In ΔABC the following relationship holds:

$$3\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\geq\sum\frac{b^2+bc+c^2}{bcm_a}$$

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Using Tereshin's inequality,

$$m_a\geq\frac{b^2+c^2}{4R},$$

$$m_b\geq\frac{c^2+a^2}{4R},$$

$$m_c\geq\frac{a^2+b^2}{4R}$$

$$\sum\frac{b^2+bc+c^2}{bcm_a}\leq 4R\left\{\frac{b^2+c^2+bc}{bc(b^2+c^2)}+\frac{c^2+a^2+ca}{ca(c^2+a^2)}+\frac{a^2+b^2+ab}{ab(a^2+b^2)}\right\}=$$

$$=4R\left\{\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right)+\left(\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{1}{a^2+b^2}\right)\right\}=$$

$$\leq 4R\left\{\frac{a+b+c}{abc}+\frac{1}{2}\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right)\right\}=$$

$$(b^2+c^2\geq 2bc, c^2+a^2\geq 2ca, a^2+b^2\geq 2ab)$$

$$=4R\left(\frac{a+b+c}{abc}+\frac{a+b+c}{2abc}\right)=4R\cdot\frac{3}{2}\left(\frac{2s}{4Rrs}\right)=\frac{3}{r}\qquad\textbf{(1)}$$

Now,

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=2R\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right)=2R\frac{2s}{abc}=\frac{4Rs}{4Rrs}=\frac{1}{r}$$

$$3\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)=\frac{3}{r} \quad (2)$$

(1) and (2) ⇒ the desired inequality