SOCIETATEA DE ȘTIINȚE MATEMATICE DIN ROMÂNIA
ROMANIAN MATHEMATICAL SOCIETY

Filiala Mehedinți - Mehedinți Branch
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Solve for real numbers:

$$\begin{cases}\cos 2x + \cot 3y = \tan 5z \\ \cot 3y + \cot 5z = \tan 2x \\ \cot 5z + \cot 2x = \tan 3y\end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Seyran Ibrahimov – Maasilli – Azerbaidjan

$$\cot 2x = u \quad \cot 3y = i \quad \cot 5z = s \quad u, i, s \neq 0$$

$$\Rightarrow u + i = \frac{1}{s} \Leftrightarrow su + si = 1 \quad (1)$$
$$\Rightarrow i + s = \frac{1}{u} \Leftrightarrow ui + su = 1 \quad (2)$$
$$\Rightarrow s + u = \frac{1}{i} \Leftrightarrow si + ui = 1 \quad (3)$$

$$(1) = (2) su + si = ui + su = 1 \Leftrightarrow s = u$$
$$(1) = (3) su + si = si + ui = 1 \Leftrightarrow s = i \quad \rightarrow u = i = s$$
$$(2) = (3) ui + su = si + ui = 1 \Leftrightarrow u = i$$

$$u(i,s) + u(i,s) = \frac{1}{u(i,s)} \quad u, i, s = \pm\frac{1}{\sqrt{2}}$$

$$\cot 2x = u = \pm\frac{1}{\sqrt{2}} \quad \cot 3y = i = \pm\frac{1}{\sqrt{2}} \quad \cot 5z = s = \pm\frac{1}{\sqrt{2}}$$

$$x = \pm\frac{1}{2}\arctan\frac{1}{\sqrt{2}} + \frac{1}{2}\pi k \quad y = \pm\frac{1}{3}\arctan\frac{1}{\sqrt{2}} + \frac{1}{3}\pi k \quad z = \pm\frac{1}{5}\arctan\frac{1}{\sqrt{2}} + \frac{1}{5}\pi k$$

$$s > u > i$$

$$\begin{matrix} su + si = 1 \\ ui + su = 1 \\ si + ui = 1 \end{matrix} \Rightarrow su + si > si + ui \text{ (no answer)}$$

$$\text{answer } s = i = u = \pm\frac{1}{\sqrt{2}}$$

Solution 2 by Soumava Chakraborty – Kolkata – India

Let $\cot 2x = a, \cot 3y = b, \cot 5z = c$

$\tan 2x, \tan 3y, \tan 5z$ are defined, $\cos 2x, \cos 3y, \cos 5z \neq 0 \Rightarrow a, b, c \neq 0$

$$a + b = \frac{1}{c} \quad (1)$$

$$b + c = \frac{1}{a} \quad (2)$$

$$c + a = \frac{1}{b} \quad (3)$$

$$(1) - (2) \Rightarrow a - c = \frac{1}{c} - \frac{1}{a} \Rightarrow (a - c)\left(1 - \frac{1}{ac}\right) = 0 \quad (4)$$
$$(2) - (3) \Rightarrow b - a = \frac{1}{a} - \frac{1}{b} \Rightarrow (b - a)\left(1 - \frac{1}{ab}\right) = 0 \quad (5)$$
$$(3) - (1) \Rightarrow c - b = \frac{1}{b} - \frac{1}{c} \Rightarrow (c - b)\left(1 - \frac{1}{bc}\right) = 0 \quad (6)$$

If $1 = \frac{1}{ac}$, then $\frac{1}{c} = a \Rightarrow a + b = a$ (form (1)) $\Rightarrow b = 0$

If $1 = \frac{1}{ab}$, then $\frac{1}{b} = a \Rightarrow c + a = a$ (from (3)) $\Rightarrow c = 0$

If $1 = \frac{1}{bc}$, then $\frac{1}{c} = b \Rightarrow a + b = b$ (from (1)) $\Rightarrow a = 0$

But $a, b, c \neq 0$, $\quad 1 \neq \frac{1}{ac}, 1 \neq \frac{1}{ab}, 1 \neq \frac{1}{bc}$

$$(4), (5), (6) \Rightarrow a = b = c$$

Putting $b = c = a$ in (1), $a + a = \frac{1}{a} \Rightarrow a^2 = \frac{1}{2}$

$$\Rightarrow a = \pm\frac{1}{\sqrt{2}}$$

$$(a, b, c) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ or } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$\cot 2x = \cot 3y = \cot 5z = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \tan 2x = \tan 3y = \tan 5z = \sqrt{2}$$

$$\Rightarrow 2x = \tan^{-1}\left(\sqrt{2}\right) + n\pi \Rightarrow x = \frac{1}{2}\left(\tan^{-1}\sqrt{2}\right) + \frac{n\pi}{2}$$

$$y = \frac{1}{3}\tan^{-1}\left(\sqrt{2}\right) + \frac{n'\pi}{3}$$

$$z = \frac{1}{5}\tan^{-1}\left(\sqrt{2}\right) + \frac{n''\pi}{5}$$

Similarly, $\cot 2x = \cot 3y = \cot 5z = -\frac{1}{\sqrt{2}}$

$$\Rightarrow x = -\frac{1}{2}\tan^{-1}\left(\sqrt{2}\right) + \frac{n\pi}{2}$$

$$y = -\frac{1}{3}\tan^{-1}\left(\sqrt{2}\right) + \frac{n'\pi}{3}$$

$$z = -\frac{1}{5}\tan^{-1}\left(\sqrt{2}\right) + \frac{n''\pi}{5}$$

solutions are:

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$$\begin{cases} x = \pm\frac{1}{2}\tan^{-1}\left(\sqrt{2}\right) + \frac{n\pi}{2} \\ y = \pm\frac{1}{3}\tan^{-1}\left(\sqrt{2}\right) + \frac{n'\pi}{3} \\ z = \pm\frac{1}{5}\tan^{-1}\left(\sqrt{2}\right) + \frac{n''\pi}{5} \end{cases}$$