In $\Delta\ ABC$:

$$\frac{R}{2r} \geq \left(\frac{64a^2b^2c^2}{(4a^2-(b-c)^2)(4b^2-(c-a)^2)(4c^2-(a-b)^2)}\right)^2$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Soumitra Mandal-Chandar Nagore-India

Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by Soumitra Mandal-Chandar Nagore-India

Applying RAVI Transformation, $a = x + y$, $b = y + z$ ***and*** $c = z + x$

$\therefore p = \frac{a+b+c}{2} = x + y + z$. ***Now,*** $\frac{R}{2r} = \frac{abcp}{4\Delta^2}$, ***where*** Δ = ***area of*** $\Delta\ ABC$

$\frac{R}{2r} = \frac{(x+y)(y+z)(z+x)}{8xyz}$. ***Again,*** $4a^2 - (b-c)^2 = (2a+b-c)(2a-b+c)$

$= (3y+x)(3x+y)$, ***similarly,*** $4b^2-(c-a)^2 = (3y+z)(3z+y)$ ***and***

$4c^2-(a-b)^2 = (3z+x)(3x+z)$. ***So, we need to prove,***

$$\frac{(x+y)(y+z)(z+x)}{8xyz} \geq \left(\frac{64\prod_{cyc}(x+y)^2}{(3x+y)(3y+x)(3y+z)(3z+y)(3z+x)(3x+z)}\right)^2$$

$$\Leftrightarrow (3x+y)^2(3y+x)^2(3y+z)^2(3z+y)^2(3z+x)^2(3x+z)^2 \geq 32^3\prod_{cyc}(x+y)^3$$

Now, $(3x+y)(3y+x) = 3(x+y)^2 + 4xy \geq 4\sqrt[4]{4xy(x+y)^6}$

Similarly, $(3y+z)(3z+y) \geq 4\sqrt[4]{4yz(y+z)^6}$ ***and***

$(3z+x)(3x+z) \geq 4\sqrt[4]{4zx(z+x)^6}$

$$\therefore (3x+y)^2(3y+x)^2(3y+z)^2(3z+y)^2(3z+x)^2(3x+z)^2 \geq 32^3\prod_{cyc}(x+y)^3$$

(established)

$$\frac{R}{2r} \geq \left(\frac{64a^2b^2c^2}{(4a^2-(b-c)^2)(4b^2-(c-a)^2)(4c^2-(a-b)^2)}\right)^2$$

(proved)

Solution 2 by Soumava Chakraborty-Kolkata-India

Given inequality $\Leftrightarrow$

$$\left\{\frac{[4a^2-(b-c)^2][4b^2-(c-a)^2][4c-(a-b)^2]}{64a^2b^2c^2}\right\}^2 \geq \frac{2r}{R} \quad \textbf{(1)}$$

Now, $4a^2-(b-c)^2 = 3a^2+a^2-(b-c)^2$

$$= 3a^2 + (a+b-c)(a-b+c) = 3a^2 + 4(s-b)(s-c)$$

$$= a^2+a^2+a^2+4(s-b)(s-c) \overset{A-G}{\geq} 4\sqrt[4]{a^6\cdot 4(s-b)(s-c)}$$

$$\Rightarrow 4a^2-(b-c)^2 \geq 4\sqrt[4]{4a^6(s-b)(s-c)} \quad \textbf{(i)}$$

Similarly, $4b^2-(c-a)^2 \geq 4\sqrt[4]{4b^6(s-c)(s-a)}$ ***(ii) and***

$$4c^2-(a-b)^2 \geq 4\sqrt[4]{4c^6(s-a)(s-b)} \quad \textbf{(iii)}$$

$$\therefore LHS\ \textbf{of (1)} \geq \left(\frac{64\sqrt[4]{64a^6b^6c^6(s-a)^2(s-b)^2(s-c)^2}}{64a^2b^2c^2}\right)^2 \ \textbf{(by (i)} \times \textbf{(ii)} \times \textbf{(iii))}$$

$$= \frac{8a^3b^3c^3(s-a)(s-b)(s-c)}{a^4b^4c^4} = \frac{8s(s-a)(s-b)(s-c)}{s(4Rrs)} = \frac{8r^2s^2}{4Rrs^2}$$

$$= \frac{2r}{R} = RHS\ \textbf{of (1)}$$

(Proved)