frac{1}{6}} + 4(a^3bc^2)^{\frac{1}{6}} + 7(a^2b^3c)^{\frac{1}{6}} \geq 15(a^{10}b^{11}c^9)^{\frac{1}{30}}.$$

Adding the three and dividing by 15 gives the required inequality. □

Proof 3 (by Alexander Bogomolny).

This is also a direct consequence of *Muirhead's inequality.* Indeed, let $\alpha = \left\{\frac{3}{6}, \frac{2}{6}; \frac{1}{6}\right\}$ **and** $\beta = \left\{\frac{11}{30}, \frac{10}{30}, \frac{9}{30}\right\}$**. Then** α **majorizes** β **which imediately implies the given inequality.** □

Acknowledgment (by Alexander Bogomolny)
The problem from his book *Math Accent* has been posted by Dan Sitaru at the *CutTheKnotMath facebook page*. He also added two solutions. Proof 1 by Dang Thanh Tung and Proof 2 by Ravi Prakash.

40. BROCARD POINT AND A RELATION OF CIRCUMRADII

Ω is the first Brocard point of ΔABC.R_a, R_b, R_c are the circumradii of triangles $\Omega BC, \Omega CA, \Omega AB$, respectively.
Prove that $R_a R_b R_c = R^3$,
where R is the circumradius of ΔABC.

Proposed by Mehmet Sahin - Ankara - Turkey

Proof 1 (by Daniel Sitaru - Romania).

Let ω denote the *Brocard angle*. Then

$$\frac{c}{2\sin(\pi - \omega - (B - \omega))} = \frac{2R\sin C}{2\sin B}.$$

Two more relations are obtained in the same manner. The product of the three solves the problem:

$$\prod_{cycl} R_a = \prod_{cycl} \frac{R\sin C}{\sin B} = R^3.$$

□

Proof 2 (by Leonard Giugiuc - Romania).

WLOG, assume $A = (-2u, 0), B = (2v, 0), C = (0, 2)$. It's well known that $R^2 = (1 + u^2)(1 + v^2)$. Let's find R_a first. The midpoint of AB is $(v - u, 0)$; the perpendicular at B to BC has the equation $vx - y = 2v^2$, so that the center of (ΩBC) is at $(v - u, v(u + v))$, implying $R_a^2 = (u + v)^2(1 + v^2)$. Similarly, $R_b^2 = \frac{(1+u^2)(1+v^2)^2}{(u+v)^2}$ and $R_c = (1 + u^2)^2$. Clearly, $R_a R_b R_c = R^3$. □

Proof 3 (by Mehmet Sahin - Ankara - Turkey).

Clearly, $\angle B\Omega C = 180°, \angle C\Omega A = 180° - A, \angle A\Omega B = 180° - B$. Using the *Law of Sines* in triangles in $\Omega BC, \Omega CA, \Omega AB$, we get

$$\frac{a}{\sin(180° - C)} = 2R_a,$$

$$\frac{b}{\sin(180° - A)} = 2R_b,$$

$$\frac{c}{\sin(180° - B)} = 2R_c,$$

so that

$$\frac{a}{\sin C} = 2R_a, \frac{b}{\sin A} = 2R_b, \frac{c}{\sin B} = 2R_c.$$

In $\Delta ABC, R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$.
Combining the finds,

$$R_aR_bR_c = \frac{a}{2\sin A} \cdot \frac{b}{2\sin B} \cdot \frac{c}{2\sin C} = R^3,$$

as desired.

Acknowledgment (by Alexander Bogomolny)
The problem has been kindly posted at the *CutTheKnotMath facebook page* by Mehmet Sahin (Turkey). Proof 1 is by Daniel Sitaru; Proof 2 is by Leo Giugiuc; Proof 3 is by Mehmet Sahin. □

41. AN INTEGRAL INEQUALITY FROM THE RMM

If $f : [0, 1] \to (0, \infty)$, f ***derivable,*** f' ***continuous,***
$f'(x) = f'(1 - x), \forall x \in [0, 1]$ ***then:***

$$\int_0^1 f(x)dx \geq \sqrt{f(0) \cdot f(1)}$$

Proposed by Daniel Sitaru - Romania

Proof 1 (by Safal Das Biswas - India).

The condition $f'(x) = f'(1-x)$**implies**$f(x)+f(1-x) = C$**, a constant, for** $x \in [0, 1]$**. This is a kind of situation that has been considered *elsewhere* on three different *occasions*.**

$$\text{If } I = \int_0^1 f(x)dx, \text{ then}$$

$$2I = \int_0^1 \Big(f(x) + f(1 - x)\Big)dx = C = f(x) + f(1 - x),$$

for any $x \in [0, 1]$**. In particular, with** $x = 0, 2I = f(0) + f(1)$**. It then follows by the AM-GM inequality that**

$$I = \frac{f(0) + f(1)}{2} \geq \sqrt{f(0)f(1)}$$

□

Proof 2 (by Soumitra Mandal - nickname Diego Alvariz - Chandar Nagore -India).

$$\text{With } I = \int_0^1 f(x)dx, \text{ and integrating by parts,}$$

$$I = \Big[xf(x)\Big]_0^1 - \int_0^1 xf'(x)dx = f(1) - \int_0^1 xf'(x)dx$$

$$f(1) + \int_0^1 xf'(1-x)d(1-x)$$

$$f(1) + \int_0^1 f'(1-x)d(1-x) - \int_0^1 (1-x)f'(1-x)d(1-x)$$

$$f(1) + \int_0^1 f'(x)dx - \int_1^0 xf'(x)dx$$

$$= f(1) + f(0) + \int_0^1 \left[\frac{d}{dx}(x)\int_0^1 f'(x)dx\right]dx$$

$$f(1) + f(0) - \int_0^1 f(x)dx = f(1) + f(0) - I,$$

$$\text{implying } I = \frac{f(0)+f(1)}{2} \geq \sqrt{f(0)f(1)}$$

□

42. A CYCLING INEQUALITY WITH INTEGRALS II

Prove that, for $a, b, c \geq 0$*,*

$$\Omega(a,b,c) + \Omega(b,c,a) + \Omega(c,a,b) \leq 1,$$

$$\textit{where } \Omega(p,q,r) = \int_0^1 \frac{x^p}{1+x^p+x^{q+r}}dx.$$

Proposed by Daniel Sitaru - Romania

Proof (by Daniel Sitaru - Romania).

Introduce $u = x^a, v = x^b, w = x^c$. **For** $x \in [0,1], u, v, w \in [0,1]$, **such that also** $1-u, 1-v, 1-w \in [0,1]$. **From here, say** $(1-u)(1-v) \geq 0$, **implying** $1 + uv \geq u + v$ **and, subsequently,**

$$1 + w + uv \geq u + v + w.$$

$$\text{Therefore, } \frac{1}{1+w+uv} \leq \frac{1}{u+v+w}$$

$$\text{and } \frac{w}{1+w+uv} \leq \frac{w}{u+v+w}$$

Similarly, $\frac{u}{1+u+uw} \leq \frac{u}{u+v+w}$ **and** $\frac{v}{1+v+wu} \leq \frac{v}{u+v+w}$. **Adding the three up we obtain**

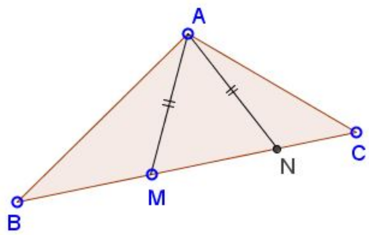
$$\sum_{cycl} \frac{u}{1+u+vw} \leq \sum_{cycl} \frac{u}{u+v+w} = 1.$$

Taking integral form 0 to 1 we obtain the required inequality. Equality is only possible for $a = b = c = 0$. □

Acknowledgment (by Alexander Bogomolny)
Daniel Sitaru has kindly posted the above problem from his book *Math Accent*, at the *CutTheKnotMath facebook page*. He later communicated privately the solution above.

43. LEO'S LEMMA, SECOND APPLIACATION

Assume in ΔABC, points M and N are on BC in this order B, M, N, C.



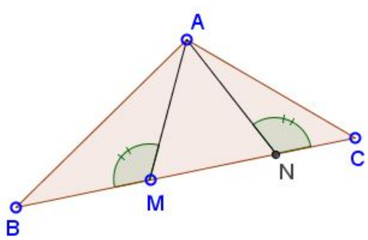
Then $(AB - AC)(BM - CN) \geq 0$.

Proposed (by Daniel Sitaru - Romania)

Proof (by Alexander Bogomolny).

Assume, without loss of generality (http://www.cut-the-knot.org/blue/WLOG.shtml), that $AB > AC$. Then $\angle ACB > \angle ABC$.
Since ΔMAN is isosceles, $\angle AMB = \angle ANC$, implying that $\angle BAM > \angle CAN$. Obviously, $\angle BAM + \angle CAN < \pi$, implying by Leo Giugiuc's Lemma http://www.cut-the-knot.org/arithmetic/algebra/LeosLemma.shtml), that

$$\sin \angle BAM > \sin \angle CAN.$$



Set $\delta = \angle AMB = \angle ANC$. By the Law of Sines (http://www.cut-the-knot.org/pythagoras/cosine2.shtml), in triangles ABM and ACN, and due to the assumption $AB > AC$,

$$\frac{BM}{\sin \angle BAM} = \frac{AB}{\sin \delta} > \frac{AC}{\sin \delta} = \frac{CN}{\sin \angle CAN}$$

such that

$$BM = \frac{\sin \angle BAM}{\sin \angle CAN} CN > CN$$

because, as we've seen, $AB > AC$ implies $\sin \angle BAM > \sin \angle CAN$.
Thus, in this case, indeed, $(AB - AC)(BM - CN) \geq 0$. The assumption $AB < AC$ - by symmetry - leads to the same result. The case $AB = AC$ is even more straightforward. □

44. AN EQUATION IN DETERMINANTS

Dan Sitaru has posted the following problem and its solution at the CutTheKnotMath facebook page:

Solve in $M_4(\mathbb{Z})$ the equation: $\det(X^4 + I_4) = 2013$

Proposed by Daniel Sitaru - Romania

Solution (by Alexander Bogomolny).

Observe that $X^4 + I_4 = (X^2 - X\sqrt{2} + I_4)(X^2 + X\sqrt{2} + I_4)$ from which

$$\det(X^4 + I_4) = (m - n\sqrt{2})(m + n\sqrt{2}) = m^2 - 2n^2; m, n \in \mathbb{Z}.$$

Now, $m^2 - 2n^2 = 2013$ is same as $m^2 - 2013 = 2n^2$. The latter implies $m \in 2\mathbb{Z} + 1$, and consequently $m^2 \in 8\mathbb{Z} + 1$. We'll show that is impossible by considering two case.

If $n \in 2\mathbb{Z}$ then $2n^2 \in 8\mathbb{Z}$ and $m^2 - 2n^2 \in 8\mathbb{Z} + 1$ which means that $2013 \in 8\mathbb{Z} + 1$. But this is not so because $2013 \mod 8 \equiv 5$.

If $n \in 2\mathbb{Z} + 1$ then $n^2 \in 8\mathbb{Z}$ so that $m^2 - 2n^2 \in 8\mathbb{Z} + 7$, or $2013 \in 8\mathbb{Z} + 5$, but this is as impossible as the preavious case.

Thus the equation has no solutions.

□

Remark (by Alexander Bogomolny)

The problem can be stated as a scaler or a polynomial equation, with only minor typographical changes.

The problem poses the question for the year 2013. The above solution will work for any year residue of division by 8 is neither 1 nor 7. Thus it will not work out (as it was pointed out in the comments) for 2015; there is enough time to investigate whether the matrix equation has or does not have a solution for a coming year. The scalar equivalent, obviously, does not dose have a solution in integers for most of the years.

45. AN INNOVATIVE RECURRENCE

Assume $a_1, a_2, a_3, a_4 \in \mathbb{N}$ and $a_{a_n} = n + 4$.

Find $a_{2012} + a_{2013} + a_{2014} + a_{2015}$.

Proposed by Daniel Sitaru, Leonard Giugiuc - Romania

Solution by proposers, comments by Alexander Bogomolny:

Let's use functional notations: $f(n) = a_n$ where $f : \mathbb{N} \to \mathbb{N}$.

Then $f(f(n)) = n + 4$. The trick is to express $f(f(f(n)))$ in two ways: one is to replace n with $f(n)$, the other is to apply f to both sides of the identity:

$$f(f(f(n))) = f(n) + 4$$
$$f(f(f(n))) = f(n + 4).$$

From this $f(n+4) = f(n) + 4$.
Repeating the steps, $f(n+4) = f(n)+4 = f(n-4)+2\cdot 4 = f(n-8)+3\cdot 4$, etc., $f(n) = f(n-4k) + 4k, k$ a positive integer. In particular,

$$f(2012) = f(0) + 2012, f(2013) = f(1) + 2012,$$
$$f(2014) = f(2) + 2012, f(2015) = f(3) + 2012.$$

In the originals terms

$$a_{2012} + a_{2013} + a_{2014} + a_{2015} = a_0 + a_1 + a_2 + a_3 + 4 \cdot 2012.$$

$$\text{In general, } a_n = \begin{cases} a_0 + n, n = 4k, \\ a_1 + n - 1, n = 4k + 1, \\ a_2 + n - 2, n = 4k + 2, \\ a_3 + n - 3, n - 4k + 3. \end{cases}$$

This could be expressed with a single formula: $a_n = a_{n \mod 4} + 4 \cdot \left\lfloor \frac{n}{4} \right\rfloor$.

□

46. TANGENTIAL CHAOS

Solve in real numbers:

$$\begin{cases} yx^4 + 4x^3 + y = 6x^2y + 4x \\ zy^4 + 4y^3 + z = 6y^2z + 4y \\ xz^4 + 4z^3 + x = 6z^2x + 4z \end{cases}$$

Proposed by Daniel Sitaru - Romania

Solution by Daniel Sitaru - Romania.
Let $x = \tan a, a \in (-\frac{\pi}{3}, \frac{\pi}{2})$. Then $y = \frac{4x-4x^3}{x^4-6x^2+1} = \tan 4a$.
Thus, $z = \tan 16a$ and $x = \tan 64a$, implying $\tan a = \tan 64a$ from which $a = \frac{k\pi}{63}$, with k an integer. But, since $a \in (-\frac{\pi}{2}, \frac{\pi}{2}), k = 0, \pm 1, \ldots \pm 31$.
It follows that

$$(x, y, z) \in \left\{ \left(\tan \frac{k\pi}{63}, \tan \frac{4k\pi}{63}, \tan \frac{16k\pi}{63} \right) : k \in \{0, \pm 1, \ldots, \pm 31\} \right\}$$

□

Note by Alexander Bogomolny
Let $f(x) = \frac{4x-4x^3}{x^4-6x^2+1}$. Then the solution to the system $y = f(x)$, $z = f(y), x = f(z)$ could be seen as having iterations on f run into 3 - cycle which, reminds (if only spuriously) of Sharkovsky's theorem (http://www.tufts.edu/as/math/Preprints/BurnsHasselblattShort.pdf)(see also Period Three Implies Chaos) (http://faculty.washington.edu/joelzy/LiYorke_period3.pdf) means that the iteration on function f have cycles of any length and are, in principle, chaotic. Dan's solution makes it obvious that the substitutions $x = \tan a$ will solve n - cycles for any $n = 2, 3, 4, \ldots$ Moreover, the union of all such solutions is the countable set of numbers in the form $\frac{k\pi}{4^n-1}$, where $|k| < 4^n/2$Iterations that start with any other

point will be chaotic.

Quite obviously the same can be said of a simpler function $f(x) = \frac{2x}{1-x^2}$, that, for example, could be converted to a system of three much simpler equations:

$$\begin{cases} y - 2x = x^2 y \\ z - 2y = y^2 z \\ x - 2z = z^2 x \end{cases}$$

47. DAN SITARU'S CYCLCIC INEQUALITY IN ONE VARIABLE

Prove that, for $x \in \mathbb{R}$,

$$\left(\sqrt{x^2 - x + 1} - \sqrt{x^2 + x + 1}\right)^2 + \left(\sqrt{x^2 - x + 1} - \sqrt{4x^2 + 3}\right)^2 +$$

$$+\left(\sqrt{x^2 + x + 1} - \sqrt{4x^2 + 3}\right)^2 < 6x^2 + 2.$$

Proposed by Daniel Sitaru - Romania

Solution 1 (by Soumava Chakraborty - Kolkata - India).
For typographic convenience, let's denote $a = x^2 - x + 1, b = x^2 - x + 1$, and $c = 4x^2 + 3$. Then, upon squaring the required inequality takes an equivalent form:

$$6x^2 + 8 = 12x^2 + 10 - (6x^2 + 2) < 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}),$$

$$\text{or, } 3x^2 + 4 < \sqrt{ab} + \sqrt{bc} + \sqrt{ca}.$$

Squaring once more give

$$3x^2 + 4)^2 - (ab + bc + ca) < 2(a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}),$$

which reduces to

$$(*) \qquad 9(x^2 + 1) < 2(a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}).$$

Now,

$$a = x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4},$$

$$b = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4},$$

$$c = 4x^2 + 3 \geq 3.$$

The right - hand side of * is then estimated as

$$RHS\ * \ > 2\left(a\frac{3}{2} + b\frac{3}{2} + c\frac{3}{4}\right) = 3(2x^2 + 2) + \frac{3}{2}(4x^2 + 3) =$$

$$= 12x^2 + \frac{21}{2} > 9(x^2 + 1) = LHS\ *.$$

□

Solution 2 (by Daniel Sitaru - Romania).
Let's denote $= \sqrt{x^2 - x + 1}, c = \sqrt{x^2 - x + 1}$, and $a = \sqrt{4x^2 + 3}$. Note that a, b, c are the sides of a triangle: $a + b > c, b + c > a, c + a > b$.
For example, square $b + c > a$ to obtain

$$2(x^2 + 1) + 2\sqrt{(x^2 + 1)^2 - x^2} > 4x^2 + 3, \text{ or } 2\sqrt{(x^2 + 1)^2 - x^2} > 2x^2 + 1;$$

squaring the second time:
$4x^2 + 4x^2 + 4 > 4x^2 + 4x^2 + 1$, which is true.
In ABC,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{-2x^2 - 1}{2\sqrt{x^4 + x^2 + 1}}$$

$$\sin A = \sqrt{1 - \left(\frac{-2x^2 - 1}{2\sqrt{x^4 + x^2 + 1}}\right)^2} = \sqrt{\frac{3}{4(x^4 + x^2 + 1)}}$$

$$S = \frac{1}{2} bc \sin A = \frac{1}{2} \cdot \sqrt{x^4 + x^2 + 1} \cdot \sqrt{\frac{3}{4(x^4 + x^2 + 1)}} = \frac{\sqrt{3}}{4}.$$

By the *Hadwiger - Finsler inequality*,

$$\sum_{cycl} (a - b)^2 + 4S\sqrt{3} < \sum_{cycl} a^2$$

such that

$$\sum_{cycl} (a - b)^2 + 4\sqrt{3} \cdot \frac{\sqrt{3}}{4} < x^2 - x + 1 + x^2 + x + 1 + 4x^2 + 3,$$

$$\text{i.e., } \sum_{cycl} (a - b)^2 < 6x^2 + 2.$$

This is exactly the required inequality. □

Acknowledgment (by Alexander Bogomolny)
Daniel Sitaru has kindly posted a problem for the *Romanian Mathematical Magazine*, with a solution (Solution 1) by Soumava Chakraborty. Solution 2 is by Daniel Sitaru.

48. DAN SITARU'S CYCLCIC INEQUALITY IN MANY VARIABLES

Prove that, for $a, b, c, d > 0$,

$$a + b + c + d \leq \frac{a^5 + b^5 + c^5 + d^4}{abcd}.$$

Solution (by Kunihiko Chikaya - Tokyo - Japan).
By the *AM-GM inequality:*

$$\frac{a^5 + (a^5 + b^5 + c^5 + d^5)}{5} \geq \sqrt[5]{a^5 \cdot a^5 b^5 c^5 d^5} = a \cdot abcd,$$

$$\frac{b^5 + (a^5 + b^5 + c^5 + d^5)}{5} \geq \sqrt[5]{b^5 \cdot a^5 b^5 c^5 d^5} = b \cdot abcd,$$

$$\frac{c^5 + (a^5 + b^5 + c^5 + d^5)}{5} \geq \sqrt[5]{c^5 \cdot a^5 b^5 c^5 d^5} = c \cdot abcd,$$

$$\frac{d^5 + (a^5 + b^5 + c^5 + d^5)}{5} \geq \sqrt[5]{d^5 \cdot a^5 b^5 c^5 d^5} = d \cdot abcd.$$

Adding up the four gives

$$\frac{5(a^5 + b^5 + c^5 + d^5)}{5} \geq (a + b + c + d)abcd,$$

which is exactly the required inequality. □

Generalization (by Alexander Bogomolny)

Prove that, for integer $n > 0$ ***and*** $a_k, k \in \overline{1, n}$,

$$\Big(\sum_{k=1}^{n} a_k\Big)\Big(\prod_{k=1}^{n} a_k\Big) \leq \sum_{k=1}^{n} a_k^{n+1}.$$

Proof (by Alexander Bogomolny).
By the AM-GM inequality,

$$\sum_{j=1}^{n}\Big(\frac{ta_j^{n+t} + \sum_{k=1}^{n} a_k^{n+t}}{n+t}\Big) \geq \sum_{k=1}^{n} \sqrt[n+t]{a_j^{t(n+t)} \cdot \prod_{k=1}^{n} a_k^{n+t}} =$$

$$= \sum_{k=1}^{n} a_j^t \cdot \prod_{k=1}^{n} a_k = \Big(\sum_{k=1}^{n} a_j^t\Big) \cdot \Big(\prod_{k=1}^{n} a_k\Big).$$

It remains only to note that

$$\sum_{j=1}^{n}\Big(\frac{ta_j^{n+t} + \sum_{k=1}^{n} a_k^{n+t}}{n+t}\Big) = \frac{t\sum_{j=1}^{n} a_j^{n+t} + \sum_{j=1}^{n}\sum_{k=1}^{n} a_k^{n+t}}{n+t}$$

$$= \frac{t\sum_{j=1}^{n} a_j^{n+t} + n\sum_{k=1}^{n} a_k^{n+t}}{n+t} = \frac{(n+t)\sum_{k=1}^{n} a_k^{n+t}}{n+t} = \sum_{kj=1}^{n} a_k^{n+t}$$

□

Acknowledgment (by Alexander Bogomolny)
Dan Sitaru has shared a problem from the *Romania Mathematical Magazine*, with a beautiful solution by Kunihiko Chikaya. Both the problem and the solution suggest a generalization.

49. DIMENSIONLESS INEQUALITY IN THE EUCLIDEAN PLANE

Given six points in the Euclidean plane: A, B, C, D, E, F. ***Prove that***

$$2(AB^2 + BC^2 + CD^2 + DE^2 + EF^2 + FA^2) \geq AD^2 + BE^2 + CF^2$$

Source: TST - Romania

Proof (by Ioan Serdean - Romania).
Let the points have coordinates $(x_k, y_k), k = 1, \ldots, 6$. **Setting** $(x_7, y_7) = (x_1, y_1)$, **the required inequality becomes**

$$2\Big(\sum_{k=1}^{6}\Big[(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2\Big]\Big) \geq \sum_{k=1}^{3}\Big[(x_{k+3} - x_k)^2 + (y_{k+3} - y_k)^2\Big]$$

This will be proved if we manage to show that

$$2\Big(\sum_{k=1}^{6}(x_{k+1}-x_k)^2\Big) \geq \sum_{k=1}^{3}(x_{k+3}-x_k)^2,$$

$$2\Big(\sum_{k=1}^{6}(y_{k+1}-y_k)^2\Big) \geq \sum_{k=1}^{3}(y_{k+3}-y_k)^2.$$

Obviously, suffice it to prove just one of these inequalities as the difference between the two is exclusively notational. Remarkably, this would mean that the problem could have been posed in any Euclidean space $\mathbb{R}^n, n \geq 1$, and not just the Euclidean plane.
Thus introduce $a_k = x_{k+1} - x_k, k = 1, \ldots, 6$. The first inequality then reduces to

$$2\Big(\sum_{k=1}^{6} a_k^2\Big) \geq \sum_{k=1}^{3}(a_{k+2}+a_{k+1}+a_k)^2.$$

Note that by the definition, $\sum_{k=1}^{6} a_k = 0$. With this contraint, the above inequality is equivalent to

$$2\Big(\sum_{k=1}^{5} a_k^2\Big) + 2\Big(\sum_{k=1}^{5} a_k\Big) \geq \sum_{k=1}^{3}(a_{k+2}+a_{k+1}+a_k)^2.$$

The latter can be transformed into

$$\Big(\sum_{k=1}^{5} a_k\Big)^5 + (a_1+a_3+a_5)^2 + (a_1+a_4)^2 + (a_2+a_5)^2 \geq 0$$

which is of course true. □

Acknowledgment (by Alexander Bogomolny)
The problem from the ***Romanian Mathematical Magazine*** has been kindly shared at the ***CutTheKnotMath facebook page*** by Daniel Sitaru, along with the beautiful solution by Ioan Serdean.

50. A TRIGONOMETRIC INEQUALITY FROM THE RMM

Prove that, for $x, y \in \Big(0, \dfrac{\pi}{2}\Big), \sin(x+y) \leq \sin x\Big(\dfrac{\sin y}{y}\Big)^3 + \sin y\Big(\dfrac{\sin x}{x}\Big)^3$

Proposed by Daniel Sitaru - Romania

Proof (by Soumava Chakrabory - Kolkata - India).
Due to the addition formula for sine, $\sin(x+y) = \sin x \cos y + \cos x \sin y$, to prove the required inequality suffice it to establish that, for
$z \in \Big(0, \frac{\pi}{2}\Big), \cos z < \Big(\frac{\sin z}{z}\Big)^3$. This is the same as $f(z) = \sin^2 z \tan z - z^3 > 0$.
Note that $f(0) = 0$. We shall differentiable repeatedly.

$$\begin{aligned} f'(z) &= \sin^2 z \sec^2 z + \tan z(2\sin z\cos z) - 3z^2 \\ &= \tan^2 z + 2\sin^2 z - 3z^2. \end{aligned}$$

Introduce $g(z) = f'(z)$ and note that $g(0) = 0$.

$$g'(z) = 2\tan z\sec^2 z + 4\sin z\cos z - 6z$$

Introduce $h(z) = \frac{1}{2}g'(z)$ and note that $h(0) = 0$.

$$\begin{aligned} h'(z) &= (\sec^2 z)^2 + (\tan z)(2\sec z)(\sec z\tan z) + 2(\cos^2 z - \sin^2 z) - 3 \\ &= (1+\tan^2 z)^2 + 2\tan^2 z(1+\tan^2 z) + 2(2\cos^2 z - 1) - 3 \\ &= (1+t^2)^2 + 2t(1+t^2) + \frac{4}{1+t} - 5 = \frac{3t^3+7t^2}{1+t}, \end{aligned}$$

$t = \tan z > 0$ and so $h'(z) > 0$. Hence, $h(z) > h(0) = 0$, so that $g'(z) > 0$, and $g(z) > g(0) = 0$, meaning $f'(z) > 0$ and $f(z) > f(0) = 0$.
This completes the proof. □

Acknowledgment (by Alexander Bogomolny)
Dan Sitaru has kindly posted the above problem form the *Romanian Mathematical Magazine* at the *CutTheKnotMath facebook page*.

Its nice to be important but more important its to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru