REGARDING PROBLEM S:L16.284 FROM SGM 11/2016 METHODS OF SOLVING AN INEQUALITY

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In Mathematical Gazette Supplement nr. 11/2016 the following problem is proposed:
Let be $n \in N, n \geq 2$. Prove that for $a_1, a_2, \ldots, a_n \in \left(0, \sqrt{n}\right]$, with
$a_1 + a_2 + \ldots a_n = n$, the following inequality holds:

$$\frac{1}{a_1^2 + (a_2 + a_3 + \ldots + a_n)} + \frac{1}{a_2^2 + (a_1 + a_3 + \ldots + a_n)} + \ldots + \frac{1}{a_n^2 + (a_1 + a_3 + \ldots + a_{n-1})} \leq 1$$

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The article presents a methodical treatment of this problem, descending it first to three variable, developing then this result and finishing with the developing of the general case.

Proof.

Case $n = 3$.
If $a, b, c > 0$ with $a + b + c = 3$, prove that $\frac{1}{a^2+b+c} + \frac{1}{b^2+c+a} + \frac{1}{c^2+a+b} \leq 1$. □

Proof. Because $a + b + c = 3$, we have $b + c = 3 - a$ and we write the inequality $\sum \frac{1}{a^2-a+3} \leq 1$.
In order to obtain this result we look for an inequality having the following form: $\frac{1}{a^2-a+3} \leq x \cdot a + y$ (Tangent Line Method) and we determine x and y such that the attached equation in the variable "a" to have double root on 1. We obtain $x = \frac{-1}{9}$ and $y = \frac{4}{9}$.
We have $\frac{1}{a^2-a+3} \leq \frac{4-a}{9} \Leftrightarrow (a-1)^2(3-a) \geq 0$, obviously from $a, b, c > 0$ and $a + b + c = 3$, with equality if and only if $a = 1$.
We obtain $\sum \frac{1}{a^2-a+3} \leq \sum \frac{4-a}{9} = \frac{12-\sum a}{9} = \frac{12-3}{9} = 1$.
The equality holds if and only if $a = b = c = 1$.

□

Development.
If $a, b, c > 0$ with $a + b + c = 3$, prove that

$$\frac{1}{a^2 + k(b+c)} + \frac{1}{b^2 + k(c+a)} + \frac{1}{c^2 + k(a+b)} \leq \frac{3}{2k+1}, \text{ where } 1 \leq k \leq 2.$$

Proof. Because $a + b + c = 3$, we have $b + c = 3 - a$ and we write the inequality

$$\sum \frac{1}{a^2 + k(3-a)} \leq \frac{3}{2k+1}$$

We look for an inequality having the form $\frac{1}{a^2+k(3-a)} \leq x \cdot a + y$ and we determine x and y such that the attached equation the variable "a" to have double root on 1.

We obtain $x = \frac{k-2}{(2k+1)^2}$ and $y = \frac{k+3}{(2k+1)^2}$.
We have $\frac{1}{a^2+k(3-a)} \leq \frac{k+3+(k-2)a}{(2k+1)^2} \Leftrightarrow (a-1)^2\Big[(k-2)a-k^2+5k-1\Big] \geq 0$, obviously from $a,b,c>0$ with $a+b+c=3$ and $1 \leq k \leq 2$, with equality if and only if $a=1$.
We obtain

$$\sum \frac{1}{a^2+k(3-a)} \leq \sum \frac{k+3+(k-2)a}{(2k+1)^2} = \frac{3(k+3)+(k-2)\sum a}{(2k+1)^2} =$$
$$= \frac{3(k+3)+(k-2)3}{(2k+1)^2} = \frac{3}{2k+1}$$

The equality holds if and only if $a=b=c=1$. □

Solving the general case.
Let be $n \in N, n \geq 2$. Prove that for $a_1, a_2, \ldots, a_n > 0$, with $a_1+a_2+\ldots+a_n = n$, then

$$\frac{1}{a_1^2+(a_2+a_3+\ldots+a_n)} + \frac{1}{a_2^2+(a_1+a_3+\ldots+a_n)} + \ldots + \frac{1}{a_n^2+(a_1+a_3+\ldots+a_{n-1})} \leq 1$$

Proof. Because $a_1+a_2+\ldots+a_n=n$, we have $a_2+a_3+\ldots+a_n = n-a_1$ and we write the inequality

$$\sum_{i=1}^{n} \frac{1}{a_i^2-a_i+n} \leq 1.$$

We look an inequality having the form $\frac{1}{a^2-a+n} \leq x \cdot a + y$, and we determine x and y such that the attached equation in variable "a" to have double root on 1.
We obtain $x = \frac{-1}{n^2}$ and $y = \frac{n+1}{n^2}$.
We have $\frac{1}{a_i^2-a+n} \leq \frac{n+1-a_i}{n^2} \Leftrightarrow (a_i-1)^2(n-a_i) \geq 0$, obviously from $\sum_{i=1}^{n} a_i = 1$ and $a_i > 0, i=\overline{1,n}$, with equality if and only if $a_i = 1, i = \overline{1,n}$.
We obtain $\sum_{i=1}^{n} \frac{1}{a_i^2-a_i+n} \leq \sum_{i=1}^{n} \frac{n+1-a_i}{n^2} = \frac{n(n+1)-n}{n^2} = 1$.
The equality holds if and only if $a_1 = a_2 = \ldots = a_n = 1$.

□

Observation.
The condition $a_1, a_2, \ldots, a_n \in \Big(0, \sqrt{n}\Big]$ is not necessary. It is sufficient to have $a_1, a_2, \ldots, a_n > 0$.
Development.
Let be $n \in N, n \geq 2$. Prove that for $a_1, a_2, \ldots, a_n > 0$, with $a_1+a_2+\ldots+a_n = n$ holds the following inequality:

$$\frac{1}{a_1^2+k(a_2+a_3+\ldots+a_n)} + \frac{1}{a_2^2+k(a_1+a_3+\ldots+a_n)} + \ldots +$$
$$+ \frac{1}{a_n^2+k(a_1+a_3+\ldots+a_{n-1})} \leq \frac{n}{1+k(n-1)}$$

where $1 \leq k \leq 2$.

Proof. Because $a_1+a_2+\ldots+a_n=n$, we have $a_2+a_3+\ldots+a_n = n-a_1$ and we write the inequality

$$\sum_{i=1}^{n} \frac{1}{a_i^2+k(n-a_i)} \leq \frac{3}{1+k(n-1)}.$$

We look for an inequality having the form $\frac{1}{a^2+k(n-a)} \le x \cdot a + y$ and we determine x and y such that the attached equation in variable "a" to have double root on 1. We obtain

$$x = \frac{k-2}{[1+k(n-1)]^2} \text{ and } y = \frac{3+k(n-2)}{[1+k(n-1)]^2}.$$

We have $\frac{1}{a_i^2+k(n-a_i)} \le \frac{kn+3-2k+(k-2)a_i}{[1+k(n-1)]^2} \Leftrightarrow (a_i-1)^2\left[(k-2)a_i+kn-(k-1)^2\right] \ge 0$, obviously from $\sum_{i=1}^n a_i = 1$ and $a_i > 0, i = \overline{1,n}$, with inequality if and only if $a_i = 1, i = \overline{1,n}$.

We obtain $\sum_{i=1}^n \frac{1}{a_i^2+k(n-a_i)} \le \sum_{i=1}^n \frac{kn+3-2k+(2-k)a_i}{[1+k(n-1)]^2} \le \frac{n(kn+3-2k)+(k-2)n}{[1+k(n-1)]^2} =$
$= \frac{n}{1+k(n-1)}$.

The equality holds if and only if $a_1 = a_2 = \ldots = a_n = 1$. □

References

[1] Andra - Mălina Cardaş, *Problem S:L16.284.*, Mathematical Gazette Supplement, nr. 11/2016.

[2] Marin Chirciu, *Algebraic Inequalities, from initiation to performance.*, Paralela 45 Publishing House, Piteşti, 2015, Problem 1.81, page 25.

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