In any triangle ABC:

$$\frac{27m_a^2m_b^2m_c^2}{S^6} \le \left(\frac{1}{(p-a)^2}+\frac{1}{(p-b)^2}+\frac{1}{(p-c)^2}\right)^3$$

m_a − median length, S − area

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{27m_a^2m_b^2m_c^2}{\Delta^6} \le \left\{\frac{1}{(p-a)^2}+\frac{1}{(p-b)^2}+\frac{1}{(p-c)^2}\right\}^3 \qquad (1)$$

$$m_a^2 = \frac{2b^2+2c^2-a^2}{4}, m_b^2 = \frac{2c^2+2a^2-b^2}{4}$$

$$m_c^2 = \frac{2a^2+2b^2-c^2}{4}$$

$$(1) \Leftrightarrow \frac{27}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2) \le$$

$$\le \left\{\frac{\Delta^2}{(p-a)^2}+\frac{\Delta^2}{(p-b)^2}+\frac{\Delta^2}{(p-c)^2}\right\}^3$$

$$\Leftrightarrow \frac{3}{4}\sqrt[3]{(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)} \le$$

$$\le \frac{\Delta^2}{(p-a)^2}+\frac{\Delta^2}{(p-b)^2}+\frac{\Delta^2}{(p-c)^2}$$

$$GM \le AM \Rightarrow \frac{3}{4}\sqrt[3]{(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)} \le$$

$$\le \frac{3}{4}\,\frac{(2b^2+2c^2-a^2)+(2c^2+2a^2-b^2)+(2a^2+2b^2-c^2)}{3}$$

$$=\frac{3}{4}(a^2+b^2+c^2)$$

I shall show that, $\frac{3}{4}(a^2+b^2+c^2)\le\frac{\Delta^2}{(p-a)^2}+\frac{\Delta^2}{(p-b)^2}+\frac{\Delta^2}{(p-c)^2}$ **(2)**

which will prove (1)

RHS **of (2)** $=\frac{p(p-a)(p-b)(p-c)}{(p-a)^2}+\frac{p(p-a)(p-b)(p-c)}{(p-b)^2}+\frac{p(p-a)(p-b)(p-c)}{(p-c)^2}$

$$=p\left\{\frac{(p-b)(p-c)}{p-a}+\frac{(p-c)(p-a)}{p-b}+\frac{(p-a)(p-b)}{p-c}\right\}$$

$$=\frac{(a+b+c)}{4}\left\{\frac{(c+a-b)(a+b-c)}{b+c-a}+\frac{(a+b-c)(b+c-a)}{c+a-b}+\frac{(b+c-a)(c+a-b)}{a+b-c}\right\}$$

(2) $\Leftrightarrow 3(a^2+b^2+c^2)\overset{(3)}{\le}(a+b+c)\left\{\frac{(c+a-b)(a+b-c)}{b+c-a}+\frac{(a+b-c)(b+c-a)}{c+a-b}+\frac{(b+c-a)(c+a-b)}{a+b-c}\right\}$

Let $a+b-c=x, b+c-a=y, c+a-b=z$

$$a+b+c=x+y+z, a=\frac{z+x}{2}, b=\frac{x+y}{2}, c=\frac{y+z}{2}$$

(3) $\Leftrightarrow\frac{3}{4}\{(z+x)^2+(x+y)^2+(y+z)^2\}\le(x+y+z)\left(\frac{xy}{2}+\frac{yz}{x}+\frac{zx}{y}\right)$

$$\Leftrightarrow 6\left(\sum x^2+\sum xy\right)xyz\le 4\left(\sum x\right)\left(\sum x^2y^2\right)$$

$$\Leftrightarrow 2(x+y+z)(x^2y^2+y^2z^2+z^2x^2)\ge 3(x^2+y^2+z^2+xy+yz+zx)xyz$$

$$\Leftrightarrow 2x^3y^2+2x^2y^3+2y^3z^2+2y^2z^3+2z^3x^2+2z^2x^3+$$
$$+2xyz(xy+yz+zx)\ge 3xyz\left(\sum x^2\right)+3xyz(3xyz+zx)$$

$$\Leftrightarrow 2x^3y^2+2x^2y^3+2y^3z^2+2y^2z^3+2z^3x^2+2z^2x^3\ge$$
$$\ge 3x^3yz+3y^3zx+3z^3xy+x^2y^2z+y^2z^2x+z^2x^2y\quad(4)$$

$$AM \geq GM \Rightarrow \frac{3}{2}(x^3y^2 + x^3z^2) \overset{i}{\geq} 3x^3yz$$

$$\frac{3}{2}(y^3x^2 + y^3z^2) \overset{ii}{\geq} 3y^3zx$$

$$\frac{3}{2}(z^3x^2 + z^3y^2) \overset{iii}{\geq} 3z^3xy$$

$$i + ii + iii \Rightarrow \frac{3}{2}(x^3y^2 + x^2y^3 + y^3z^2 + y^2z^3 + z^3x^2 + z^2x^3) \geq$$
$$\geq 3x^3yz + 3y^3zx + 3z^3xy \quad (A)$$

$$\left.\begin{cases} x^3y^2 + y^2z^3 + y^2z^3 \geq 3xy^2z^2 \\ y^3x^2 + x^2z^3 + x^2z^3 \geq 3yx^2z^2 \\ y^3z^2 + z^2x^3 + z^2x^3 \geq 3yz^2x^2 \\ z^3y^2 + y^2x^3 + y^2x^3 \geq 3zy^2x^2 \\ z^3x^2 + x^2y^3 + x^2y^3 \geq 3zx^2y^2 \\ x^3z^2 + z^2y^3 + z^2y^3 \geq 3xy^2z^2 \end{cases}\right\} AM \geq GM$$

Adding, $3(x^3y^2 + x^2y^3 + y^3z^2 + y^2z^3 + z^3x^2 + z^2x^3) \geq$
$$\geq 6(x^2y^2z + y^2z^2x + z^2x^3y)$$

$$\Rightarrow \frac{1}{2}(x^3y^2 + x^2y^3 + y^3z^2 + y^2z^3 + z^3x^2 + z^2x^3) \geq x^2y^2z + y^2z^2x + z^3x^3y \ (B)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{27m_a^2m_b^2m_c^2}{S^6} \leq \left\{\frac{1}{(p-a)^2} + \frac{1}{(p-b)^2} + \frac{1}{(p-c)^2}\right\}^3$$

$$\Leftrightarrow 3\sqrt[3]{m_a^2m_b^2m_c^2} \leq \frac{S^2}{(p-a)^2} + \frac{S^2}{(p-b)^2} + \frac{S^2}{(p-c)^2} = \sum r_a^2$$

Now, $3\sqrt[3]{m_a^2m_b^2m_c^2} \overset{AM-GM}{\leq} \sum m_a^2 \leq \frac{27}{4}R^2 \leq \sum r_a^2$