An Useful Technique in Proving Inequalities

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Abstract

There are a lot of distinct ways to prove inequalities. This paper mentions a simple and useful technique in proving inequalities through problems.

1 Basic knowledge

A number α is called a root of the polynomial $f(x)$ of multiplicity k ($k \geq 2, k \in \mathbb{N}$) if the following conditions are satisfied

(i) $f(x)$ is divisible by $(x-\alpha)^k$,

(ii) $f(x)$ is not divisible by $(x-\alpha)^{k+1}$.

In the other words $f(x)$ may be written in form $f(x) = (x-\alpha)^k g(x)$, where $g(x)$ is not divisible by $x - \alpha$.

In the particular case if α is a root of $f(x)$ of multiplicity 2, we say that $f(x)$ has a double root $x = \alpha$. We have the following result

Theorem 1. *The necessary and sufficient condition so that α is a root of $f(x)$ of multiplicity k as follows*

$$\begin{cases} f^{(i)}(\alpha) = 0, \ \forall i \in \{0, 1, \ldots, k-1\}, \\ f^{(k)}(\alpha) \neq 0, \end{cases}$$

where $f^{(i)}(x) = \dfrac{d^i f(x)}{(dx)^i}$ is the derivative of degree i of $f(x)$ with convention $f^{(0)}(x) = f(x)$.

2 Examples

Example 1. Prove that the following inequality holds for all positive real numbers a, b, c

$$\frac{a^3}{a^2+ab+2b^2} + \frac{b^3}{b^2+bc+2c^2} + \frac{c^3}{c^2+ca+2a^2} \geq \frac{a+b+c}{4}.$$

Analysis. Firstly we guess the equality holds when $a = b = c$. We want to find a result in form as follows

$$\frac{a^3}{a^2+ab+2b^2} \geq ma + nb.$$

This inequality is equivalent to

$$a^3 \geq (ma+nb)(a^2+ab+2b^2),$$

or

$$(1-m)a^3 - (m+n)a^2b - (2m+n)ab^2 - 2nb^3 \geq 0.$$

Consider the polynomial $f(a) = (1-m)a^3 - (m+n)a^2b - (2m+n)ab^2 - 2nb^3$. We will find two real numbers m, n for which $a = b$ is a double root of $f(a)$. This happens when

$$\begin{cases} f(b) = 0 \\ f'(b) = 0 \end{cases} \Leftrightarrow \begin{cases} (1-m)b^3 - (m+n)b^3 - (2m+n)b^3 - 2nb^3 = 0 \\ 3(1-m)b^2 - 2(m+n)b^2 - (2m+n)b^2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 4m + 4n = 1 \\ 7m + 3n = 3 \end{cases} \Leftrightarrow \begin{cases} m = 9/16, \\ n = -5/16. \end{cases}$$

Thus we give the solution as follows

Solution. We will show that

$$\frac{a^3}{a^2+ab+2b^2} \geq \frac{9a-5b}{16}.$$

Indeed, this inequality is equivalent to

$$\frac{(a-b)^2(7a+10b)}{16(a^2+ab+2b^2)} \geq 0$$

which is clearly. Similarly, we have

$$\frac{b^3}{b^2+bc+2c^2} \geq \frac{9b-5c}{16}, \quad \frac{c^3}{c^2+ca+2a^2} \geq \frac{9c-5a}{16}.$$

Summing up these relations we obtain

$$\frac{a^3}{a^2+ab+2b^2} + \frac{b^3}{b^2+bc+2c^2} + \frac{c^3}{c^2+ca+2a^2} \geq \frac{9a-5b}{16} + \frac{9b-5c}{16} + \frac{9c-5a}{16} = \frac{a+b+c}{4}.$$

The proof is complete. □

Example 2. Let a, b, c be positive real numbers. Prove that

$$\frac{5a^3-ab^2}{a+b} + \frac{5b^3-bc^2}{b+c} + \frac{5c^3-ca^2}{c+a} \geq 2(a^2+b^2+c^2).$$

Analysis. We observe that the equality occurs when $a = b = c$. Hence we want to find a result in form as follows

$$\frac{5a^3-ab^2}{a+b} \geq ma^2 + nb^2$$

which is equivalent to

$$(5-m)a^3 - ma^2b - (1+n)ab^2 - nb^3 \geq 0.$$

Consider the polynomial $f(a) = (5-m)a^3 - ma^2b - (1+n)ab^2 - nb^3$. Two real numbers m, n need to be chosen for which $f(a)$ receives $a = b$ as a double root. This happens if $f(b) = f'(b) = 0$, i.e. m, n are roots of the system of equations

$$\begin{cases} (5-m) - m - (1+n) - n = 0 \\ 3(5-m) - 2m - (1+n) = 0 \end{cases} \Leftrightarrow \begin{cases} m = 3, \\ n = -1. \end{cases}$$

So we have the following solution

Solution. We will prove that
$$\frac{5a^3 - ab^2}{a+b} \geq 3a^2 - b^2.$$
Indeed, this result is equivalent to
$$\frac{(a-b)^2(2a+b)}{a+b} \geq 0$$
which is obviously true. Similarly we also have
$$\frac{5b^3 - bc^2}{b+c} \geq 3b^2 - c^2, \quad \frac{5c^3 - ca^2}{c+a} \geq 3c^2 - a^2.$$
Therefore
$$\begin{aligned}\frac{5a^3 - ab^2}{a+b} + \frac{5b^3 - bc^2}{b+c} + \frac{5c^3 - ca^2}{c+a} &\geq (3a^2 - b^2) + (3b^2 - c^2) + (3c^2 - a^2) \\ &= 2(a^2 + b^2 + c^2)\end{aligned}$$
and we are done. □

Example 3. Let a, b, c be positive real numbers such that $a + b + c = 9$. Prove that
$$\frac{a^3 + b^3}{ab + 9} + \frac{b^3 + c^3}{bc + 9} + \frac{c^3 + a^3}{ca + 9} \geq 9.$$

Analysis. We noticed that when $a = b = c = 3$ the equality holds. Thus we need to find a result in form
$$\frac{a^3 + b^3}{ab + 9} \geq m(a + b) + n$$
which is equivalent to
$$a^3 + b^3 - (ab + 9)(m(a + b) + n) \geq 0.$$
Consider the polynomial $f(a) = a^3 + b^3 - (ab+9)(m(a+b)+n)$. We must find two real numbers m, n so that $f(a)$ receives $a = b = 3$ as a double root. This occurs if $f(b) = f'(b) = 0$, where $f'(a) = 3a^2 - b(m(a + b) + n) - m(ab + 9)$. I.e. we have
$$\begin{cases} 2b^3 - (b^2 + 9)(2mb + n) = 0, \\ 3b^2 - b(2mb + n) - m(b^2 + 9) = 0. \end{cases}$$
From this system, choosing $b = 3$ we get
$$\begin{cases} 54 - 18(6m + n) = 0, \\ 27 - 3(6m + n) - 18m = 0, \end{cases} \quad \Leftrightarrow \begin{cases} m = 1, \\ n = -3. \end{cases}$$
This analysis leads us to a solution below

Solution. We will show that
$$\frac{a^3 + b^3}{ab + 9} \geq a + b - 3. \tag{1}$$
Indeed, we have
$$\frac{a^3 + b^3}{ab + 9} \geq \frac{4(a^3 + b^3)}{(a + b)^2 + 36} \geq \frac{(a + b)^3}{(a + b)^2 + 36}.$$
Thus we only need to prove
$$\frac{(a + b)^3}{(a + b)^2 + 36} \geq a + b - 3,$$

or

$$(a+b)^3 \geq (a+b-3)((a+b)^2+36),$$

or

$$(a+b-6)^2 \geq 0$$

which is trivially true. So (1) has been proved. Similar as above, we also have

$$\frac{b^3+c^3}{bc+9} \geq b+c-3. \tag{2}$$

$$\frac{c^3+a^3}{ca+9} \geq c+a-3. \tag{3}$$

Summing up (1), (2) and (3) side by side we get the desired inequality. □

Example 4 (Moldova, 2005)**.** Let a, b, c be positive real numbers such that $a^4+b^4+c^4=3$. Prove that

$$\frac{1}{4-ab}+\frac{1}{4-bc}+\frac{1}{4-ca} \leq 1.$$

Analysis. We hope that there exists a result with the following form

$$\frac{1}{4-ab} \leq m(ab)^2+n,$$

or

$$m(ab)^3-4m(ab)^2+n(ab)-4n+1 \leq 0.$$

Consider the polynomial $f(t)=mt^3-4mt^2+nt-4n+1$ (with $t=ab$). We want to find the numbers m, n so that $f(t)$ receives $t=1$ as a double root. This happens if $f(1)=f'(1)=0$. That is

$$\begin{cases} m-4m+n-4n+1=0, \\ 3m-8m+n=0, \end{cases} \Leftrightarrow \begin{cases} m=1/18, \\ n=5/18. \end{cases}$$

Solution. We will show that

$$\frac{1}{4-ab} \leq \frac{(ab)^2+5}{18}. \tag{4}$$

Indeed, this inequality may be rewritten as

$$(ab)^3-4(ab)^2+5(ab)-2 \leq 0,$$

or

$$(ab-1)^2(ab-2) \leq 0.$$

This result is true because

$$3=a^4+b^4+c^4 > a^4+b^4 \geq 2(ab)^2 \Rightarrow ab < \sqrt{\frac{3}{2}} < 2.$$

Thus (4) has been proved. Similarly we also have

$$\frac{1}{4-bc} \leq \frac{(bc)^2+5}{18}. \tag{5}$$

$$\frac{1}{4-ca} \leq \frac{(ca)^2+5}{18}. \tag{6}$$

Adding up the relations (4), (5) and (6) we obtain

$$\begin{aligned}\frac{1}{4-ab}+\frac{1}{4-bc}+\frac{1}{4-ca} &\le \frac{(ab)^2+(bc)^2+(ca)^2+15}{18}\\ &\le \frac{a^4+b^4+c^4+15}{18}\\ &=1\end{aligned}$$

as desired. □

Example 5. Let a, b, c be positive real numbers such that $ab^2+bc^2+ca^2=3$. Prove that

$$\frac{2a^5+3b^5}{ab}+\frac{2b^5+3c^5}{bc}+\frac{2c^5+3a^5}{ca}\ge 15(a^3+b^3+c^3-2).$$

Analysis. By the given condition, we can guess that

$$\frac{2a^5+3b^5}{ab}\ge ma^3+nab^2+pb^3.$$

This inequality is equivalent to

$$2a^5+3b^5-ab(ma^3+nab^2+pb^3)\ge 0.$$

Consider the polynomial of degree 5 below

$$f(a)=2a^5+3b^5-ab(ma^3+nab^2+pb^3).$$

We have

$$\begin{aligned}f'(a)&=10a^4-4ma^3b-2nab^3-pb^4,\\ f''(a)&=40a^3-12ma^2b-2nb^3,\\ f'''(a)&=120a^2-24mab.\end{aligned}$$

We have to find the numbers m, n, p so that $f(a)$ receives $a=b$ as a root of multiplicity 4. This occurs when $f(b)=f'(b)=f''(b)=f'''(b)=0$. That is

$$\begin{cases}5b^5-b^2(mb^3+nb^3+pb^3)=0,\\ 10b^4-4mb^4-2nb^4-pb^4=0,\\ 40b^3-12mb^3-2nb^3=0,\\ 120b^2-24mb^2=0,\end{cases}\Leftrightarrow\begin{cases}m+n+p=5,\\ 4m+2n+p=10,\\ 12m+2n=40,\\ m=5,\end{cases}\Leftrightarrow\begin{cases}m=5,\\ n=-10,\\ p=10.\end{cases}$$

This analysis leads us to the following solution

Solution. We will show that

$$\frac{2a^5+3b^5}{ab}\ge 5a^3-10ab^2+10b^3.$$

Indeed, this result is equivalent to $(a-b)^4(2a+3b)\ge 0$ which is clearly true. Similarly

$$\frac{2b^5+3c^5}{bc}\ge 5b^3-10bc^2+10c^3,$$

and

$$\frac{2c^5+3a^5}{ca}\ge 5c^3-10ca^2+10a^3.$$

Adding up these relations we obtain

$$\frac{2a^5+3b^5}{ab}+\frac{2b^5+3c^5}{bc}+\frac{2c^5+3a^5}{ca} \geq 15(a^3+b^3+c^3)-10(ab^2+bc^2+ca^2)$$
$$= 15(a^3+b^3+c^3-2).$$

The proof is complete. □

Example 6. Prove that for all positive real numbers a, b, c

$$\frac{4a^3+5b^3-3a^2b+10ab^2}{3a+b}+\frac{4b^3+5c^3-3b^2c+10bc^2}{3b+c}+\frac{4c^3+5a^3-3c^2a+10ca^2}{3c+a}$$
$$\geq 5(a^2+b^2+c^2)-(ab+bc+ca).$$

Analysis. We believe that there exists a correct result in form

$$\frac{4a^3+5b^3-3a^2b+10ab^2}{3a+b} \geq ma^2+nb^2-ab$$

This is rewritten as

$$f(a) = (4-3m)a^3 - ma^2b + (11-3n)ab^2 + (5-n)b^3 \geq 0.$$

We want to choose m, n for which $f(a)$ receives $a = b$ as a double root. That is $f(b) = f'(b) = 0$,

$$\Leftrightarrow \begin{cases} m+n=5, \\ 11m+3n=23, \end{cases} \quad \Leftrightarrow \begin{cases} m=1, \\ n=4. \end{cases}$$

Solution. We will check that

$$\frac{4a^3+5b^3-3a^2b+10ab^2}{3a+b} \geq a^2+4b^2-ab.$$

Indeed, this inequality is equivalent to

$$a^3-a^2b-ab^2+b^3 \geq 0,$$

$$\Leftrightarrow (a-b)^2(a+b) \geq 0$$

which is trivially true. Similarly

$$\frac{4b^3+5c^3-3b^2c+10bc^2}{3b+c} \geq b^2+4c^2-bc,$$

$$\frac{4c^3+5a^3-3c^2a+10ca^2}{3c+a} \geq c^2+4a^2-ca.$$

Adding up these relations we get the required inequality. □

Example 7 (Crux)**.** Let a, b, c be positive real numbers such that $a^2+b^2+c^2 = 1$. Find the minimum value of

$$E = \frac{a}{b^2+c^2}+\frac{b}{c^2+a^2}+\frac{c}{a^2+b^2}.$$

Analysis. We guess that E minimizes at $a = b = c = \frac{1}{\sqrt{3}}$ and we want to find a common form of inequalities as follows

$$\frac{a}{b^2+c^2} = \frac{a}{1-a^2} \geq ma^2 + n$$

which is rewritten as

$$a \geq (1-a^2)(ma^2+n)$$

or

$$f(a) = ma^4 - (m-n)a^2 + a - n \geq 0.$$

The numbers m, n need to choose so that $a = \frac{1}{\sqrt{3}}$ is a double root of $f(a)$. Namely $f(\frac{1}{\sqrt{3}}) = f'(\frac{1}{\sqrt{3}}) = 0$,

$$\Leftrightarrow \begin{cases} 2m + 6n = 3\sqrt{3}, \\ 2m - 6n = 3\sqrt{3}, \end{cases} \quad \Leftrightarrow \begin{cases} m = \frac{3\sqrt{3}}{2}, \\ n = 0. \end{cases}$$

Thus we go to the following solution

Solution. Firstly we will show that

$$\frac{a}{1-a^2} \geq \frac{3\sqrt{3}}{2}a^2.$$

Indeed this is equivalent to

$$3\sqrt{3}a^3 - 3\sqrt{3}a + 2 \geq 0$$

or

$$(\sqrt{3}a - 1)^2(\sqrt{3}a + 2) \geq 0$$

which is clearly true. Similarly

$$\frac{b}{1-b^2} \geq \frac{3\sqrt{3}}{2}b^2, \ \frac{c}{1-c^2} \geq \frac{3\sqrt{3}}{2}c^2.$$

Hence

$$E \geq \frac{3\sqrt{3}}{2}(a^2+b^2+c^2) = \frac{3\sqrt{3}}{2}.$$

Thus we conclude that $\min E = \frac{3\sqrt{3}}{2}$, khi $a = b = c = \frac{1}{\sqrt{3}}$. □

Example 8 (Crux)**.** Let a, b, c be positive real numbers such that

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2.$$

Prove that

$$\frac{1}{4a+1} + \frac{1}{4b+1} + \frac{1}{4c+1} \geq 1.$$

Analysis. We want to find two numbers m, n so that the following inequality holds

$$\frac{1}{4a+1} \geq \frac{m}{a+1} + n$$

or

$$4na^2 + (4m + 5n - 1)a + m + n - 1 \leq 0.$$

Let $f(a) = 4na^2 + (4m + 5n - 1)a + m + n - 1$. Note that if $a = b = c = 1/2$ the equality holds. So we must choose m, n so that $a = 1/2$ is a double root of $f(a)$. This happens if $f(1/2) = f'(1/2) = 0$. That is

$$\begin{cases} n + \frac{4m+5n-1}{2} + m + n - 1 = 0, \\ 4n + 4m + 5n - 1 = 0, \end{cases} \Leftrightarrow \begin{cases} m = 1, \\ n = -\frac{1}{3}. \end{cases}$$

This analysis gives us a solution below

Solution. We will check that

$$\frac{1}{4a+1} \geq \frac{1}{a+1} - \frac{1}{3}$$

Indeed, this inequality is equivalent to

$$(2a-1)^2 \geq 0$$

which is obviously true. Similarly

$$\frac{1}{4b+1} \geq \frac{1}{b+1} - \frac{1}{3}, \ \frac{1}{4c+1} \geq \frac{1}{c+1} - \frac{1}{3}.$$

Summing up these relations we get

$$\frac{1}{4a+1} + \frac{1}{4b+1} + \frac{1}{4c+1} \geq \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} - 1 = 1.$$

The proof is complete. □

Remark 1. We have a more general result as follows: If $a_i > 0$ $(i = 1, 2, \ldots, n)$ satisfy

$$\frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_n+1} = n - 1$$

then

$$\frac{1}{4a_1+1} + \frac{1}{4a_2+1} + \cdots + \frac{1}{4a_n+1} \geq \frac{2n-3}{3}.$$

Example 9 (Japan 1997)**.** Prove that the following inequality holds for all $a, b, c > 0$

$$\frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} + \frac{(a+b)^2-c^2}{(a+b)^2+c^2} \geq \frac{3}{5}.$$

Solution. Because the inequality is homogeneous, hence without loss of generality, we can assume that (normalization) $a + b + c = 3$. Then the desired inequality reduces to

$$\frac{(3-2a)^2}{2a^2-6a+9} + \frac{(3-2b)^2}{2b^2-6b+9} + \frac{(3-2c)^2}{2c^2-6c+9} \geq \frac{3}{5}.$$

We want to find an inequality having the following type

$$\frac{(3-2a)^2}{2a^2-6a+9} \geq ma + n,$$

or

$$f(a) = (3-2a)^2 - (ma+n)(2a^2-6a+9) \geq 0.$$

The polynomial of degree three $f(a)$ receives $a = 1$ as a double root, the following conditions have to be satisfied $f(1) = f'(1) = 0$. I.e.

$$\begin{cases} 1 - 5(m+n) = 0, \\ 3m - 2n + 4 = 0, \end{cases} \Leftrightarrow \begin{cases} m = -18/25, \\ n = 23/25. \end{cases}$$

So we will check that

$$\frac{(3-2a)^2}{2a^2 - 6a + 9} \geq \frac{-18a + 23}{25}.$$

This result is equivalent to

$$(a-1)^2(2a+1) \geq 0$$

which is trivially true. Similarly

$$\frac{(3-2b)^2}{2b^2 - 6b + 9} \geq \frac{-18b + 23}{25},$$

$$\frac{(3-2c)^2}{2c^2 - 6c + 9} \geq \frac{-18c + 23}{25}.$$

These imply

$$\frac{(3-2a)^2}{2a^2 - 6a + 9} + \frac{(3-2b)^2}{2b^2 - 6b + 9} + \frac{(3-2c)^2}{2c^2 - 6c + 9} \geq \frac{-18(a+b+c) + 69}{25} = \frac{3}{5}$$

and we are done. □

Example 10. Let a, b, c be non-negative real numbers such that $a+b+c = 3$. Find the maximum and minimum value of

$$E = \sqrt{a^2 + a + 4} + \sqrt{b^2 + b + 4} + \sqrt{c^2 + c + 4}.$$

Analysis. We guess that the minimum value happens at the "center," i.e. at $a = b = c = 1$. Therefore we will find two numbers m, n for which

$$\sqrt{a^2 + a + 4} \geq ma + n.$$

We need the following system of equations (these are conditions to two graphs tangent each other)

$$\begin{cases} \sqrt{a^2 + a + 4} = ma + n, \\ \frac{2a+1}{2\sqrt{a^2+a+4}} = m, \end{cases}$$

satisfied when $a = 1$. By this way we find $m = \frac{\sqrt{6}}{4}, n = \frac{3\sqrt{6}}{4}$.

Similar as above we also believe that the maximum value happens at the "boundary," i.e. at $(a, b, c) = (3, 0, 0)$ and its permutions. Therefore, maybe the common inequality will be as

$$\sqrt{a^2 + a + 4} \leq \alpha a + \beta$$

where α, β are two numbers that we have to find so that the equality holds when $a = 3$ and $a = 0$. By this way we obtain $\alpha = 2/3$ and $\beta = 2$. And we go to the following solution

Solution. (a) Find the minimum value: Firstly we will show that

$$\sqrt{a^2+a+4} \geq \frac{\sqrt{6}}{4}a + \frac{3\sqrt{6}}{4}.$$

In deed, this result is equivalent to $(a-1)^2 \geq 0$ which is obviously. Similarly

$$\sqrt{b^2+b+4} \geq \frac{\sqrt{6}}{4}b + \frac{3\sqrt{6}}{4}, \quad \sqrt{c^2+c+4} \geq \frac{\sqrt{6}}{4}c + \frac{3\sqrt{6}}{4}.$$

Therefore

$$E \geq \frac{\sqrt{6}}{4}(a+b+c) + \frac{9\sqrt{6}}{4} = 3\sqrt{6}.$$

Thus $\min E = 3\sqrt{6}$, khi $a = b = c = 1$.

(b) Find the maximum value: We will check that

$$\sqrt{a^2+a+4} \leq \frac{2}{3}a + 2.$$

This inequality is equivalent to $a(a-3) \leq 0$ which is true because $0 \leq a \leq 3$. Similarly

$$\sqrt{b^2+b+4} \leq \frac{2}{3}b + 2, \qquad \sqrt{c^2+c+4} \leq \frac{2}{3}c + 2.$$

Thus

$$E \leq \frac{2}{3}(a+b+c) + 6 = 8.$$

So $\max E = 8$, when $(a,b,c) = (3,0,0)$ and its permutions.

□

Example 11. Let a, b, c be positive real numbers such that $a^2+b^2+c^2 = 3$. Prove that

$$\frac{1}{2-a} + \frac{1}{2-b} + \frac{1}{2-c} \geq 3.$$

Solution. By the similar way as above, we need to check the following inequality

$$\frac{1}{2-a} \geq \frac{a^2+1}{2}.$$

But this result is equivalent to $a(a-1)^2 \geq 0$, which is trivial. We also have similar relations

$$\frac{1}{2-b} \geq \frac{b^2+1}{2}, \qquad \frac{1}{2-c} \geq \frac{c^2+1}{2}.$$

Hence

$$\frac{1}{2-a} + \frac{1}{2-b} + \frac{1}{2-c} \geq \frac{a^2+b^2+c^2+3}{2} = 3$$

as desired. □

Example 12 (Mathematics and Youth magazine)**.** Let a, b, c be positive real numbers such that $a+b+c = 1$. Prove that

$$(1+a^2)(1+b^2)(1+c^2) \geq \left(\frac{10}{9}\right)^3.$$

Solution. The required inequality is equivalent to

$$\ln(1+a^2)+\ln(1+b^2)+\ln(1+c^2)\geq 3\ln\frac{10}{9}$$

Now we find again two numbers α,β for which

$$\ln(1+a^2)\geq \alpha a+\beta.$$

Note that the equality holds when $a=b=c=1/3$, hence we want two graphs of the functions $f(a)=\ln(1+a^2)$ and $g(a)=\alpha a+\beta$ are tangent each other at point $a=1/3$. This means that

$$\begin{cases} f(1/3)=g(1/3) \\ f'(1/3)=g'(1/3) \end{cases} \Leftrightarrow \begin{cases} \ln\frac{10}{9}=\frac{\alpha}{3}+\beta \\ 3/5=\alpha \end{cases} \Leftrightarrow \begin{cases} \alpha=3/5, \\ \beta=\ln\frac{10}{9}-\frac{1}{5}. \end{cases}$$

So we will go to proving

$$\ln(1+a^2)\geq \frac{3a}{5}+\ln\frac{10}{9}-\frac{1}{5}. \tag{7}$$

Indeed, we consider the function

$$h(a)=\ln(1+a^2)-\frac{3a}{5}-\ln\frac{10}{9}+\frac{1}{5}$$

with $a\in[0,1]$. We have $h'(a)=\frac{2a}{1+a^2}-\frac{3}{5}=\frac{-3a^2+10a-3}{5(1+a^2)}$. The equation $h'(a)=0$ has a root $a=1/3\in[0,1]$. By establishing a table of increase and decrease of the function $h(a)$ in the interval $[0,1]$, we easily see that $h(a)\geq h(1/3)=0$. So we conclude that (7) is true. Similar as above we also have

$$\ln(1+b^2)\geq \frac{3b}{5}+\ln\frac{10}{9}-\frac{1}{5} \tag{8}$$

$$\ln(1+c^2)\geq \frac{3c}{5}+\ln\frac{10}{9}-\frac{1}{5} \tag{9}$$

Adding up (7), (8) and (9) we get the desired result. The proof is complete. □

Remark 2. We have a more general result as follows: For $x_1,x_2,\ldots,x_n\geq 0$ satisfy $x_1+x_2+\cdots+x_n=1$ then

$$(1+x_1^2)(1+x_2^2)\cdots(1+x_n^2)\geq\left(1+\frac{1}{n^2}\right)^n.$$

Finally, we give a few problems for reader's practice

3 Proposed problems

1. Let a,b,c,d be real numbers such that $a^2+b^2+c^2+d^2=4$. Prove that

$$a^3+b^3+c^3+d^3\leq 8.$$

2. Let $x,y,z\leq 1$ be real numbers such that $x+y+z=1$. Prove that

$$\frac{1}{1+x^2}+\frac{1}{1+y^2}+\frac{1}{1+z^2}\leq\frac{27}{10}.$$

3. (Poland 1996) Let $a,b,c\geq -\frac{3}{4}$ be real numbers such that $a+b+c=1$. Prove that

$$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}\leq\frac{9}{10}.$$

4. (USA 2003) Prove that the following inequality holds for all positive real numbers a, b, c
$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\le 8.$$

5. (China 2006) Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that
$$\frac{a^2+9}{2a^2+(b+c)^2}+\frac{b^2+9}{2b^2+(c+a)^2}+\frac{c^2+9}{2c^2+(a+b)^2}\le 5.$$

6. (France 2007) Let a, b, c, d be positive real numbers such that $a+b+c+d=1$. Prove that
$$6(a^3+b^3+c^3+d^3)\ge a^2+b^2+c^2+d^2+\frac{1}{8}.$$

7. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that
$$\frac{1}{a^2+b+c}+\frac{1}{b^2+c+a}+\frac{1}{c^2+a+b}\le 1.$$

8. Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Find the minimum value of
$$E=3(a+b+c)+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

9. Prove that for all positive real numbers a, b, c,
$$\frac{3a^3+7b^3}{2a+3b}+\frac{3b^3+7c^3}{2b+3c}+\frac{3c^3+7a^3}{2c+3a}+ab+bc+ca\ge 3(a^2+b^2+c^2).$$

10. (Crux) Let a, b, c be positive real numbers such that $a^2+b^2+c^2=1$. Prove that
$$a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 4\sqrt{3}.$$

11. (Mathematics and Youth magazine) Let x, y, z be positive real numbers such that $x+2y+3z=\frac{1}{4}$. Find the maximum value of
$$E=\frac{232y^3-x^3}{2xy+24y^2}+\frac{783z^3-8y^3}{6yz+54z^2}+\frac{29x^3-27z^3}{3zx+6x^2}.$$

12. (Crux) Let $a, b, c>1$ be real numbers such that
$$\frac{1}{a^2-1}+\frac{1}{b^2-1}+\frac{1}{c^2-1}=1.$$
Prove that
$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\le 1.$$

13. Let a, b, c, d, e be positive real numbers such that
$$\frac{1}{4+a}+\frac{1}{4+b}+\frac{1}{4+c}+\frac{1}{4+d}+\frac{1}{4+e}=1.$$
Prove that
$$\frac{a}{4+a^2}+\frac{b}{4+b^2}+\frac{c}{4+c^2}+\frac{d}{4+d^2}+\frac{e}{4+e^2}\le 1.$$

14. Prove that for all positive real numbers a, b, c

$$\frac{a^3+b^3}{a+2b}+\frac{b^3+c^3}{b+2c}+\frac{c^3+a^3}{c+2a}\geq\frac{2}{3}(a^2+b^2+c^2).$$

15. Let a, b, c be non-negative real numbers such that $a+b+c=3$. Prove that

$$(a^2+a+1)(b^2+b+1)(c^2+c+1)\leq 27.$$

16. Let a, b, c be non-negative real numbers such that $a+b+c=3$. Prove that

$$\sqrt[3]{\frac{a^3+4}{a^2+4}}+\sqrt[3]{\frac{b^3+4}{b^2+4}}+\sqrt[3]{\frac{c^3+4}{c^2+4}}\geq 3.$$

17. Let a, b, c be non-negative real numbers such that $a^2+b^2+c^2=12$. Prove that

$$\frac{a^3+1}{a^2+2}+\frac{b^3+1}{b^2+2}+\frac{c^3+1}{c^2+2}\leq\frac{9}{2}.$$

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