TWO REFINEMENTS OF IONESCU WEITZENBÖCK INEQUALITY

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If m_a, m_b, m_c are the lengths of the medians, w_a, w_b, w_c are the lengths of the internal bisectors and h_a, h_b, h_c are the lengths of the altitudes from the vertex A, B, C of a triangle ABC with the sides of lengths a, b, c, respectively, and the area S then holds the following inequalities:
a) $a^2 + b^2 + c^2 \geq 2\sqrt{3} \cdot a \cdot m_a$, and other two analogously;
b) $\frac{w_a}{h_a} + \frac{w_b}{h_b} + \frac{w_c}{h_c} \leq \frac{a^2+b^2+c^2}{4S}\sqrt{3}$.

Proof.

a) $a^2 + b^2 + c^2 \geq 2\sqrt{3} \cdot a \cdot m_a \Leftrightarrow (a^2 + b^2 + c^2)^2 \geq 12a^2 m_a^2 = 3a^2(4m_a^2) =$
$= 3a^2(2b^2 + 2c^2 - a^2) = 6a^2b^2 + 6c^2a^2 - 3a^4 \Leftrightarrow a^4 + b^4 + c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2 \geq$
$\geq 6a^2b^2 + 6c^2a^2 - 3a^4 \Leftrightarrow 4a^2 + b^4 + c^4 - 4a^2b^2 - 4c^2a^2 + 2b^2c^2 \geq 0 \Leftrightarrow$
$\Leftrightarrow (b^2 + c^2 - 2a^2)^2 \geq 0$, which is true.

b) WLOG we can assume that $a \leq b \leq c$ and then $w_a \geq w_b \geq w_c, h_a \geq h_b \geq h_c$.
So by Chebyshev's inequality, we have

$$\frac{w_a}{h_a} + \frac{w_b}{h_b} + \frac{w_c}{h_c} \leq \frac{1}{3}(w_a + w_b + w_c)\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right), \tag{0.1}$$

But,

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{a}{a \cdot h_a} + \frac{b}{b \cdot h_b} + \frac{c}{c \cdot h_c} = \frac{a+b+c}{2S} = \frac{2s}{2S} = \frac{s}{S}, \tag{0.2}$$

Also we have

$$w_a = \frac{2bc}{b+c}\cos\frac{A}{2} = \frac{2bc}{b+c}\sqrt{\frac{s(s-a)}{bc}} = \frac{2\sqrt{bc}}{b+c} \cdot \sqrt{s(s-a)} \overbrace{\leq}^{\text{AM-GM}}$$

$$\overbrace{\leq}^{\text{AM-GM}} \frac{2\sqrt{bc}}{2\sqrt{bc}}\sqrt{s(s-a)} = \sqrt{s(s-a)}, \text{ and other two similar,} \tag{0.3}$$

Therefore, we deduce that
(0.4)
$$w_a + w_b + w_c \leq \sqrt{s(s-a)} + \sqrt{s(s-b)} + \sqrt{s(s-c)} = \sqrt{s}(\sqrt{s-a} + \sqrt{s-b} + \sqrt{s-c}),$$

Since the function $f : \mathbb{R}_+^* \to \mathbb{R}_+^*, f(x) = x^2$, is convex on $\mathbb{R}_+^*$, applying Jensen's inequality we obtain that
(0.5)
$$x^2 + y^2 + z^2 \geq 3\left(\frac{x+y+z}{3}\right)^2 = \frac{(x+y+z)^2}{3} \Leftrightarrow x + y + z \leq \sqrt{3(x^2 + y^2 + z^2)},$$

where if we take $x = \sqrt{s-a}, y = \sqrt{s-b}, z = \sqrt{s-c}$ we deduce that

$$(s-a)+(s-b)+(s-c) \geq \frac{(\sqrt{s-a}+\sqrt{s-b}+\sqrt{s-c})}{3} \Leftrightarrow 3s \geq (\sqrt{s-a}+\sqrt{s-b}+\sqrt{s-c})^2 \Leftrightarrow$$

$$\Leftrightarrow \sqrt{s}\cdot\sqrt{3} \geq \sqrt{s-a}+\sqrt{s-b}+\sqrt{s-c}, \tag{0.6}$$

By 0.4 and 0.6 we infer that

$$w_a + w_b + w_c \leq \sqrt{s}\cdot\sqrt{s}\cdot\sqrt{3} = s\sqrt{3} \tag{0.7}$$

Hence, by 0.1, 0.2 and 0.7 yields that

$$\frac{w_a}{h_a}+\frac{w_b}{h_b}+\frac{w_c}{h_c} \leq \frac{1}{3}\cdot s\sqrt{3}\cdot\frac{s}{S} = \frac{s^2}{S\sqrt{3}} = \frac{(a+b+c)^2}{4S\sqrt{3}} \leq \frac{a^2+b^2+c^2}{4S}\sqrt{3},$$

and we are done!

□

Remark 0.1. The inequalities form (a) and (b) are in facts the refinements of Ionescu - Weitzenböck inequality.
Indeed:
• If in (a) we taking account by $m_a \geq h_a$, we obtain that

$$a^2+b^2+c^2 \geq 2\sqrt{3}\cdot a\cdot m_a \geq 2\sqrt{3}\cdot a\cdot h_a = 2\sqrt{3}\cdot 2S = 4S\sqrt{3},$$

i.e. Ionescu - Weitzenböck inequality.
• If in (b) we taking account by $w_a \geq h_a, w_b \geq h_b, w_c \geq h_c \Leftrightarrow \frac{w_a}{h_a} \geq 1, \frac{w_b}{h_b} \geq 1, \frac{w_c}{h_c} \geq 1$, yields that

$$3 \leq \frac{w_a}{h_a}+\frac{w_b}{h_b}+\frac{w_c}{h_c} \leq \frac{a^2+b^2+c^2}{4S}\sqrt{3},$$

so, we obtain $2\sqrt{3}S \leq a^2+b^2+c^2$, i.e. Ionescu - Weitzenböck inequality.