ELEMENTARY PROBLEMS TREATED NON-ELEMENTARY

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Abstract. Without entering into details regarding Fermat's theorem for functions having multiple variables, the theorem of bilinear forms and Fubini's formula for calculus of double and triple integrals we can consider known their results and we can apply them to the class or to the circle of pupils to the following types of problems:

Problem 1:

Find the minimum of the expression:

$$E(x,y) = x^2 + y^2 - 2x - 4y + 10; x, y \in \mathbb{R}$$

Proof.

$$\begin{cases} E'_x = 2x - 2 \\ E'_y = 2y - 4 \end{cases}; \begin{cases} E'_x = 0 \\ E'_y = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 2 \end{cases}$$

$$\min E(x,y) = E(1,2) = 1 + 4 - 2 - 8 + 10 = 5$$

$$E''_{x^2} = 2; E''_{y^2} = 2; E''_{xy} = 0$$

$$H_E(1,2) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\Delta_1 = 2 > 0; \Delta_2 = 4 > 0 \Rightarrow (1,2) \text{ minimum point}$$

□

Problem 2

Find the minimum of the expression:

$$E(x,y,z) = x^2 + y^2 + z^2 - 2x - 4y + 6z + 30$$

Proof.

$$\begin{cases} E'_x = 2x - 2 \\ E'_y = 2y - 4 \\ E'_z = 2z + 6 \end{cases}; \begin{cases} E'_x = 0 \\ E'_y = 0 \\ E'_z = 0 \end{cases}; \begin{cases} x = 1 \\ y = 2 \\ z = -3 \end{cases}$$

$$E''_{x^2} = 2; E''_{xy} = 0; E''_{xz} = 0$$

$$E''_{yx} = 0; E''_{y^2} = 2; E''_{yz} = 0$$

$$E''_{zx} = 0; E''_{zy} = 0; E''_{z^2} = 2$$

$$H_E(1,2,-3) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\Delta_1 = 2 > 0; \Delta_2 = 4 > 0; \Delta_3 = 8 > 0$$

$$(1,2,-3) \text{ minimum point}$$

$$\min E(x,y,z) = E(1,2,-3) = 16$$

□

Problem 3
For which value $m > 0$, the area of the set:

$$A = \left\{ (x,y) \middle| m \le x \le 2m; 0 \le y \le x + \frac{6}{x^2} \right\}$$

is minimum? Find the minimum of the area.

Proof. Let be $f : (0,\infty) \to \mathbb{R}$;

$$f(m) = area(A) = \int\int dxdy = \int_m^{2m} \left(\int_0^{x+\frac{6}{x^2}} dy \right) dx = \frac{3m^2}{2} + \frac{3}{m}$$

$$f'(m) = 0 \Rightarrow m = 1 \Rightarrow \min(A) = f(1) = \frac{9}{2}$$

□

Problem 4
Prove that the volume of the cube having the side "a" is $V = a^3$.

Proof. Let be $C(x,y,z) = \{(x,y,z) | 0 \le x \le a, 0 \le y \le a, 0 \le z \le a\}$.

$$V = \int\int\int dxdydz = \int_0^a \left(\int_0^a \left(\int_0^a dz \right) dy \right) dx = \int_0^a \left(\int_0^a ady \right) dx =$$

$$= \int_0^a a^2 dx = a^3$$

□

Problem 5
Prove that the volume of the cuboid having the sides a, b, c is $V = abc$.

Proof. Let be $PD(x,y,z) = \{(x,y,z) | 0 \le x \le a, 0 \le y \le b, 0 \le z \le c\}$

$$V = \int\int\int dxdydz = \int_0^a \left(\int_0^b \left(\int_0^c dz \right) dy \right) dx = \int_0^a \left(\int_0^b cdy \right) dx =$$

$$= \int_0^a bcdx = abc$$

□

Problem 6
Prove that the volume of regular rectangular prism having the base side "a" and the height "h" is $V = a^2 h$.

Proof. Let be $P(x,y,z) = \{(x,y,z) | 0 \le x \le a, 0 \le y \le a, 0 \le z \le h\}$

$$V = \int\int\int dxdydz = \int_0^a \left(\int_0^a \left(\int_0^h dz \right) dy \right) dx = \int_0^a \left(\int_0^a hdy \right) dx =$$

$$= \int_0^a ahdx = a^2 h$$

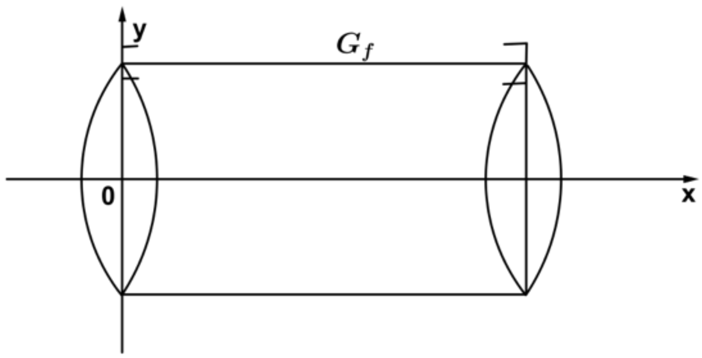
□

Problem 7

Prove that the lateral area and the volume of the right circular cylinder having the radius "r" and height "h" are given by the formulas:

$$A_l = 2\pi rg; V = \pi r^2 h$$

Proof.



$$f : [0, h] \to \mathbb{R}; f(x) = r; f'(x) = 0$$

$$V_f = \pi \int_0^h f^2(x)dx = \pi \int_0^h r^2 dx = \pi r^2 x\Big|_0^h = \pi r^2 h$$

$$A_f = 2\pi \int_0^h f(x)\sqrt{1 + f'^2(x)}dx = 2\pi \int_0^h r\sqrt{1 + 0^2}dx = 2\pi rx\Big|_0^h = 2\pi rh = 2\pi rg$$

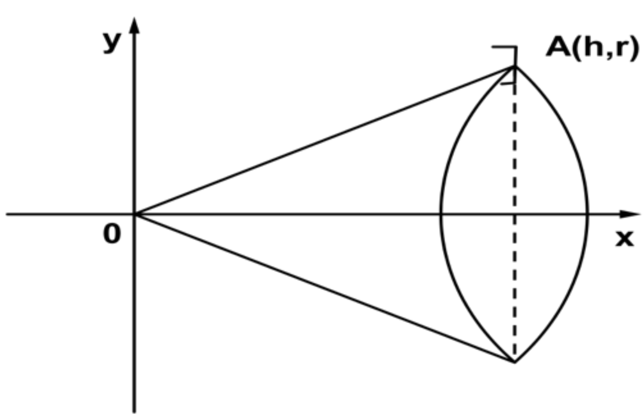
□

Problem 8

Prove that the lateral area and the volume of the right circular cone having the radius "r", "h", generator "g" are given by the formulas:

$$A_l = \pi rg; V = \frac{\pi r^2 h}{3}$$

Proof.



$$OA:\begin{vmatrix} x & y & 1\\ 0 & 0 & 1\\ h & r & 1\end{vmatrix}=0; OA: ya-xr=0; OA: y=\frac{r}{h}x; f:[0,h]\to\mathbb{R}; f(x)=\frac{r}{h}x$$

$$f'(x)=\frac{r}{h}; 1+f'^2(x)=1+\frac{r^2}{h^2}=\frac{h^2+r^2}{h^2}=\frac{g^2}{h^2}$$

$$V_f=\pi\int_0^h f^2(x)dx=\pi\int_0^h \frac{r^2x^2}{h^2}dx=\frac{\pi r^2}{h^2}\cdot\frac{x^3}{3}\Big|_0^h=\frac{\pi r^2}{h^2}\cdot\frac{h^3}{3}=\frac{\pi r^2 h}{3}$$

$$A_f=2\pi\int_0^h f(x)\sqrt{1+f'^2x}dx=2\pi\int_0^h\frac{rx}{h}\cdot\sqrt{\frac{g^2}{h^2}}dx=\frac{2\pi r}{h}\cdot\frac{g}{h}\cdot\frac{x^2}{2}\Big|_0^h=\pi rg$$

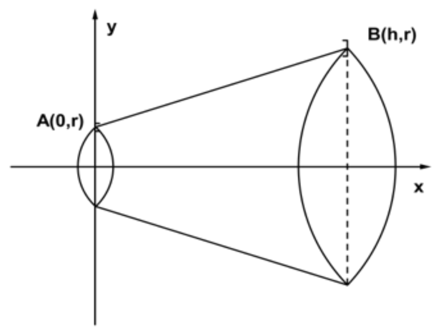
□

Problem 9

Prove that the lateral area and the volume of the circular truncated cone having the radii "R", r", height "h", generator "g" are given by the formulas:

$$A_l=\pi g(R+r); V=\frac{\pi h}{3}(R^2+r^2+Rr)$$

Proof.



$$AB:\begin{vmatrix} x & y & 1\\ 0 & r & 1\\ h & R & 1\end{vmatrix}=0$$

$$AB: xr+yh-hr-Rx=0; AB: yh=x(R-r)+hr; AB: y=\frac{R-r}{h}x+r$$

$$f:[0,h]\to\mathbb{R}; f(x)=\frac{R-r}{h}x+r; f'(x)=\frac{R-r}{h}; 1+f'^2(x)=1+\frac{(R-r)^2}{h^2}=\frac{g^2}{h^2}$$

$$V_f=\pi\int_0^h f^2(x)dx=\pi\int_0^h\left[\left(\frac{R-r}{h}\right)^2x^2+\frac{2(R-r)r}{h}x+r^2\right]=$$

$$=\pi\frac{(R-r)^2}{h^2}\cdot\frac{x^3}{3}\Big|_0^h+\frac{2\pi(R-r)r}{h}\cdot\frac{x^2}{2}\Big|_0^h+\pi r^2x|_0^h=\frac{\pi(R-r)^2h}{3}+\pi(R-r)rh+\pi r^2h$$

$$=\frac{\pi h}{3}(R^2-2Rr+r^2+3Rr-3r^2+3r^2)=\frac{\pi h}{3}(R^2+Rr+r^2)$$

$$A_f=2\pi\int_0^h f(x)\sqrt{1+f'^2(x)}dx=2\pi\int_0^h\left(\frac{R-r}{h}x+r\right)\cdot\sqrt{\frac{g^2}{h^2}}dx=$$

$$= \frac{2\pi g}{h}\Big(\frac{R-r}{h}\cdot\frac{x^2}{2}\Big|_0^h + rx\Big|_0^h\Big) = \frac{2\pi g}{h}\Big(\frac{R-r}{h}\cdot\frac{h^2}{2} + rh\Big) = 2\pi h\Big(\frac{R-r}{h} + r\Big) = \pi g(R+r)$$

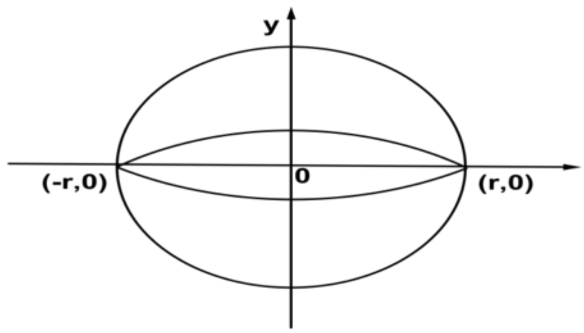
□

Problem 10

Prove that the area and the volume of the sphere having the radius "R" are given by the formulas:

$$A = 4\pi R^2; V = \frac{4\pi R^3}{3}$$

Proof.



$$f : [-r, r] \to \mathbb{R}; f(x) = \sqrt{r^2 - x^2}; f'(x) = \frac{-x}{\sqrt{r^2 - x^2}}; 1 + f'^2(x) = \frac{r^2}{r^2 - x^2}$$

$$V_f = \pi \int_{-r}^{r} f^2(x)dx = \pi \int_{-r}^{r} (r^2 - x^2)dx = \pi r^2 x\Big|_{-r}^{r} - \pi\frac{x^3}{3} =$$

$$= \pi r^2(r + r) - \frac{\pi}{3}\Big(r^3 - (-r)^3\Big) = 2\pi r^3 - \frac{2\pi}{3}r^3 = \frac{4\pi r^3}{3}$$

$$A_f = 2\pi \int_{-r}^{r} f(x)\sqrt{1 + f'^2(x)}dx = 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \cdot \frac{r}{\sqrt{r^2 - x^2}}dx =$$

$$= 2\pi r x\Big|_{-r}^{r} = 2\pi r\Big(r - (-r)\Big) = 4\pi r^2$$

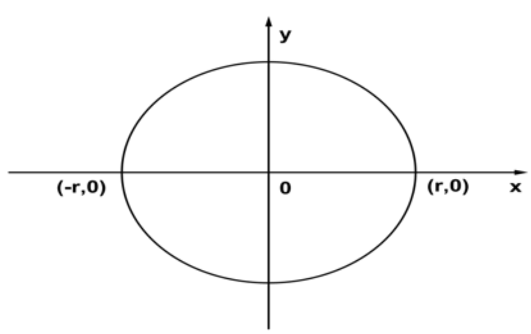
□

Problem 11

Prove that the length of the circle having the radius "R" is given by the formula:

$$L = 2\pi R$$

Proof.



$$f:[-r,r]\to\mathbb{R}; f(x)=\sqrt{r^2-x^2}; 1+f'^2(x)=1+\frac{x^2}{r^2-x^2}=\frac{r^2}{r^2-x^2}$$
$$L_f=\int_{-r}^{r}\sqrt{1+f'^2(x)}dx=\int_{-r}^{r}\frac{r}{\sqrt{r^2-x^2}}dx=r\arcsin\frac{x}{r}\Big|_{-r}^{r}=r\Big(\arcsin 1-\arcsin(-1)\Big)=$$
$$=r\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=\pi r; L_{\text{semicircle}}=\pi r\Rightarrow L_{\text{circle}}=2\pi r$$

□

Problem 12

Prove that the aria of the disc having the radius "R" is given by the formula:

$$A=\pi R^2$$

Proof.

$$A_{\text{semidisc}}=\int_{-r}^{r}f(x)dx=\int_{-r}^{r}\sqrt{r^2-x^2}dx=\frac{\pi r^2}{2}; A_{\text{disc}}=\pi r^2$$

□

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