Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a^3+8abc}+\frac{1}{b^3+8abc}+\frac{1}{c^3+8abc}\le\frac{1}{3abc}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Vietnam

Siendo: a, b, c números $\mathbb{R}^+$. Probar que:

$$\frac{1}{a^3+8abc}+\frac{1}{b^3+8abc}+\frac{1}{c^3+8abc}\le\frac{1}{3abc}$$

$$\Rightarrow\frac{bc}{a^2+8bc}+\frac{ac}{b^2+8ac}+\frac{ab}{c^2+8ab}\le\frac{1}{3}$$

$$\Rightarrow\frac{bc}{a^2+8bc}=\frac{1}{8}-\frac{a^2}{8(a^2+8bc)}$$

$$\left(\frac{1}{8}-\frac{a^2}{8(a^2+8bc)}\right)+\left(\frac{1}{8}-\frac{b^2}{8(b^2+8ac)}\right)+\left(\frac{1}{8}-\frac{c^2}{8(c^2+8ab)}\right)\le\frac{1}{3}$$

Demostraremos que: $\frac{a^2}{(a^2+8bc)}+\frac{b^2}{(b^2+8ac)}+\frac{c^2}{(c^2+8ab)}\ge\frac{1}{3}$

Por desigualdad de Cauchy:

$$\frac{a^2}{(a^2+8bc)}+\frac{b^2}{(b^2+8ac)}+\frac{c^2}{(c^2+8ab)}\ge\frac{(a+b+c)^2}{\sum a^2+8\sum ab}\ge\frac{1}{3}\Rightarrow$$

$$\Rightarrow 3(a+b+c)^2\ge\sum a^2+8\sum ab$$

$\Leftrightarrow \sum a^2\ge\sum ab$ **(LQQD). Por lo tanto:**

$$\left(\frac{1}{8}-\frac{a^2}{8(a^2+8bc)}\right)+\left(\frac{1}{8}-\frac{b^2}{8(b^2+8ac)}\right)+\left(\frac{1}{8}-\frac{c^2}{8(c^2+8ab)}\right)\le\frac{3}{8}-\frac{1}{24}\le\frac{1}{3}$$

(LQQD)

Solution 2 by Soumitra Moukherjee - Chandar Nagore – India

$$\frac{1}{3abc}\ge\sum_{cyc}\frac{1}{a^3+8abc}\Leftrightarrow\sum_{cyc}\left(\frac{1}{9abc}-\frac{1}{a^3+8abc}\right)\ge 0$$

$$\Leftrightarrow\sum_{cyc}\frac{a^2-bc}{9abc(a^2+8bc)}\ge 0\Leftrightarrow\sum_{cyc}\frac{(a-b)(a+c)+(a-b)(a+c)}{18abc(a^2+8bc)}\ge 0$$

$$\Leftrightarrow\sum_{cyc}\frac{1}{18abc}\left(\frac{(a-b)(a+c)}{a^2+8bc}+\frac{(a+b)(a-c)}{a^2+8bc}\right)\ge 0$$

$$\Leftrightarrow\sum_{cyc}\frac{1}{18abc}\left\{\frac{(a-b)(a+c)}{a^2+8bc}+\frac{(b-a)(b+c)}{b^2+8ac}\right\}\ge 0$$

$$\Leftrightarrow\sum_{cyc}\frac{(a-b)}{18abc}\left\{\frac{7c(a^2-b^2)+8c^2(a-b)-ab(a-b)}{(a^2+8bc)(a^2+8ac)}\right\}\ge 0$$

$$\Leftrightarrow\sum\nolimits_{cyc}\frac{(a-b)^2}{18abc}(7ac+7bc+8c^2-ab)\ge 0,\ \text{which is true}$$

$$\text{so, }\frac{1}{3abc}\ge\frac{1}{a^3+8abc}\ \text{(proved)}$$