SOCIETATEA DE ŞTIINŢE MATEMATICE DIN ROMÂNIA
ROMANIAN MATHEMATICAL SOCIETY

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Prove that in any triangle ABC,

$$\frac{h_a h_b}{r_a r_b}+\frac{h_b h_c}{r_b r_c}+\frac{h_c h_a}{r_c r_a}\ge 3$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC:

$$\frac{h_a h_b}{r_a r_b}+\frac{h_b h_c}{r_b r_c}+\frac{h_a h_c}{r_a r_c}\ge 3$$

Recordar lo siguiente:

$$h_a=\frac{2S}{a},h_b=\frac{2S}{b},h_c=\frac{2S}{c}$$

$$r_a=\frac{S}{p-a},r_b=\frac{S}{p-b},r_c=\frac{S}{p-c}$$

La desigualdad es equivalente:

$$\frac{2(p-a)}{a}\cdot\frac{2(p-b)}{b}+\frac{2(p-b)}{b}\cdot\frac{2(p-c)}{c}+\frac{2(p-a)}{a}\cdot\frac{2(p-c)}{c}\ge 3$$

$$\frac{(b+c-a)(a+c-b)}{ab}+\frac{(a+c-b)(a+b-c)}{bc}+\frac{(b+c-a)(a+b-c)}{ac}\ge 3$$

Sean: $b=z+x, c=x+y, a=y+z$

Por la tanto: $b+c-a=2x\ge 0, a+c-b=2y\ge 0, a+b-c=2z\ge 0$

$$\frac{xy}{(z+y)(z+x)}+\frac{yz}{(x+z)(x+y)}+\frac{xz}{(y+z)(y+z)}\geq\frac{3}{4}$$

Por desigualdad de Cauchy:

$$\frac{(xy)^2}{x(y+z)y(z+x)}+\frac{(yz)^2}{y(z+x)z(x+y)}+\frac{(xz)^2}{x(y+z)z(x+y)}\geq$$

$$\geq\frac{(xy+yz+xz)^2}{\sum xy(z+x)(z+y)}\geq\frac{\sum(xy)^2+2xyz(x+y+z)}{\sum z^2xy+\sum x^2yz+\sum y^2xz+\sum(xy)^2}$$

$$\Rightarrow\frac{\sum(xy)^2+2xyz(x+y+z)}{3xyz(x+y+z)+\sum(xy)^2}\geq\frac{3}{4}\Leftrightarrow\sum(xy)^2\geq xyz(x+y+z)$$