SOCIETATEA DE ȘTIINȚE MATEMATICE DIN ROMÂNIA
ROMANIAN MATHEMATICAL SOCIETY

Filiala Mehedinți - Mehedinți Branch
www.ssmrmh.ro

Let ABC be an arbitrary triangle, I_a, I_b, I_c are excenters, r is inradius, and R is circumradius of ABC. Prove that:

$$12r\sqrt{3} \le P(I_aI_bI_c) \le 6R\sqrt{3}$$

where $P(I_aI_bI_c)$ is perimeter of ABC.

Proposed by Mehmet Şahin – Ankara – Turkey

Solution by Adil Abdullayev – Baku – Azerbaidjian

Lemma 1.

$$I_AI_B = 4R\cos\frac{C}{2} = 4R\sqrt{\frac{p(p-c)}{ab}}$$

Lemma 2.

$$\sum \cos\frac{A}{2} \le \frac{3\sqrt{3}}{2}$$

$$LHS = P(I_AI_BI_C) = 4R\sum_{cyc}\cos\frac{A}{2} \le 6R\sqrt{3} \leftrightarrow \sum_{cyc}\cos\frac{A}{2} \le \frac{3\sqrt{3}}{2}$$

$$LHS = 4R\sum_{cyc}\sqrt{\frac{p(p-a)}{bc}} \ge 12r\sqrt{3} \leftrightarrow \sum_{cyc}\sqrt{\frac{p(p-a)}{bc}} \ge \frac{3r\sqrt{3}}{R}$$

$$AM-GM \rightarrow \sum_{cyc}\sqrt{\frac{p(p-a)}{bc}} \ge 3\sqrt[6]{\frac{p^3(p-a)(p-b)(p-c)}{a^2b^2c^2}} \ge \frac{3r\sqrt{3}}{R} \leftrightarrow$$

$$\leftrightarrow p^2 \ge \frac{16\cdot 27r^6}{R^4}$$

GERRETSEN $\rightarrow p^2 \ge 16Rr - 5r^2 \ge 27r^2 \ge \frac{16\cdot 27r^6}{R^4} \leftrightarrow$

$$\leftrightarrow R^4 \ge 16r^4 \leftrightarrow R \ge 2r$$