SUMMARY OF THE INEQUALITIES PROBLEM

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Problem 1 : Let $a_k, (k = 1, 2, 3, ..., n)$ be real number such that : $\sum_{k=1}^{n} a_k = \frac{n(n+1)}{2}$.

Prove that : $\sum_{k=1}^{n} \frac{(k^2-1)a_k + k^2 + 2k}{a_k^2 + a_k + 1} \geq \frac{n(n+1)}{2}$.

(Source : Kunihiko Chikaya from Japan)

My solution

We known : $2 + 4 + 6 + ... + 2n = n(n+1)$, with $n \in \mathbb{N}$.

We have : $\sum_{k=1}^{n} \frac{(k^2-1)a_k + k^2 + 2k}{a_k^2 + a_k + 1} = \frac{3}{a_1^2 + a_1 + 1} + \frac{3a_2 + 8}{a_2^2 + a_2 + 1} + ... + \frac{(n^2-1)a_n + n^2 + 2n}{a_n^2 + a_n + 1}$.

We prove that : $\frac{(n^2-1)a_n + n^2 + 2n}{a_n^2 + a_n + 1} \geq -a_n + 2n, (*)$. Indeed ,

$(*) \Leftrightarrow (n^2-1)a_n + n^2 + 2n \geq (a_n^2 + a_n + 1)(-a_n + 2n)$

$\Leftrightarrow (n^2-1)a_n + n^2 + 2n \geq -a_n^3 + (2n-1)a_n^2 + (2n-1)a_n + 2n$

$\Leftrightarrow a_n^3 - (2n-1)a_n^2 + (n^2 - 2n)a_n + n^2 \geq 0 \Leftrightarrow (a_n - n)^2 (a_n + 1) \geq 0 \, (True)$

$\Rightarrow \sum_{k=1}^{n} \frac{(k^2-1)a_k + k^2 + 2k}{a_k^2 + a_k + 1} \geq (2 + 4 + 6 + ... + 2n) - \sum_{k=1}^{n} a_k = n(n+1) - \frac{n(n+1)}{2} = \frac{n(n+1)}{2}$

Equality occurs when $a_k = k, (k = 1, 2, 3.., n)$.

Problem 2 : Let $a_1, a_2, ..., a_n$ be positive real number with $a_1 + a_2 + ... + a_n = 1$.

Prove that : $\sum_{k=1}^{n} \frac{a_k}{2 - a_k} \geq \frac{n}{2n-1}$.

(Source : Srinivas Das)

My solution

Use Cauhcy – Schwarz , we have : $\sum_{k=1}^{n} \frac{a_k}{2 - a_k} = \sum_{k=1}^{n} \frac{a_k^2}{2a_k - a_k^2} \geq \frac{\left(\sum_{k=1}^{n} a_k\right)^2}{2\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_k^2} = \frac{1}{2 - \sum_{k=1}^{n} a_k^2}$

And $-\sum_{k=1}^{n} a_k^2 \le -\frac{\left(\sum_{k=1}^{n} a_k\right)^2}{n} = -\frac{1}{n} \Rightarrow \sum_{k=1}^{n} \frac{a_k}{2-a_k} \ge \frac{1}{2-\sum_{k=1}^{n} a_k^2} \ge \frac{1}{2-\frac{1}{n}} = \frac{n}{2n-1}$.

Equality occurs when $a_1 = a_2 = ... = a_n = \frac{1}{n}$.

Solution by Nguyen Viet Hung – Ha Noi – Viet Nam

We have : $\sum_{k=1}^{n} \frac{a_k}{2-a_k} = \sum_{k=1}^{n} \left(\frac{2}{2-a_k} - 1\right) \ge \frac{2n^2}{\sum_{k=1}^{n}\left(2-x_k\right)} - n = \frac{n}{2n-1}$.

Equality occurs when $a_1 = a_2 = ... = a_n = \frac{1}{n}$.

My Solution

Considering function : $f\left(a\right) = \frac{a}{2-a}, \forall a \in \left[0;1\right]$.

We have : $f'\left(a\right) = \frac{2}{\left(2-a\right)^2} \Rightarrow f''\left(a\right) = \frac{4}{\left(2-a\right)^3} > 0, \forall a \in \left[0;1\right]$

$$\Rightarrow f\left(\frac{\sum_{k=1}^{n} a_k}{n}\right) \le \frac{1}{n} f\left(\sum_{k=1}^{n} a_k\right) \Leftrightarrow \sum_{k=1}^{n} \frac{a_k}{2-a_k} \ge nf\left(\frac{1}{n}\right) = \frac{n}{2n-1}$$

Equality occurs when $a_1 = a_2 = ... = a_n = \frac{1}{n}$.

My solution

Considering function : $f\left(a\right) = \frac{a}{2-a}, \forall a \in \left[0;1\right]$.

We have : $f'\left(a\right) = \frac{2}{\left(2-a\right)^2} \Rightarrow f''\left(a\right) = \frac{4}{\left(2-a\right)^3} > 0, \forall a \in \left[0;1\right]$

. $f\left(a_1\right) \ge f'\left(\frac{1}{n}\right)\left(a_1 - \frac{1}{n}\right) + f\left(\frac{1}{n}\right) = \frac{2n^2}{\left(2n-1\right)^2}\left(a_1 - \frac{1}{n}\right) + \frac{1}{2n-1}$

. $f\left(a_2\right) \ge f'\left(\frac{1}{n}\right)\left(a_2 - \frac{1}{n}\right) + f\left(\frac{1}{n}\right) = \frac{2n^2}{\left(2n-1\right)^2}\left(a_2 - \frac{1}{n}\right) + \frac{1}{2n-1}$

. ...

. $f\left(a_n\right) \geq f'\left(\frac{1}{n}\right)\left(a_n - \frac{1}{n}\right) + f\left(\frac{1}{n}\right) = \frac{2n^2}{\left(2n-1\right)^2}\left(a_n - \frac{1}{n}\right) + \frac{1}{2n-1}$

$$\Rightarrow \sum_{k=1}^{n} f\left(a_k\right) \geq \frac{2n^2}{\left(2n-1\right)^2}\left(\sum_{k=1}^{n} a_k - 1\right) + \frac{n}{2n-1} = \frac{n}{2n-1} \text{ (with } \sum_{k=1}^{n} a_k = 1 \text{)}$$

Equality occurs when $a_1 = a_2 = ... = a_n = \frac{1}{n}$.

<u>Problem 3</u> : Let $a_1, a_2, ..., a_k$ be positive real number such that $\prod_{k=1}^{n} a_k = 1$.

Prove that : $\sum_{k=1}^{n} \frac{1}{1+a_k} \leq \frac{1}{4}\sum_{k=1}^{n} a_k, (*)$

My solution

We have : $(*) \Leftrightarrow \frac{1}{4}\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} \frac{1}{1+a_k} \geq 0 \Leftrightarrow \frac{1}{4}\sum_{k=1}^{n} a_k - \sum_{k=1}^{n}\left(1 - \frac{a_k}{1+a_k}\right) \geq 0$

$$\Leftrightarrow \frac{1}{4}\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} \frac{a_k}{1+a_k} - \frac{3n}{4} \geq 0 \Leftrightarrow \sum_{k=1}^{n} a_k + 4\sum_{k=1}^{n} \frac{a_k}{1+a_k} \geq 3n, (1).$$

Use $AM-GM$ inequality , $a + \frac{4a}{a+1} = (a+1) + \frac{4a}{a+1} - 1 \geq 2\sqrt{(a+1).\frac{4a}{a+1}} - 1 = 4\sqrt{a} - 1, \forall a > 0$

$$\Rightarrow \sum_{k=1}^{n} a_k + 4\sum_{k=1}^{n} \frac{a_k}{1+a_k} = \left(a_1 + \frac{4a_1}{1+a_1}\right) + \left(a_2 + \frac{4a_2}{1+a_2}\right) + ... + \left(a_n + \frac{4a_n}{1+a_n}\right) \geq 4\sum_{k=1}^{n}\sqrt{a_k} - n$$

Use $AM-GM$ inequality , $4\sum_{k=1}^{n}\sqrt{a_k} \geq 4n\sqrt[n]{\prod_{k=1}^{n}\sqrt{a_k}} = 4n \Rightarrow 4\sum_{k=1}^{n}\sqrt{a_k} - n \geq 3n \Rightarrow LHS(1) \geq RHS(1)$

<u>Problem 4</u> : Let $a_1, a_2, ..., a_n$ $(n \geq 2)$ be positive real number .

Prove that : $\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \geq \left(n^2 - 2n + 2\right) + \frac{(n-1)^2 \sum_{k=1}^{n} a_k^2}{\sum_{1 \leq i < j \leq n} a_i a_j}, (*)$

My solution

The homogeneous inequality , normoliza : $\sum_{k=1}^{n} a_k = n$

Let $\sum_{k=1}^{n} a_k^2 = n + n(n-1)t^2$, with t be real number , $0 \le t < 1$.

. $2\sum_{1\le i<j\le n} a_i a_j = \left(\sum_{k=1}^{n} a_k\right)^2 - \left(\sum_{k=1}^{n} a_k^2\right) = n^2 - n - n(n-1)t^2 = n(n-1)(1-t^2)$

Case 1 : If $t = 0$

We have : $LHS(*) = \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) = n\sum_{k=1}^{n}\frac{1}{a_k} \ge \frac{n.n^2}{\sum_{k=1}^{n} a_k} = n^2$.

And $RHS(*) = (n^2 - 2n + 2) + \frac{(n-1)^2 \sum_{k=1}^{n} a_k^2}{\sum_{1\le i<j\le k} a_i a_j} = n^2 - 2n + 2 + \frac{2n(n-1)^2}{n^2 - n} = n^2$

$\Rightarrow LHS(*) \ge n^2 = RHS(*)$.

Case 2 : If $0 < t < 1$

We have : $\sum_{k=1}^{n}\frac{1}{a_k} = \frac{1}{1+(n-1)t}\sum_{k=1}^{n}\frac{1+(n-1)t - a_k}{a_k} + \frac{n}{1+(n-1)t}$

Use Cauchy – Schwarz inequality:

$$\sum_{k=1}^{n}\frac{1+(n-1)t-a_k}{a_k} \ge \frac{\left(\sum_{k=1}^{n}\left[1+(n-1)t-a_k\right]\right)^2}{\sum_{k=1}^{n} a_k + (n-1)t\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_k^2} = \frac{n(n-1)t}{1-t}$$

$$\Rightarrow \sum_{k=1}^{n}\frac{1}{a_k} \ge \frac{n(n-1)t}{\left[1+(n-1)t\right](1-t)} + \frac{n}{1+(n-1)t} = \frac{n(nt-2t+1)}{\left[1+(n-1)t\right](1-t)}$$

We need to prove that : $\frac{n^2(nt-2t+1)}{\left[1+(n-1)t\right](1-t)} \ge \frac{n^2t^2 - 2nt^2 + n^2}{1-t^2}, (**)$

$$(**) \Leftrightarrow \frac{n(nt-2t+1)}{1+(n-1)t} \ge \frac{(nt^2 - 2t^2 + n)}{1+t} \Leftrightarrow n(nt-2t+1)(t+1) \ge (nt^2 - 2t^2 + n)(1+(n-1)t)$$

$$\Leftrightarrow n^2t^2 + n^2t - 2nt^2 - nt + n \ge n^2t^3 - 3nt^3 + 2t^3 + nt^2 - 2t^2 + n^2t - nt + n$$

$$\Leftrightarrow (n^2 - 3n + 2)t^3 - (n^2 - 3n + 2)t^2 \le 0 \Leftrightarrow t^2(n-1)(n-2)(t-1) \le 0 (True)$$

Problem 5 : Let $a_1, a_2, ..., a_n$ $(n \ge 2)$ be positive real number such that $\sum_{k=1}^{n} a_k = n$.

With $\alpha, \beta > 0$ are real number shrink before : $4\alpha(n-1)(2\alpha n\sqrt{n} + \beta) > \beta^2\sqrt{n}$.

Prove that : $\alpha\sum_{k=1}^{n}\frac{1}{a_k} + \frac{\beta}{\sqrt{\sum_{k=1}^{n} a_k^2}} \ge n\alpha + \frac{\beta}{\sqrt{n}}$.

My solution

Let $\sum_{k=1}^{n} a_k^2 = n + n(n-1)t^2$, with t be real number , $0 \le t < 1$.

We have : $\alpha \sum_{k=1}^{n} \frac{1}{a_k} = \frac{\alpha}{1+(n-1)t} \sum_{k=1}^{n} \frac{1+(n-1)t - a_k}{a_k} + \frac{n\alpha}{1+(n-1)t}$.

Use Cauchy – Schwarz inequality:

$$\sum_{k=1}^{n} \frac{1+(n-1)t - a_k}{a_k} \ge \frac{\left(\sum_{k=1}^{n}\left[1+(n-1)t - a_k\right]\right)^2}{\sum_{k=1}^{n} a_k + (n-1)t\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_k^2} = \frac{n(n-1)t}{1-t}$$

$$\Rightarrow \alpha \sum_{k=1}^{n} \frac{1}{a_k} \ge \frac{n\alpha(n-1)t}{\left[1+(n-1)t\right](1-t)} + \frac{n\alpha}{1+(n-1)t} = \frac{n\alpha(nt-2t+1)}{\left[1+(n-1)t\right](1-t)}$$

$$\Rightarrow \alpha \sum_{k=1}^{n} \frac{1}{a_k} + \frac{\beta}{\sqrt{\sum_{k=1}^{n} a_k^2}} \ge \frac{n\alpha(nt-2t+1)}{\left[1+(n-1)t\right](1-t)} + \frac{\beta}{\sqrt{n+n(n-1)t^2}}.$$

We need to prove that : $\frac{n\alpha(nt-2t+1)}{\left[1+(n-1)t\right](1-t)} + \frac{\beta}{\sqrt{n+n(n-1)t^2}} \ge n\alpha + \frac{\beta}{\sqrt{n}}, (*)$

$$(*) \Leftrightarrow \frac{n\alpha(n-1)t^2}{\left[1+(n-1)t\right](1-t)} - \frac{\beta(n-1)t^2}{\sqrt{n+n(n-1)t^2}.\left(1+\sqrt{1+(n-1)t^2}\right)} \ge 0$$

$$\Leftrightarrow (n-1)\left(\alpha n\sqrt{n} + \beta\right)t^2 - \beta(n-2)t + n\alpha\sqrt{n+n(n-1)t^2} + \alpha n\sqrt{n} - \beta \ge 0$$

Considering function :

$f(t) = (n-1)\left(\alpha n\sqrt{n} + \beta\right)t^2 - \beta(n-2)t + n\alpha\sqrt{n+n(n-1)t^2} + \alpha n\sqrt{n} - \beta$.

We have : $f(t) \ge g(t) = (n-1)\left(\alpha n\sqrt{n} + \beta\right)t^2 - \beta(n-2)t + 2\alpha n\sqrt{n} - \beta$.

Considering function : $g(t) = (n-1)(\alpha n\sqrt{n} + \beta)t^2 - \beta(n-2)t + 2\alpha n\sqrt{n} - \beta, \forall t \in [0;1]$

Three quadractic formala $g(t)$ reaches the minimum value when $t = \dfrac{\beta(n-2)}{2(n-1)(n\sqrt{n}\alpha + \beta)}$

$$\Rightarrow Ming(t) = 2\alpha n\sqrt{n} - \beta - \frac{\beta^2(n-2)^2}{4(n-1)(n\sqrt{n}\alpha + \beta)} = \frac{4\alpha(n-1)(2\alpha n\sqrt{n} + \beta) - \beta^2\sqrt{n}}{4(n-1)(n\sqrt{n}\alpha + \beta)} > 0$$

$\Rightarrow f(t) \ge g(t) \ge Ming(t) > 0$. Equality occurs when $a_1 = a_2 = ... = a_n = 1$.

<u>Problem 6</u> : Let $a_1, a_2, ..., a_n$ $(n \ge 3)$ be real number .

Prove that : $\left(\sum_{k=1}^{n} a_k\right)^2 \le \sum_{i,j=1}^{n} \frac{ij}{i+j-1} a_i a_j$

(China MO)

Solution

We have : $\sum_{i,j=1}^{n} \frac{ij}{i+j-1} a_i a_j = \sum_{i.j=1}^{n} i a_i a_j \int_0^1 t^{i+j-2} dt = \int_0^1 \left(\sum_{i.j=1}^{n} i a_i a_j t^{i-1+j-1}\right) dt = \int_0^1 \left(i a_i t^{i-1}\right)^2 dt$

Use Cauchy – Schwarz inequality :

$$\int_0^1 \left(\sum_{i=1}^{n} i a_i t^{i-1}\right)^2 dt \ge \left(\int_0^1 \left(\sum_{i=1}^{n} i a_i t^{i-1}\right) dt\right)^2 = \left(\sum_{k=1}^{n} a_k\right)^2 \Rightarrow \sum_{k=1}^{n} a_k \le \sum_{i,j=1}^{n} \frac{ij}{i+j-1} a_i a_j .$$

<u>Problem 7</u> : Let $a_1, a_2, ..., a_n \ge 0$ and $b_1, b_2, ..., b_n > 0$.

Prove that : $\sum_{1 \le i \le j \le n} \left(|a_i - a_j| + |b_i - b_j|\right) \le \sum_{1 \le i \le j \le n} |a_i - a_j|$.

(Poland – 1999)

Solution

Considering function : $f_i, g_i : [0;+\infty) \to$ □

With $f_i(x) = \begin{cases} 1 \ if \ x \in [0, a_i] \\ 0 \ if \ x > a_i \end{cases}$ and $g_i(x) = \begin{cases} 1 \ if \ x \in [0, b_i] \\ 0 \ if \ x > b_i \end{cases}$

Let $f(x) = \sum_{i=1}^{n} f_i(x)$ and $g(x) = \sum_{j=1}^{n} g_j(x)$.

We have : $\int_0^{+\infty} f(x)g(x)dx = \int_0^{+\infty}\left(\sum_{1\le i\le j} f_i(x)g_j(x)\right)dx = \sum_{1\le i\le j}\int_0^{+\infty} f_i(x)g_j(x)dx = \sum_{1\le i\le j\le n} min(a_i - b_j)$

Similarly, $\int_0^{+\infty} f^2(x)dx = \sum_{1\le i\le j\le n} min(a_i - a_j)$ and $\int_0^{+\infty} g^2(x)dx = \sum_{1\le i\le j\le n} min(b_i - b_j)$

We have : $\sum_{1\le i\le j\le n} min(a_i - a_j) + \sum_{1\le i\le j\le n} min(b_i - b_j) \ge 2\sum_{1\le i\le j\le n} min(a_i, b_j)$

We known : $2min(x,y) = x + y - |x - y| \Rightarrow \sum_{1\le i\le j\le n}\left(|a_i - a_j| + |b_i - b_j|\right) \le \sum_{1\le i\le j\le n}|a_i - a_j|$

<u>Problem 8</u> : Let $a_1, a_2, ..., a_n$ be positive real number such that $\sum_{1\le i,j\le n}|1 - a_i a_j| = \sum_{1\le i,j\le n}|a_i - a_j|$

Prove that : $\sum_{i=1}^n a_i = n$

(Source : Gabriel Dospinescu)

My solution

We have lemma : $\sum_{1\le i,j\le n} min(a_i a_j, b_i b_j) \le \sum_{1\le i,j\le n} min(a_i b_j, b_i a_j), \forall a_i, b_i \ge 0, i = \overline{1,2,..,n}.$ $(*)$

Use lemma $(*)$ with $b_i = 1 \Rightarrow \sum_{1\le i,j\le n}(a_i, a_j) \ge \sum_{1\le i,j\le n}(1 - a_i, a_j)$ $(**)$

Use $2min(u,v) = u + v - |u - v|$,

$$(**) \Leftrightarrow 2n\sum_{i=1}^n a_i - \sum_{1\le i,j\le n}|a_i - a_j| \ge n^2 + \left(\sum_{i=1}^n a_i\right)^2 - \sum_{1\le i,j\le n}|1 - a_i a_j| \Leftrightarrow \left(\sum_{i=1}^n a_i - n\right)^2 \le 0 \Leftrightarrow \sum_{i=1}^n a_i = n$$

<u>Problem 9</u> : Let $a_1, a_2, ..., a_n$ $(n \ge 2)$ be real number such that $\sum_{k=1}^n a_k = 1$.

Prove that : $\sum_{k=1}^n \frac{x_k}{x_{k+1} - x_{k+1}^3} \ge \frac{n^3}{n^2 - 1}$.

(China South East MO 2014)

Solution

Use Cauchy – Schwarz inequality $n\sum_{k=1}^n a_k^2 \ge \left(\sum_{k=1}^n a_k\right)^2 \Rightarrow \sum_{k=1}^n a_k^2 \ge \frac{1}{n}$.

Use $AM-GM$ inequality : $\sum_{k=1}^{n}\frac{a_k}{a_{k+1}-a_{k+1}^3}\ge n\left(\prod_{k=1}^{n}\frac{a_k}{a_{k+1}-a_{k+1}^3}\right)^{\frac{1}{n}}=\frac{n}{\left(\prod_{k=1}^{n}\left(1-a_k^2\right)\right)^{\frac{1}{n}}}$

Besides we have : $\left(\prod_{k=1}^{n}\left(1-a_k^2\right)\right)^{\frac{1}{n}}\le\frac{\sum_{k=1}^{n}\left(1-a_k^2\right)}{n}=\frac{n-\sum_{k=1}^{n}a_k^2}{n}=\le\frac{n^2}{n^2-1}$

$\Rightarrow\sum_{k=1}^{3}\frac{a_k}{a_{k+1}-a_{k+1}^3}\ge\frac{n^3}{n^2-1}$.

Problem 10 : Let $a_1,a_2,...,a_n$ be real number . Prove that $\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_ia_j}{i+i}\ge 0$

(Polish MO)

Solution

We known $\frac{a_ia_j}{i+j}=\int_0^1 a_ia_jt^{i+j-1}dt\Rightarrow\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_ia_j}{i+j}=\sum_{i,j=1}^{n}\int_0^1 a_ia_jt^{i+j-1}dt=\int_0^1\left(\sum_{i,j=1}^{n}a_ia_jt^{i+j-1}\right)dt$

Besides we have : $\left(\sum_{i=1}^{n}a_ix_i\right)^2=\sum_{i,j=1}^{n}a_ia_jx_ix_j\Rightarrow\sum_{i,j=1}^{n}a_ia_jt^{i+j-1}=\left(\sum_{i=1}^{n}a_it^{i-\frac{1}{2}}\right)^2$

$\Rightarrow\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_ia_j}{i+j}=\int_0^1\left(\sum_{i,j=1}^{n}a_ia_jt^{i+j-1}\right)dt=\int_0^1\left(\sum_{i=1}^{n}a_it^{i-\frac{1}{2}}\right)^2dt\ge 0$.