TRIGONOMETRIC SUBSTITUTIONS

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Abstract. In the following we will use trigonometric substitutions for solving some equations and for the calculus of the general term of some series.

1. Solve the following equation:

$$x^3 - 3x + a(1 - 3x^2) = 0; a \in \mathbb{R}$$

Proof.

$$x \in \mathbb{R} \Rightarrow \exists b \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right); x = \tan b$$

$$a(1 - 3\tan^2 b) = 3\tan b - \tan^3 b \Rightarrow a = \frac{3\tan b - \tan^3 b}{1 - 3\tan^2 b}$$

$$a = \tan 3b \Rightarrow 3b = \arctan a + k\pi; k \in \mathbb{Z} \Rightarrow b = \frac{1}{3}\arctan a + \frac{k\pi}{3}; k \in \mathbb{Z}$$

$$x = \tan b = \tan\left(\frac{1}{3}\arctan a + \frac{k\pi}{3}\right); k \in \{-2, 0, 2\}$$

If $a = 1$ the equation: $x^3 - 3x^2 - 3x + 1 = 0$ has the solutions:

$$x = \tan\left(\frac{\pi}{12} + \frac{k\pi}{3}\right); k \in \{-2, 0, 2\}, x_1 = \tan\frac{\pi}{12}; x_2 = \tan\frac{\pi}{4}; x_3 = \tan\left(-\frac{7\pi}{4}\right)$$

If $a = 2$ the equation: $x^3 - 6x^2 - 3x + 2 = 0$ has the solutions:

$$x = \tan\left(\frac{1}{3}\arctan 2 + \frac{k\pi}{3}\right)$$

$$k \in \{-2, 0, 2\}, x_1 = \tan\left(\frac{1}{3}\arctan 2\right); x_2 = \tan\left(\frac{1}{3}\arctan 2 + \frac{2\pi}{3}\right); x_3 = \tan\left(\frac{1}{3}\arctan 2 - \frac{2\pi}{3}\right)$$

□

2. Solve the following equation:

$$4x^3 - 4x + a(x^4 - 6x^2 + 1) = 0; a \in \mathbb{R}$$

Proof.

$$x \in \mathbb{R} \Rightarrow \exists b \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right); x = \tan b; a(\tan^4 b - 6\tan^2 b + 1) = 4\tan b - 4\tan^3 b$$

$$a = \frac{4\tan b - 4\tan^3 b}{\tan^4 b - 6\tan^2 + 1} \Rightarrow a = \tan 4b \Rightarrow b = \frac{1}{4}\arctan a; x = \tan\left(\frac{1}{4}\arctan a + \frac{k\pi}{4}\right); k \in \mathbb{Z}$$

If $a = 1$ the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ has the solutions:

$$x_1 = \tan\frac{\pi}{16}; x_2 = \tan\frac{5\pi}{16}; x_3 = \tan\left(-\frac{3\pi}{16}\right); x_4 = \tan\frac{9\pi}{16}$$

□

3. Find the formula of the general term of the series given by the relationships:

$$x_1 = a; a \in [-1,1], x_{n+1} = 2x_n^2 - 1; n \geq 1$$

Proof.

$$x_1 = a \in [-1,1] \Rightarrow (\exists) b \in [0, 2\pi]; a = \cos b; b = \arccos a$$

$$x_2 = 2x_1^2 - 1 = 2\cos^2 b - 1 = \cos(2b); x_3 = 2x_2^2 - 1 = 2\cos^2(2b) - 1 = \cos(2^2 b)$$

$$x_4 = 2x_3^2 - 1 = 2\cos^2(2^2 b) - 1 = \cos(2^3 b).$$

Through induction we can prove that:

$$x_n = \cos(2^{n-1} b) = \cos(2^{n-1} \arccos a)$$

□

4. Find the formula of the general term of the series given by the relationships:

$$x_1 = a; a \in [-1,1]; x_{n+1} = 1 - 2x_n^2; n \geq 1$$

$$x_2 = 1 - 2x_1^2 = 1 - 2\cos^2 b = \cos(2b); x_3 = 1 - 2x_2^2 = 1 - 2\cos^2(2b) = \cos(2^2 b)$$

$$x_4 = 1 - 2x_3^2 = 1 - 2\cos^2(2^2 b) = \cos(2^3 b)$$

Through induction we can prove that:

$$x_n = \cos(2^{n-1} b) = \cos(2^{n-1} \arcsin a)$$

5. Find the formula of the general term of the series given by the relationships:

$$x_1 = a; a \in [-1,1]; x_{n+1} = x_n(3 - 4x_n^2); n \geq 1$$

Proof.

$$x_1 = a \in [-1,1] \Rightarrow (\exists) b \in [0, 2\pi]; a = \sin b; b = \arcsin a$$

$$x_2 = x_1(3 - 4x_1^2) = \sin b(3 - 4\sin^2 b) = \sin(3b)$$

$$x_3 = x_2(3 - 4x_2^2) = \sin 3b(3 - 4\sin^2 3b) = \sin(3^2 b)$$

$$x_4 = x_3(3 - 4x_3^2) = \sin(3^2 b)\Big(2 - 4\sin^4(3^2 b)\Big) = \sin(3^3 b)$$

Through induction we can prove that:

$$x_n = \cos(3^{n-1} b) = \cos(3^{n-1} \arccos a)$$

□

6. Find the formula of the general term of the series given by the relationships:

$$x_1 = a; a \in [-1,1]; x_{n+1} = x_n(4x_n^2 - 3); n \geq 1$$

Proof.

$$x_1 = a; a \in [-1,1] \Rightarrow (\exists) b \in [0, 2\pi]; a = \cos b; b = \arccos a$$

$$x_2 = x_1(4x_1^2 - 3) = \cos b(4\cos^2 b - 3) = \cos(3b)$$

$$x_3 = x_2(4x_2^2 - 3) = \cos 3b(4\cos^2 3b - 3) = \cos(3^2 b)$$

Through induction we can prove that:

$$x_n = \cos(3^{n-1} b) = \cos(3^{n-1} \arccos a)$$

□

7. Find the formula of the general term of the series given by the relationships:

$$x_1 = a; a \in \mathbb{R}; x_{n+1} = \frac{2x_n}{1 - x_n^2}; n \geq 1$$

Proof.

$$x_1 = a \in [-1, 1] \Rightarrow (\exists) b \in \Big(-\frac{\pi}{2}, \frac{\pi}{2} \Big); a = \tan b; b = \arctan a$$

$$x_2 = \frac{2x_1}{1 - x_1^2} = \frac{2 \tan b}{1 - \tan^2 b} = \tan(2b); x_3 = \frac{2x_2}{1 - x_2^2} = \frac{2 \tan(2b)}{1 - \tan^2(2b)} = \tan(2^2 b)$$

Through induction we can prove that:

$$x_n = \tan(2^{n-1} b) = \tan(2^{n-1} \arctan a)$$

□

8. Find the formula of the general term of the series given by the relationships:

$$x_1 = a; a \in \mathbb{R}; x_{n+1} = \frac{3x_n - x_n^3}{1 - 3x_n^2}; n \geq 1$$

Proof.

$$x_1 = a \in [-1, 1] \Rightarrow (\exists) b \in \Big(-\frac{\pi}{2}, \frac{\pi}{2} \Big); a = \tan b; b = \arctan a$$

$$x_2 = \frac{3x_1 - x_1^3}{1 - 3x_1^2} = \frac{3 \tan b - \tan^3 b}{1 - 3 \tan^2 b} = \tan(3b); x_3 = \frac{3x_2 - x_2^3}{1 - 3x_2^2} = \frac{3 \tan(3b) - \tan^3(3b)}{1 - 3 \tan^2(3b)} = \tan(3^2 b)$$

Through induction we prove that:

$$x_n = \tan(3^{n-1} b) = \tan(3^{n-1} \arctan a)$$

□

References

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