Functional Reciprocal Theory of degree 2

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1 Introduction

If for a function $f : R^+ \to R^+$ over some subset $S \subseteq R^+$ we have
• $\forall x \in S, \forall y \in S$ the pairs $(f(x) \cdot x, f(y) \cdot y)$ and $(\frac{f(x)}{x}, \frac{f(y)}{y})$ are similarly sorted then we say that $f :$ is p-monotone over S.
And if ,
• $\forall x \in S, \forall y \in S$ the pairs $(f(x) \cdot x, f(y) \cdot y)$ and $(\frac{f(x)}{x}, \frac{f(y)}{y})$ are oppositely sorted then we say that $f :$ is i-monotone over S.

2 Functional Reciprocal Sum inequalities

With this we have some lemmas :

Lemma 1:(First Functional Reciprocal Sum Inequality (FRS 1)
If $f : R^+ \to R^+$ is p-monotone over $S \subseteq R^+$ and $g : R^+ \to R^+$ is i-monotone over S then, we have the following inequality :
$2 \leq \frac{g(x)}{g(y)} + \frac{g(y)}{g(x)} \leq \frac{x}{y} + \frac{y}{x} \leq \frac{f(x)}{f(y)} + \frac{f(y)}{f(x)}$

PROOF
See the pairs $(f(x) \cdot x^2 y, f(y) \cdot y^2 x)$ and $(f(x) \cdot x, f(y) \cdot y)$ are simlarly sorted because $xy > 0$ and now since $f :$ is p-monotone over S therefore the pairs $(\frac{f(x)}{x}, \frac{f(y)}{y})$ and $(f(x) \cdot x, f(y) \cdot y)$ are similarly sorted and thus the pairs
$(f(x) \cdot x^2 y, f(y) \cdot y^2 x)$ and $(\frac{f(x)}{x}, \frac{f(y)}{y})$ are similarly sorted.
Thus by Rearrangement inequality
$f(x) \cdot x^2 y \cdot \frac{f(x)}{x} + f(y) \cdot y^2 x \cdot \frac{f(y)}{y} \geq f(x) \cdot x^2 y \cdot \frac{f(y)}{y} + f(y) \cdot y^2 x \cdot \frac{f(x)}{x}$ and thus we get
$xy \cdot ((f(x))^2 + (f(y))^2) \geq f(x)f(y) \cdot (x^2 + y^2)$ and thus we get the required inequality on dividing both sides by $f(x)f(y)xy$ (and this is greater than 0)

Similalry we get the other inequality by using $g :$ is i-monotone over S

3 Problems

Problem 1

For all positive reals a, b and all non-negative real numbers c prove the following:
$\frac{a+c}{b+c} + \frac{b+c}{a+c} \leq \frac{a}{b} + \frac{b}{a}$

Problem 2

For all positive a, b and all nonnegative c prove that:
$\frac{a^{1+c}}{b^{1+c}} + \frac{b^{1+c}}{a^{1+c}} \geq \frac{a}{b} + \frac{b}{a}$

Problem 3 (*Ceu Azul*)

For all acute angles A, B prove that:
$\frac{\sin A}{\sin B} + \frac{\sin B}{\sin A} \leq \frac{A}{B} + \frac{B}{A} \leq \frac{\tan A}{\tan B} + \frac{\tan B}{\tan A}$

Problem 4

For all $x > 1, y > 1$ show that :
$\frac{x^x}{y^y} + \frac{y^y}{x^x} \geq \frac{x}{y} + \frac{y}{x}$

4 FRS sequence convergence

FRS convergence for i-type monotone functions

If $f : R^+ \to R^+$ is i-monotone over R^+ and $f_1(x) = f(x) \forall x \in R^+$ and $\forall n \in N f_{n+1}(x) = f(f_n(x))$ then if the sequence $F_n(x,y)_{n \in N}$ has the following definition that : $\forall n \in N F_n(x,y) = \frac{f_n(x)}{f_n(y)} + \frac{f_n(y)}{f_n(x)}$ then, the sequence $F_n(x,y)_{n \in N}$ is convergent

(Sketched)Proof

Note that the sequence $F_n(x,y)_{n \in N}$ is a monotone decreasing sequence (from FRS 1) and is bounded below (2 is the lower bound) and therefore it must converge since every monotone decreasing sequence of real numbers with a real lower bound must converge.

For p-monotone functions

Note that there are p-monotone functions $f : R^+ \to R^+$ such that the sequence $F_n(x,y)_{n \in N}$ diverges

For example: Consider the function $f : R^+ \to R^+$ defined by $\forall x \in R^+$ $f(x) = x^2$ is a *p-monotone* function over positive real numberss and the sequence $F_n(x,y)_{n \in N}$ diverges.

But however not always the sequence $F_n(x,y)_{n \in N}$ diverges.

For example: Consider the function $f(x) = x$ defined over all positive real numbers.

5 FRS series divergence

With same notations as we used for FRS sequence convergence if
For a function $f : R^+ \to R^+$ we define a sequence $H_n(x,y)_{n \in N}$ by $\forall n \in N$ $H_n(x,y) = \sum_{i=1}^{n} F_n(x,y)$ then the sequence $H_n(x,y)_{n \in N}$ diverges.
(Sketched)<u>Proof</u>
Note that $\forall n \in N$ we must have :
$H_n(x,y) \geq 2$ and thus the sequence $H_n(x,y)_{n \in N}$ diverges.

6 References:

http://www.artofproblemsolving.com/community/q1h1342252p7292979
http://www.artofproblemsolving.com/community/q1h1342252p7293009
http://www.artofproblemsolving.com/community/q1h1342252p7302439

Solution to Problem 3 can be found by visiting the following:
http://www.artofproblemsolving.com/community/c6h1299029p6910573

7 Solution and Hints to some problems

<u>For Problem 1</u>
Note that the function $f : R^+ \to R^+$ defined by $f(x) = x + c$ is *i-monotone* over the set of all positive real numbers.
<u>For Problem 2</u>
Note that the function $f : R^+ \to R^+$ defined by $f(x) = x^{1+c}$ is *p-monotone* over the set of all positive real numbers.