

Number 2

Autumn 2016

R M M

ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099

ROMANIAN MATHEMATICAL MAGAZINE

PROBLEMS FOR JUNIORS

JP.016. Find all triplets (m, n, p) where m, n are two natural numbers and p is a prime number, satisfying the equation:

$$m^4 = 4(p^n - 1).$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.017. Prove the following inequality holds for all positive real numbers a, b, c

$$a^2 + b^2 + c^2 \geq \frac{1}{2}(ab + bc + ca) + \sqrt{\frac{2(a + b + c)(a^3b^3 + b^3c^3 + c^3a^3)}{(a + b)(b + c)(c + a)}}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.018. Let ABC be a triangle with the known normal notations. Prove that for any point P moving on the incircle,

$$5r \leq \frac{PA^2}{h_a} + \frac{PB^2}{h_b} + \frac{PC^2}{h_c} \leq \frac{5}{2}R.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.019. If $a, b, c > 0$ and $x, y, z \geq 1$ then:

$$x^{\frac{4a^3}{a^2+bc}} y^{\frac{4b^3}{b^2+ca}} z^{\frac{4c^3}{c^2+ab}} \geq \left(\frac{x^4}{yz}\right)^a \left(\frac{y^4}{zx}\right)^b \left(\frac{z^4}{xy}\right)^c$$

Proposed by Mihály Bencze - Romania

JP.020. Given x_1, x_2, \dots, x_n be positive real numbers such that:

$$\sum_{k=1}^n x_k = n.$$

If $\alpha, \beta > 0$, $4\alpha(n-1)(2\alpha n\sqrt{n} + \beta) > \beta^2\sqrt{n}$. then:

$$\alpha \sum_{i=k}^n \frac{1}{x_k} + \frac{\beta}{\sqrt{\sum_{k=1}^n x_k^2}} \geq n\alpha + \frac{\beta}{\sqrt{n}}$$

Proposed by Ngo Minh Ngoc Bao - Vietnam

JP.021. Prove that if $x, y, z > 0$, $xyz = 8$ then:

$$x^3 + y^3 + z^3 \geq 2x\sqrt{y+z} + 2y\sqrt{z+x} + 2z\sqrt{x+y}$$

Proposed by Iuliana Trașcă - Romania

JP.022. Let ABC be an acute triangle with the orthocenter H , inradius r , and circumradius R . Prove that

$$\frac{HA}{\sqrt{bc}} + \frac{HB}{\sqrt{ca}} + \frac{HC}{\sqrt{ab}} \leq \sqrt{2\left(1 + \frac{r}{R}\right)}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.023. Prove that for all positive real numbers a, b, c, d

$$\frac{a}{bc} + \frac{b}{cd} + \frac{c}{da} + \frac{d}{ab} \geq \frac{8}{\sqrt{a^2 + b^2 + c^2 + d^2}}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.024. Given a triangle ABC and let P be any point in its plane. Prove that

$$\frac{PB \cdot PC}{bc} + \frac{PC \cdot PA}{ca} + \frac{PA \cdot PB}{ab} \leq \frac{1}{4} \left(\frac{PA}{h_a} + \frac{PB}{h_b} + \frac{PC}{h_c} \right)^2.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.025. Let $n \geq 2$ be an integer and let a, b, c be positive real numbers such that $ab + bc + ca \leq 1$. Prove that

$$\frac{bc}{(2a^2 + bc)^n} + \frac{ca}{(2b^2 + ca)^n} + \frac{ab}{(2c^2 + ab)^n} \geq 1$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.026. Let a, b, c be non-negative real numbers and let x, y, z be real numbers different from 0, such that $by + cz = x$,

$cz + ax = y$, $ax + by = z$. Prove that

a. $abc \leq \frac{1}{8}$.

b. $\frac{1}{2+a+b} + \frac{1}{2+b+c} + \frac{1}{2+c+a} \leq 1$.

c. $a + b + c \geq 2(ab + bc + ca)$.

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.027. Find all real numbers x satisfying the following equation

$$(x + \{x\})^2 - (x + \{x\}) = 6\lfloor x \rfloor \{x\} - 1$$

where $\lfloor x \rfloor$ and $\{x\}$ denote the integer part and fractional part of x , respectively.

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.028. Prove the following inequality holds for any triangle ABC ,

$$\frac{r_b}{r_a} + \frac{r_c}{r_b} + \frac{r_a}{r_c} \geq \sqrt{1 + \frac{4R}{r}}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.029. In acute angled ΔABC ; L - Nagel's point

$$M, M' \in (AB); N, N' \in (AC);$$

$(M, L, N); (M', L, N')$ - collinear points

Prove that:

$$(a+c-b)\left(\frac{MB}{MA} + \frac{M'B}{M'A}\right) + (a+b-c)\left(\frac{NC}{NA} + \frac{N'C}{N'A}\right) > b+c-a$$

Proposed by Daniel Sitaru - Romania

JP.030. If $x_k \in [0, 1] (k = 1, 2, \dots, n)$ then:

$$3 \sum_{k=1}^n x_k^2 \leq 2n + x_1x_2x_3 + x_2x_3x_4 + \dots + x_nx_1x_2$$

Proposed by Mihály Bencze - Romania

PROBLEMS FOR SENIORS

SP.016. Let a, b, c, s, t, u be positive real numbers such that $a + b + c = 1$. Prove that:

$$\frac{sa^2 + tb^2 + uc^2}{sa + tb + uc} + \frac{sb^2 + tc^2 + ua^2}{sb + tc + ua} + \frac{sc^2 + ta^2 + ub^2}{sc + ta + ub} \geq 1$$

Proposed by Kunihiko Chikaya - Tokyo - Japan

SP.017. Let $a_k (k = 1, 2, \dots, n)$ be a positive real numbers such that

$$\sum_{k=1}^n a_k = \frac{n(n+1)}{2}$$

Prove that:

$$\sum_{k=1}^n \frac{(k^2 - 1)a_k + k^2 + 2k}{a_k^2 + a_k + 1} \geq \frac{n(n+1)}{2}$$

Proposed by Kunihiko Chikaya - Tokyo - Japan

SP.018. If $a, b, c > 0$ and $x, y, z \geq 1$ then:

$$x^{\frac{8a^3}{a+b}}y^{\frac{8b^3}{b+c}}z^{\frac{8c^3}{c+a}} \geq \left(\frac{x^5}{z}\right)^{a^2} \left(\frac{y^5}{x}\right)^{b^2} \left(\frac{z^5}{y}\right)^{c^2}$$

Proposed by Mihály Bencze - Romania

SP.019. Prove that:

$$1) \sum_{k=1}^n (2k+1)(2k^2+2k+5)(k^2+k)^4 = \frac{1}{3}(n^3+3n^2+2n)^4$$

$$2) \sum_{k=1}^n (2k+1)(k^2+k+1)(k^2+k+7)(k^2+k)^6 = \frac{1}{9}(n^3+3n^2+2n)^6$$

Proposed by Mihály Bencze - Romania

SP.020. If $x, y, z \in (0, \frac{\pi}{2})$, then prove that:

$$\frac{\tan^2 x}{(y+x)^2} + \frac{\tan^2 y}{(z+x)^2} + \frac{\tan^2 z}{(x+y)^2} > \frac{3}{4}$$

Proposed by D. M. Bătinețu - Giurgiu, Neculai Stanciu - Romania

SP.021. If $x, y, z > 0$, then prove that:

$$(x^3y^3 + y^3z^3 + z^3x^3) \left(\frac{1}{(x+y)^5z} + \frac{1}{(y+z)^5x} + \frac{1}{(z+x)^5y} \right) \geq \frac{9}{32}$$

Proposed by D. M. Bătinețu - Giurgiu, Neculai Stanciu - Romania

SP.022. Prove that if $n \in \mathbb{N}; n \geq 2; 0 < a \leq b$ then:

$$\frac{b^{n+1} - a^{n+1}}{n+1} + \frac{ab(b^{n-1} - a^{n-1})}{n-1} \leq (b-a)\sqrt{2(a^{2n} + b^{2n})}$$

Proposed by Daniel Sitaru - Romania

SP.023. Let $A, B \in M_n(C)$ such that $\det A = \det B \neq 0$. Prove that:

$\det(AB + xy(AB)^{-1} + (x+y)I_n) = \det(BA + xy(BA)^{-1} + (x+y)I_n)$
for all $x, y \in C$.

Proposed by Mihály Bencze - Romania

SP.024. Let ABC be a triangle with the centroid G and denote by S_{ABC} its area. Prove that for any point P in the plane:

$$\frac{PA \cdot GA^2}{BC} + \frac{PB \cdot GB^2}{CA} + \frac{PC \cdot GC^2}{AB} \geq \frac{4}{3}S_{ABC}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.025. If $a_i > 0$ ($i = 1, 2, \dots, n$) and $k \geq 1$ then:

$$\left(\sum_{i=1}^n a_i \right)^k \leq \sum_{i_1=1, \dots, i_k=1}^n \frac{i_1 \dots i_k}{i_1 + \dots + i_k - k + 1} a_{i_1} \dots a_{i_k}$$

Proposed by Mihály Bencze - Romania

SP.026. Let be $n \in \mathbb{N}^*$. Compute:

$$I = \int_n^{n+1} \frac{\sqrt{n+1-x}}{\sqrt{n+1-x} + \sqrt{x-n} e^{2x-2n-1}} dx$$

Proposed by Daniel Sitaru - Romania

SP.027. Solve the following equation in set of real numbers

$$8^x + 27^{\frac{1}{x}} + 2^{x+1} \cdot 3^{\frac{x+1}{x}} + 2^x \cdot 3^{\frac{2x+1}{x}} = 125$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.028. Compute:

$$L = \lim_{n \rightarrow \infty} \frac{1}{n} \int_1^n \frac{x^4 + 4x^3 + 12x^2 + 9x}{(x+3)^5 - x^5 - 243} dx$$

Proposed by Daniel Sitaru - Romania

SP.029. Compute:

$$L = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \int_0^1 \frac{x \sin \pi x}{x + (1-x)k^{1-2x}} dx$$

Proposed by Daniel Sitaru - Romania

SP.030. Prove that:

$$I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos^2 2015x - \cos^2 2016x}{\sin x} dx > 0,0001$$

Proposed by Daniel Sitaru - Romania

UNDERGRADUATE PROBLEMS

UP.016. Compute the limit:

$$\lim_{n \rightarrow \infty} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \ln \left(1 + \frac{\sin \theta \sec^2 \theta}{n} \right)^{\cos \theta} \left(1 + \frac{\cos \theta}{n} \right)^{\cot \theta} \left(1 + \frac{\cot \theta}{n} \right)^{\sin \theta \sec^2 \theta} d\theta.$$

Proposed by Kunihiko Chikaya - Tokyo - Japan

UP.017. Let $\{a_n\}$ be a sequence defined inductively by

$$a_1 = 1, a_{n+1} = \frac{1}{2}a_n + \frac{n^2 - 2n - 1}{n^2(n+1)^2} (n = 1, 2, 3, \dots).$$

Find the greatest value of n such that $a_1 + a_2 + \dots + a_n$ is minimized.

Proposed by Kunihiko Chikaya - Tokyo - Japan

UP.018. Prove that:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\cos x}}{x} dx \leq \frac{\pi}{24} \left(e^{\frac{\pi}{12}} - 1 \right)^2 + \frac{\sqrt{2} - 1}{4} + \frac{152}{\pi^3}$$

Proposed by Soumitra Mukherjee - Chandar Nagore - India

UP.019. Let $E = (n, k, p)$ the total number of (x_1, x_2, \dots, x_p) for which

$x_1 + x_2 + \dots + x_p$ is a perfect k power when the integers x_1, x_2, \dots, x_p are selected independently at random from the set $\{1, 2, \dots, n\}$.

Compute: $\lim_{n \rightarrow \infty} \frac{E(n, k, p)}{n^{\frac{k}{k-1}}}$ for $p = 2; * p \geq 3$.

Proposed by Mihály Bencze - Romania

UP.020. If $x_k \in \mathbb{R} (k = 1, 2, \dots, n)$ then:

$$\left(\sum_{k=1}^n \frac{x_k}{k(k+1)} \right)^2 \leq \frac{n}{n+1} \sum_{k=1}^n \frac{x_k^2}{k(k+1)}$$

Proposed by Mihály Bencze - Romania

UP.021. Prove that:

$$1 \leq \int_0^1 \frac{dx}{\sqrt{1-x^2+x^{2015}-x^{2016}}} \leq \frac{\pi}{2}$$

Proposed by Soumitra Mukherjee - Chandar Nagore - India

UP.022. Let ABC be a triangle with the area S and denote by r, r_a, r_b, r_c inradius, exradii respectively. Prove that:

$$(r^2 + r_a r_b)(r^2 + r_b r_c)(r^2 + r_c r_a) \geq \left(\frac{10}{3}\right)^3 (rS)^2$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

UP.023. If $x, y, z, a, b, c > 0$ then:

$$\begin{aligned} \left(\frac{x+y}{2x}\right)^{b+c} \left(\frac{y+z}{2y}\right)^{c+a} \left(\frac{z+x}{2z}\right)^{a+b} &\geq \\ &\geq (x+y)^{b-c}(y+z)^{c-a}(z+x)^{a-b} \end{aligned}$$

Proposed by Mihály Bencze - Romania

UP.024. Calculate

$$\sum_{n=2}^{\infty} \operatorname{arctanh}\left(\frac{1}{F_{2n}}\right),$$

where F_n is the n th Fibonacci number.

Proposed by Cornel Ioan Valean - Romania

UP.025. Compute:

$$\lim_{n \rightarrow \infty} \left(\sqrt[3n+3]{(n+1)!} - \sqrt[3n]{n!} \right) \cdot \sqrt[3]{n^2}$$

Proposed by D. M. Bătinețu - Giurgiu, Neculai Stanciu - Romania

UP.026. Compute:

$$\lim_{n \rightarrow \infty} \left(\frac{n+1}{\sqrt[2n+2]{(2n+1)!!}} - \frac{n}{\sqrt[2n]{(2n-1)!!}} \right)^{\sqrt{n}}$$

Proposed by D. M. Bătinețu - Giurgiu, Neculai Stanciu - Romania

UP.027. If $\Gamma : (0, \infty) \rightarrow (0, \infty)$ is Euler's function, compute:

$$\lim_{x \rightarrow \infty} \left(\frac{x+1}{(\Gamma(x+2))^{\frac{1}{2x+2}}} - \frac{x}{(\Gamma(x+1))^{\frac{1}{2x}}} \right)^{\sqrt{x}}$$

Proposed by D. M. Bătinețu - Giurgiu, Neculai Stanciu - Romania

UP.028. Let be $f : (0, \infty) \rightarrow (0, \infty)$,

$f(x) = (x+1)^{\frac{(m+1)(x+2)}{x+1}} - x^{\frac{(m+1)(x+1)}{x}} ; m \in [0, \infty)$. Compute:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x^m}$$

Proposed by D. M. Bătinețu - Giurgiu, Neculai Stanciu - Romania

UP.029. If $x, y \in (0, \infty)$ then:

$$\frac{2}{\pi} \tan^{-1}(x+y) \tan^{-1}\left(\frac{1}{x+y}\right) < \frac{x+y}{4xy+1}$$

Proposed by Daniel Sitaru - Romania

UP.030. Prove that:

$$\begin{aligned} \sum_{k=1}^n \frac{k^{m-1} + (k+1)^{m-1} + (k+2)^{m-1}}{(2k+3)k^{m+1} + 2(k+1)^{m+2} + (2k+1)(k+2)^{m+1}} &\leq \\ &\leq \frac{1}{8} - \frac{1}{4(n+1)(n+2)} \end{aligned}$$

for all $n, m \in \mathbb{N}^*$.

Proposed by Mihály Bencze - Romania

MATHEMATICS DEPARTMENT, "THEODOR COSTESCU" NATIONAL ECONOMIC, COLLEGE DROBETA TURNU - SEVERIN, MEHEDINTI, ROMANIA