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CONTENT

A SIMPLE PROOF FOR PRESTIN'S INEQUALITY

By Daniel Sitaru-Romania

Abstract: *In this paper is presented a simple proof for Prestin's inequality and a few applications.*

PRESTIN'S INEQUALITY:

$$\textbf{If } 0 < |x| < \frac{\pi}{2} \textbf{ then: } \left|\frac{1}{\sin x} - \frac{1}{x}\right| \le 1 - \frac{2}{\pi}$$

Proof. We will us the well known Jordan's and Kober's inequalities.

JORDAN'S INEQUALITY:

$$\textbf{If } 0 < |x| < \frac{\pi}{2}\textbf{, then: } \frac{2}{\pi} \le \frac{\sin x}{x} < 1$$

Equality holds for $x = \frac{\pi}{2}$.

KOBER'S INEQUALITY:

$$\textbf{If } x \in \left[0, \frac{\pi}{2}\right] \textbf{ then: } 1 - \frac{2}{\pi}x \le \cos x \le 1 - \frac{x^2}{\pi}$$

Equality holds for $x = 0$.

Observation: If $x \in \left[-\frac{\pi}{2}, 0\right]$ then $(-x) \in \left[0, \frac{\pi}{2}\right]$ and hence

$$1 - \frac{2}{\pi}(-x) \le \cos(-x) \le 1 - \frac{(-x)^2}{\pi}$$

$$1 + \frac{2}{\pi}x \le \cos x \le 1 - \frac{x^2}{\pi}$$

Case 1) $x \in \left(0, \frac{\pi}{2}\right]$. Let be $f: \left(0, \frac{\pi}{2}\right] \to \mathbb{R}, f(x) = \frac{1}{\sin x} - \frac{1}{x} - 1 + \frac{2}{\pi}$, then

$$f'(x) = \frac{-\cos x}{\sin^2 x} + \frac{1}{x^2} = \frac{1}{x^2 \sin^2 x}(\sin^2 x - x^2 \cos x) \overset{Jordan's}{\ge}$$

$$\ge \frac{1}{x^2 \sin^2 x}\left(\frac{4}{\pi^2}x^2 - x^2 \cos x\right) = \frac{1}{\sin^2 x}\left(\frac{4}{\pi^2} - \cos x\right) \overset{Kober's}{\ge}$$

$$\geq \frac{1}{\sin^2 x}\left(\frac{4}{\pi^2}-\left(\frac{2}{\pi}x-1\right)\right)=\frac{1}{\sin^2 x}\left(\frac{4}{\pi^2}+1-\frac{2}{\pi}x\right)\geq$$

$$\geq \frac{1}{\sin^2 x}\left(\frac{4}{\pi^2}+1-1\right)=\frac{4}{\pi^2\sin^2 x}>0;\left(\because 0<x\leq\frac{\pi}{2}\Rightarrow 0<-\frac{2}{\pi x}>-1\right)$$

$$f\text{ −increasing, then }\max f(x)=f\left(\frac{\pi}{2}\right)=1-\frac{2}{\pi}-1+\frac{2}{\pi}=0$$

$$f(x)\leq 0;\forall x\in\left(0,\frac{\pi}{2}\right]$$

$$\frac{1}{\sin x}-\frac{1}{x}-1+\frac{2}{\pi}\leq 0$$

$$\frac{1}{\sin x}-\frac{1}{x}\leq 1-\frac{2}{\pi}$$

Case 2) $x\in\left[-\frac{\pi}{2},0\right)$. Let be $g:\left[-\frac{\pi}{2},0\right)\to\mathbb{R}, g(x)=\frac{1}{\sin x}-\frac{1}{x}+1-\frac{2}{\pi}$, then:

$$g'(x)=\frac{-\cos x}{\sin^2 x}+\frac{1}{x^2}=\frac{1}{x^2\sin^2 x}(\sin^2 x-x^2\cos^2 x)=$$

$$=\frac{1}{(-x)^2\sin^2(-x)}(\sin^2(-x)-(-x)^2\cos(-x))\geq 0, x\in\left[-\frac{\pi}{2},0\right)\Rightarrow -x\in\left(0,\frac{\pi}{2}\right]$$

$$g\text{ −increasing , then }\min g(x)=g\left(-\frac{\pi}{2}\right)=-1+\frac{2}{\pi}+1-\frac{2}{\pi}=0$$

$$g(x)\geq 0;\forall x\in\left[-\frac{\pi}{2},0\right)$$

$$\frac{1}{\sin x}-\frac{1}{x}+1-\frac{2}{\pi}\geq 0$$

$$\frac{1}{\sin x}-\frac{1}{x}\geq -\left(1-\frac{2}{\pi}\right)$$

From $\frac{1}{\sin x}-\frac{1}{x}\leq 1-\frac{2}{\pi}$ and $\frac{1}{\sin x}-\frac{1}{x}\geq -\left(1-\frac{2}{\pi}\right)$ we obtain:

$$\left|\frac{1}{\sin x}-\frac{1}{x}\right|\leq 1-\frac{2}{\pi}$$

Application 1: If $\mathbf{0}<|\boldsymbol{x}|,|\boldsymbol{y}|\leq\frac{\boldsymbol{\pi}}{\mathbf{2}}$ then:

$$\frac{1}{\sin x \sin y} + \frac{1}{xy} \le \frac{1}{x \sin y} + \frac{1}{y \sin x} + \left(1 - \frac{2}{\pi}\right)^2$$

Proof. By Prestin's inequality:

$$\left|\frac{1}{\sin x} - \frac{1}{x}\right| \le 1 - \frac{2}{\pi} \text{ and } \left|\frac{1}{\sin y} - \frac{1}{y}\right| \le 1 - \frac{2}{\pi}$$

By multiplying:

$$\left|\frac{1}{\sin x} - \frac{1}{x}\right| \cdot \left|\frac{1}{\sin y} - \frac{1}{y}\right| \le \left(1 - \frac{2}{\pi}\right)^2$$

$$\left|\frac{1}{\sin x \sin y} - \frac{1}{x \sin y} - \frac{1}{y \sin x} + \frac{1}{xy}\right| \le \left(1 - \frac{2}{\pi}\right)^2$$

$$\frac{1}{\sin x \sin y} - \frac{1}{x \sin y} - \frac{1}{y \sin x} + \frac{1}{xy} \le \left(1 - \frac{2}{\pi}\right)^2$$

$$\frac{1}{\sin x \sin y} + \frac{1}{xy} \le \frac{1}{x \sin y} + \frac{1}{y \sin x} + \left(1 - \frac{2}{\pi}\right)^2$$

Equality holds for $x = y = \frac{\pi}{2}$.

Application 2: If $0 < |x| \le \frac{\pi}{2}$ then:

$$\frac{1}{\sin x} + \frac{2}{\pi x} \le \frac{1}{x} + \frac{2}{\pi \sin x} + \left(1 - \frac{2}{\pi}\right)^2$$

Proof. In application 1 we take $y = \frac{\pi}{2}$.

$$\frac{1}{\sin x \sin \frac{\pi}{2}} + \frac{1}{x \cdot \frac{\pi}{2}} \le \frac{1}{x \sin \frac{\pi}{2}} + \frac{1}{\frac{\pi}{2} \sin x} + \left(1 - \frac{2}{\pi}\right)^2$$

$$\frac{1}{\sin x} + \frac{2}{\pi x} \le \frac{1}{x} + \frac{2}{\pi \sin x} + \left(1 - \frac{2}{\pi}\right)^2$$

Equality holds for $x = \frac{\pi}{2}$.

Application 3: Prove without any software:

$$\boldsymbol{\pi^2\sqrt{2}+4<2\pi\sqrt{2}+\pi^2}$$

Proof. In application 2 we take $x=\frac{\pi}{4}$.

$$\frac{1}{\sin\frac{\pi}{4}}+\frac{2}{\pi\cdot\frac{\pi}{4}}<\frac{1}{\frac{\pi}{4}}+\frac{2}{\pi\sin\frac{\pi}{4}}+\left(1-\frac{2}{\pi}\right)^2$$

$$\frac{1}{\frac{\sqrt{2}}{2}}+\frac{8}{\pi^2}<\frac{4}{\pi}+\frac{2}{\pi\cdot\frac{\sqrt{2}}{2}}+1-\frac{4}{\pi}+\frac{4}{\pi^2}$$

$$\sqrt{2}+\frac{4}{\pi^2}<\frac{2\sqrt{2}}{\pi}+1$$

$$\pi^2\sqrt{2}+4<2\pi\sqrt{2}+\pi^2$$

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ABOUT A FEW ALGEBRAIC INEQUALITIES

By Florică Anastase-Romania

Abstract: *In this paper are presented a few non-standard inequalities and a few applications.*

App. 1) If $\boldsymbol{x_1,x_2,\dots,x_n>0}$ such that $\boldsymbol{\frac{1}{1+x_1}+\frac{1}{1+x_2}+\dots+\frac{1}{1+x_n}=1}$, then:

$$\boldsymbol{x_1\cdot x_2\cdot\ldots\cdot x_n\ge(n-1)^n}$$

Proof. Let us denote $y_i=\frac{1}{1+x_i}, i\in\overline{1,n}$, then $y_1+y_2+\dots+y_n=1$ and $y_i+y_ix_i=1$.

$$x_i=\frac{1-y_i}{y_i}=\frac{y_1+y_2+\dots+y_{i-1}+y_{i+1}+\dots+y_n}{y_i}$$

Using AM-GM inequality, we have:

$$x_i\ge\frac{(n-1)\sqrt[n-1]{\frac{y_1y_2\cdot\ldots\cdot y_n}{y_i}}}{y_i}$$

$$x_1\cdot x_2\cdot\ldots\cdot x_n\ge(n-1)^n\cdot\frac{\sqrt[n-1]{(y_1\cdot y_2\cdot\ldots\cdot y_n)^n}\cdot\frac{1}{\sqrt[n-1]{y_1\cdot y_2\cdot\ldots\cdot y_n}}}{y_1\cdot y_2\cdot\ldots\cdot y_n}$$

$$x_1 \cdot x_2 \cdot \ldots \cdot x_n \geq (n-1)^n \cdot \frac{\sqrt[n-1]{(y_1 \cdot y_2 \cdot \ldots \cdot y_n)^n}}{y_1 \cdot y_2 \cdot \ldots \cdot y_n \sqrt[n-1]{y_1 \cdot y_2 \cdot \ldots \cdot y_n}}$$

$$x_1 \cdot x_2 \cdot \ldots \cdot x_n \geq (n-1)^n$$

App.2) If $\boldsymbol{a, b, c, d} > 0$, then

$$\prod_{cyc} \frac{\mathbf{1} + \boldsymbol{a}^{\mathbf{3}}}{\mathbf{1} + \boldsymbol{a}^{\mathbf{2}}} \geq \frac{\mathbf{1} + \boldsymbol{abcd}}{\mathbf{2}}$$

Proof. If $a > 0$ we have:

$$\left(\frac{a^3+1}{a^2+1}\right)^2 \geq \frac{1+a^3}{1+a}; (1) \Leftrightarrow a(a-1)^2 \geq 0. (\text{Equality for } a = 1).$$

$$\left(\frac{a^3+1}{a+1}\right)^2 \geq \frac{1+a^4}{2}; (2) \Leftrightarrow 2(a^2-a+1)^2 \geq (1+a^4) \Leftrightarrow$$

$$(a^2-2a+1)^2 \geq 0. (\text{Equality for } a = 1).$$

From (1) and (2) we have:

$$\left(\frac{1+a^3}{1+a^2}\right)^4 \geq \left(\frac{1+a^3}{1+a}\right)^2 \geq \frac{1+a^4}{2} \Leftrightarrow \left(\frac{1+a^3}{1+a^2}\right)^4 \geq \frac{1+a^4}{2}$$

$$\frac{1+a^3}{1+a^2} \geq \sqrt[4]{\frac{1+a^4}{2}}; (3)$$

$$\prod_{cyc} \frac{1+a^3}{1+a^2} \geq \frac{1}{2} \cdot \sqrt[4]{\prod_{cyc}(1+a^4)} = \frac{1}{2} \cdot \sqrt[4]{(1+a^4)(1+b^4)(1+c^4)(1+d^4)} \overset{CBS}{\geq}$$

$$\geq \frac{1}{2} \cdot \sqrt[4]{(1+a^2b^2)^2 \cdot (1+c^2d^2)^2} = \frac{1}{2} \cdot \sqrt{(1+a^2b^2) \cdot (1+c^2d^2)} \overset{CBS}{\geq} \frac{1+abcd}{2}$$

Equality holds for $a = b = c = d = 1$.

App. 3) If $\boldsymbol{a, b, c, d} > 0$ such that $\frac{\mathbf{1}}{\mathbf{1}+\boldsymbol{a}} + \frac{\mathbf{1}}{\mathbf{1}+\boldsymbol{b}} + \frac{\mathbf{1}}{\mathbf{1}+\boldsymbol{c}} + \frac{\mathbf{1}}{\mathbf{1}+\boldsymbol{d}} = \mathbf{1}$ then:

$$\prod_{cyc} \frac{\mathbf{1} + \boldsymbol{a}^{\mathbf{3}}}{\mathbf{1} + \boldsymbol{a}^{\mathbf{2}}} \geq \mathbf{41}$$

Proof. Using App.1) we get $abcd \geq 3^4$ and using App. 2) we have

$$\prod_{cyc} \frac{1+a^3}{1+a^2} \geq \frac{1+abcd}{2} \geq \frac{1+3^4}{2} = 41$$

App. 4) If $a,b,c,d>0$, then

$$\sum_{cyc}\frac{1+a^3}{1+a^2}\ge 4\cdot\sqrt[4]{\frac{1+abcd}{2}}$$

Proof. Using App.1) and AM-GM inequality, we have:

$$\sum_{cyc}\frac{1+a^3}{1+a^2}\overset{AM-GM}{\ge}4\cdot\sqrt[4]{\prod_{cyc}\frac{1+a^3}{1+a^2}}\overset{(App.1)}{\ge}4\cdot\sqrt[4]{\frac{1+abcd}{2}}$$

App. 6) If $x,y,z,t\ge 1$ then:

$$\frac{1+\log^3 x}{1+\log^2 x}\cdot\frac{1+\log^3 y}{1+\log^2 y}\cdot\frac{1+\log^3 z}{1+\log^2 z}\cdot\frac{1+\log^3 t}{1+\log^2 t}\ge\frac{1+\log x\cdot\log y\cdot\log z\cdot\log t}{2}$$

Proof. For $x,y,z,t>1$, let us denote: $\log x=a,\log y=b,\log z=c,\log t=d$, $a,b,c,d>0$ and using App.1) we get the desired inequality.

Equality holds for $x=y=z=t=e$.

App. 7) If $x,y,z,t\ge 1$ then:

$$\frac{1+\log^3 x}{1+\log^2 x}+\frac{1+\log^3 y}{1+\log^2 y}+\frac{1+\log^3 z}{1+\log^2 z}+\frac{1+\log^3 t}{1+\log^2 t}\ge 4\cdot\sqrt[4]{\frac{1+\log x\cdot\log y\cdot\log z\cdot\log t}{2}}$$

Proof. For $x,y,z,t>1$, let us denote: $\log x=a,\log y=b,\log z=c,\log t=d$, $a,b,c,d>0$.

Using AM-GM and App.1) we have

$$\frac{1+\log^3 x}{1+\log^2 x}+\frac{1+\log^3 y}{1+\log^2 y}+\frac{1+\log^3 z}{1+\log^2 z}+\frac{1+\log^3 t}{1+\log^2 t}=$$

$$=\frac{1+a^3}{1+a^2}+\frac{1+b^3}{1+b^2}+\frac{1+c^3}{1+c^2}+\frac{1+d^3}{1+d^2}\overset{AM-GM}{\ge}$$

$$\ge 4\cdot\sqrt[4]{\frac{1+a^3}{1+a^2}\cdot\frac{1+b^3}{1+b^2}\cdot\frac{1+c^3}{1+c^2}\cdot\frac{1+d^3}{1+d^2}}\overset{App.1)}{\ge}4\cdot\sqrt[4]{\frac{1+abcd}{2}}=$$

$$=4\cdot\sqrt[4]{\frac{1+\log x\cdot\log y\cdot\log z\cdot\log t}{2}}$$

Equality holds for $x=y=z=t=e$.

App. 8) If $x,y,z,t,m>1$ such that

$$\log_{mx}m+\log_{my}m+\log_{mz}m+\log_{mt}m=1 \text{ then:}$$

$$\frac{\mathbf{1+\log_m^3 x}}{\mathbf{1+\log_m^2 x}}\cdot\frac{\mathbf{1+\log_m^3 y}}{\mathbf{1+\log_m^2 y}}\cdot\frac{\mathbf{1+\log_m^3 z}}{\mathbf{1+\log_m^2 z}}\cdot\frac{\mathbf{1+\log_m^3 t}}{\mathbf{1+\log_m^2 t}}\geq\mathbf{41}$$

Proof. Let us denote: $a=\log_m x$, $b=\log_m y$, $c=\log_m z$, $d=\log_m t\Rightarrow a,b,c,d>0$ and we have: $1=\log_{mx} m+\log_{my} m+\log_{mz} z+\log_{mt} t=$

$$=\frac{1}{\log_m(mx)}+\frac{1}{\log_m(my)}+\frac{1}{\log_m(mz)}+\frac{1}{\log_m(mt)}=$$

$$=\frac{1}{1+\log_m x}+\frac{1}{1+\log_m y}+\frac{1}{1+\log_m z}+\frac{1}{1+\log_m t}=$$

$$=\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}+\frac{1}{1+d}\overset{(App.1)}{\Longrightarrow}abcd\geq 3^4\Rightarrow$$

$$\log_m x\cdot\log_m y\cdot\log_m z\cdot\log_m t\geq 3^4$$

$$\prod_{cyc}\frac{1+a^3}{1+a^2}\geq\frac{1+abcd}{2}\geq 41$$

$$\prod_{cyc}\frac{1+\log_m^3 x}{1+\log_m^2 x}\geq\frac{1+\log_m x\cdot\log_m y\cdot\log_m z\cdot\log_m t}{2}=41$$

Equality holds for $x=y=z=t$.

App.9) If $a,b,c\in(0,1)$ or $a,b,c\in(1,\infty)$, $ab+bc+ca=abc$ then:

$$\mathbf{2}\cdot\sqrt[4]{\left(\prod_{cyc}\boldsymbol{tan^{-1}a}\right)\left(\sum_{cyc}\boldsymbol{tan^{-1}a}\right)}\leq\boldsymbol{tan^{-1}}\left(\frac{\sqrt{\left(\sum_{cyc}\boldsymbol{a^2}\right)\left(\sum_{cyc}(\mathbf{1}-\boldsymbol{a})^2\right)}}{\mathbf{1}-\boldsymbol{abc}}\right)$$

Proof: $2\sqrt[4]{tan^{-1}a\cdot tan^{-1}b\cdot tan^{-1}c(tan^{-1}a+tan^{-1}b+tan^{-1}c)}$

$$\overset{Am-Gm}{\leq}2\cdot\frac{2(tan^{-1}a+tan^{-1}b+tan^{-1}c)}{4}$$

$$=tan^{-1}\left(\frac{a+b+c-abc}{1-ab-bc-ca}\right)=tan^{-1}\left(\frac{a+b+c-ab-bc-ca}{1-abc}\right)$$

$$=tan^{-1}\left(\frac{a(1-b)+b(1-c)+c(1-a)}{1-abc}\right)$$

Let be the function: $f(x)=tan^{-1}x, f'(x)=\frac{1}{1+x^2}>0\forall x\in\mathbb{R}\Rightarrow f$ —increasing

$$a(1-b)+b(1-c)+c(1-a)\overset{C.B.S}{\leq}\sqrt{(a^2+b^2+c^2)((1-a)^2+(1-b)^2+(1-c)^2)}$$

If $a,b,c\in(0,1)\Rightarrow\begin{cases}1-a>0\\1-b>0\\1-c>0\end{cases}$ and $1-abc>0\Rightarrow\frac{\sum a(1-b)}{1-abc}>0$

If $a, b, c \in (1, \infty)$) $\Rightarrow \begin{cases} 1-a<0 \\ 1-b<0 \\ 1-c<0 \end{cases}$ and $1-abc<0 \Rightarrow \frac{\sum a(1-b)}{1-abc} > 0$

App.10) If $a, b, c \in (1,2)$, $f: (2,3) \to (0, \infty)$ continuous with $f'(x) < 0$ and $f''(x) < 0, \forall x \in (2,3)$ then:

$$\sum_{cyc} f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \geq 2 \cdot \sqrt[4]{\prod_{cyc} f(a+1) \cdot \sum_{cyc} f(a+1)}$$

Proof: $a, b, c \in (1,2) \to (1+a), (1+b), (1+c) \in (2,3)$

$$\frac{(a+1)(b+1)}{1+\sqrt{ab}} \leq \frac{a+b+2}{2} \leftrightarrow \left(\sqrt{a}-\sqrt{b}\right)^2\left(\sqrt{ab}-1\right) \geq 0, \forall a, b \in (1,2) \text{ and analogs} \dots \dots (1)$$

$$\frac{(a+1)(b+1)}{1+\sqrt{ab}} \geq \frac{(1+\sqrt{ab})^2}{1+\sqrt{ab}} = 1+\sqrt{ab} > 2$$

$$\frac{(a+1)(b+1)}{1+\sqrt{ab}} < 3 \leftrightarrow \left(\sqrt{ab}-1\right)\left(\sqrt{ab}-2\right) < 0 \text{ true.}$$

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \geq 2\sqrt[4]{xyz(x+y+z)}, \forall x, y, z > 0 \dots \dots (2)$$

$$\text{Let } z = max\{x, y, z\} \text{ and } a = \frac{x}{z}, b = \frac{y}{z}, a, b \in [0,1] \overset{(2)}{\Leftrightarrow} (a+b+ab)^2 \geq 4ab\sqrt{a^2+b^2+1}$$

$$\text{But } (a+b+ab)^2 = a^2b^2 + (a+b)^2 + 2ab(a+b) \geq 2ab(a+b+2), \text{then we have:}$$

$$(a+b+2)^2 \geq 4(a+b+1), \text{true from } (a+b+2)^2 = (a+b)^2 + 4 + 4(a+b) \geq 4(a+b+1) \geq 4(a^2+b^2+1), \forall a, b \in [0,1]$$

From (1) and (2) we have:

$$f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \geq f\left(\frac{(a+1)+(b+1)}{2}\right) \geq \frac{f(a+1)+f(b+1)}{2} \geq \sqrt{f(a+1) \cdot f(b+1)}$$

$$\sum_{cyc} f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \geq \sum_{cyc} \sqrt{f(a+1) \cdot f(b+1)} \geq$$

$$\geq 2 \cdot \sqrt[4]{\left(\prod_{cyc} f(a+1)\right) \cdot \left(\sum_{cyc} f(a+1)\right)}$$

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A SIMPLE PROOF FOR YOUNG'S INEQUALITY AND APPLICATIONS

By Daniel Sitaru-Romania

Abstract: *In this paper is presented a simple proof for Young's inequality and a few applications.*

YOUNG'S INEQUALITY $(\boldsymbol{n}=\mathbf{2})$:

If $\boldsymbol{x_1}, \boldsymbol{x_2} \geq \mathbf{0}; \boldsymbol{p}, \boldsymbol{q} > 1, \frac{\mathbf{1}}{\boldsymbol{p}} + \frac{\mathbf{1}}{\boldsymbol{q}} = \mathbf{1}$ **then:**

$$\boldsymbol{x_1 x_1 \leq \frac{x_1^p}{p} + \frac{x_2^q}{q}; (1)}$$

Equality holds for $\boldsymbol{x_1^p = x_2^q}$**.**

Proof. Let be $f:(0,\infty) \to \mathbb{R}, f(x) = e^x$, then $f'(x) = e^x, f''(x) = e^x > 0 \Rightarrow f$ −convex function, hence $f(\lambda_1 a + \lambda_2 b) \leq \lambda_1 f(a) + \lambda_2 f(b); a, b > 0; \lambda_1, \lambda_2 > 0, \lambda_1 + \lambda_2 = 1$.

For $\lambda_1 = \frac{1}{p}; \lambda_2 = \frac{1}{q} \Rightarrow \lambda_1 + \lambda_2 = \frac{1}{p} + \frac{1}{q} = 1$ and hence

$$f\left(\frac{1}{p}a + \frac{1}{q}b\right) \leq \frac{1}{p}f(a) + \frac{1}{q}f(b) \Leftrightarrow e^{\frac{1}{p}a + \frac{1}{q}b} \leq \frac{1}{p}e^a + \frac{1}{q}e^b$$

$$e^{\frac{1}{p}a} \cdot e^{\frac{1}{q}b} \leq \frac{1}{p}e^a + \frac{1}{q}e^b$$

For $x_1 = e^{\frac{1}{p}a}, x_2 = e^{\frac{1}{q}b} \Rightarrow \frac{a}{p} = \log x_1, \frac{b}{q} = \log x_2$ or $a = p \log x_1, b = q \log x_2$.

$$e^{\frac{1}{p} \cdot p \log x_1} \cdot e^{\frac{1}{q} \cdot q \log x_2} \leq \frac{1}{p}e^a + \frac{1}{q}e^b$$

$$e^{\log x_1} \cdot e^{\log x_2} \leq \frac{1}{p}e^{p \log x_1} + \frac{1}{q}e^{q \log x_2}$$

$$x_1 \cdot x_2 \leq \frac{1}{p}\left(e^{\log x_1}\right)^p + \frac{1}{q}\left(e^{\log x_2}\right)^q$$

$$x_1 \cdot x_2 \leq \frac{1}{p} \cdot x_1^p + \frac{1}{q} \cdot x_2^q$$

If $x_1^p = x_2^q$ then:

$$\frac{1}{p}\cdot x_1^p + \frac{1}{q}\cdot x_2^q = \frac{1}{p}\cdot x_1^p + \frac{1}{q}\cdot x_1^p = \left(\frac{1}{p}+\frac{1}{q}\right)x_1^p = x_1^p =$$

$$= x_1^{p\cdot 1} = x_1^{p\left(\frac{1}{p}+\frac{1}{q}\right)} = x_1^{1+\frac{p}{q}} = x_1 \cdot x_1^{\frac{p}{q}} = x_1 \cdot \left(x_1^p\right)^{\frac{1}{q}} = x_1 \cdot \left(x_1^q\right)^{\frac{1}{q}} = x_1 \cdot x_2$$

YOUNG'S INEQUALITY ($\boldsymbol{n = 3}$):

If $\boldsymbol{x_1, x_2, x_3 \geq 0; p, q, r} > 1; \boldsymbol{\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1}$ then:

$$\boldsymbol{x_1x_2x_3 = \frac{x_1^p}{p} + \frac{x_2^q}{q} + \frac{x_3^r}{r}; (2)}$$

Equality holds for $\boldsymbol{x_1^p = x_2^q = x_3^r}$.

Proof. Let be $f:(0,\infty)\to\mathbb{R}, f(x) = e^x$, then $f'(x) = e^x, f''(x) = e^x > 0 \Rightarrow f$ −convex function, hence $f(\lambda_1 a + \lambda_2 b + \lambda_3 c) \leq \lambda_1 f(a) + \lambda_2 f(b) + \lambda_3 f(c); a, b > 0;$

$\lambda_1, \lambda_2, \lambda_3 > 0, \lambda_1 + \lambda_2 + \lambda_3 = 1.$

For $\lambda_1 = \frac{1}{p}; \lambda_2 = \frac{1}{q}; \lambda_3 = \frac{1}{r} \Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = \frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$ and

$$f\left(\frac{1}{p}a + \frac{1}{q}b + \frac{1}{r}c\right) \leq \frac{1}{p}f(a) + \frac{1}{q}f(b) + \frac{1}{r}f(c)$$

$$e^{\frac{1}{p}a+\frac{1}{q}b+\frac{1}{r}c} \leq \frac{1}{p}e^a + \frac{1}{q}e^b + \frac{1}{r}e^c$$

$$e^{\frac{1}{p}a}\cdot e^{\frac{1}{q}b}\cdot e^{\frac{1}{r}c} \leq \frac{1}{p}e^a + \frac{1}{q}e^b + \frac{1}{r}e^c$$

For $x_1 = e^{\frac{1}{p}a}; x_2 = e^{\frac{1}{q}b}; x_3 = e^{\frac{1}{r}c} \Rightarrow \frac{a}{p} = \log x_1; \frac{b}{q} = \log x_2; \frac{c}{r} = \log x_3$, or

$$a = p\log x_1; b = q\log x_2; c = r\log x_3$$

$$e^{\frac{1}{p}\cdot p\log x_1}\cdot e^{\frac{1}{q}\cdot q\log x_2}\cdot e^{\frac{1}{r}\cdot r\log x_3} \leq \frac{1}{p}e^{p\log x_1} + \frac{1}{q}e^{q\log x_2} + \frac{1}{r}e^{r\log x_3}$$

$$e^{\log x_1} \cdot e^{\log x_2} \cdot e^{\log x_3} \le \frac{1}{p}\left(e^{\log x_1}\right)^p + \frac{1}{q}\left(e^{\log x_2}\right)^q + \frac{1}{r}\left(e^{\log x_3}\right)^r$$

$$x_1 \cdot x_2 \cdot x_3 \le \frac{1}{p} \cdot x_1^p + \frac{1}{q} \cdot x_2^q + \frac{1}{r} \cdot x_3^r$$

If $x_1^p = x_2^q = x_3^r$ then:

$$\frac{1}{p} \cdot x_1^p + \frac{1}{q} \cdot x_2^q + \frac{1}{r} \cdot x_3^r = \frac{1}{p} x_1^p + \frac{1}{q} x_1^p + \frac{1}{q} x_1^p = \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) x_1^p =$$

$$= 1 \cdot x_1^p = x_1^{p \cdot 1} = x_1^{p\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)} = x_1^{1+\frac{p}{q}+\frac{p}{r}} = x_1 \cdot x_1^{\frac{p}{q}} \cdot x_1^{\frac{p}{r}} =$$

$$= x_1 \cdot \left(x_1^p\right)^{\frac{1}{q}} \cdot \left(x_2^p\right)^{\frac{1}{r}} = x_1 \cdot \left(x_2^q\right)^{\frac{1}{q}} \cdot (x_3^r)^{\frac{1}{r}} = x_1 \cdot x_2 \cdot x_3$$

GENERAL YOUNG'S INEQUALITY:

If $\boldsymbol{x_1, x_2, \dots, x_n \ge 0; p_1, p_2, \dots, p_n} > 1; n \in \mathbb{N}; \boldsymbol{n \ge 2, \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_n} = 1}$ **then:**

$$\boldsymbol{x_1 \cdot x_2 \cdot \ldots \cdot x_n \le \frac{x_1^{p_1}}{p_1} + \frac{x_2^{p_2}}{p_2} + \dots + \frac{x_n^{p_n}}{p_n}; (3)}$$

Equality holds for $\boldsymbol{x_1^{p_1} = x_2^{p_2} = \dots = x_n^{p_n}}$**.**

Proof. Let be $f: (0, \infty) \to \mathbb{R}, f(x) = e^x$, then $f'(x) = e^x, f''(x) = e^x > 0 \Rightarrow f$ −convex function, hence $f(\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n) \le \lambda_1 f(a_1) + \lambda_2 f(a_2) + \dots + \lambda_n f(a_n)$;

$$a_1, a_2, \dots, a_n > 0; \lambda_1, \lambda_2, \dots, \lambda_n > 0, \lambda_1 + \lambda_2 + \dots + \lambda_n = 1.$$

For $\lambda_1 = \frac{1}{p_1}, \lambda_2 = \frac{1}{p_2}, \dots, \lambda_3 = \frac{1}{r} \Rightarrow \lambda_1 + \lambda_2 + \dots + \lambda_n = \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_n} = 1$ and

$$f\left(\frac{1}{p_1} a_1 + \frac{1}{p_2} a_2 + \dots + \frac{1}{p_n} a_n\right) \le \frac{1}{p_1} f(a_1) + \frac{1}{p_2} f(a_2) + \dots + \frac{1}{p_n} f(a_n)$$

$$e^{\frac{1}{p_1} a_1 + \frac{1}{p_2} a_2 + \dots + \frac{1}{p_n} a_n} \le \frac{1}{p_1} e^{a_1} + \frac{1}{p_2} e^{a_2} + \dots + \frac{1}{p_n} e^{a_n}$$

$$e^{\frac{1}{p_1} a_1} \cdot e^{\frac{1}{p_2} a_2} \cdot \ldots \cdot e^{\frac{1}{p_n} a_n} \le \frac{1}{p_1} e^{a_1} + \frac{1}{p_2} e^{a_2} + \dots + \frac{1}{p_n} e^{a_n}$$

For $x_1 = e^{\frac{1}{p_1}a_1}, x_2 = e^{\frac{1}{p_2}a_2}, \dots x_n = e^{\frac{1}{p_n}a_n}$ or $a_1 = p_1 \log x_1, a_2 = p_2 \log x_2, \dots, a_n = p_n \log x_n$ we have:

$$e^{\frac{1}{p_1}p_1 \log x_1} \cdot e^{\frac{1}{p_2}p_2 \log x_2} \cdot \dots \cdot e^{\frac{1}{p_n}p_n \log x_n} \le \frac{1}{p_1}e^{p_1 \log x_1} + \frac{1}{p_2}e^{p_2 \log x_2} + \dots + \frac{1}{p_n}e^{p_n \log x_n}$$

$$e^{\log x_1} \cdot e^{\log x_2} \cdot \dots \cdot e^{\log x_n} \le \frac{1}{p_1}\left(e^{\log x_1}\right)^{p_1} + \frac{1}{p_2}\left(e^{\log x_2}\right)^{p_2} + \dots + \frac{1}{p_n}\left(e^{\log x_n}\right)^{p_n}$$

$$x_1 \cdot x_2 \cdot \dots \cdot x_n \le \frac{x_1^{p_1}}{p_1} + \frac{x_2^{p_2}}{p_2} + \dots + \frac{x_n^{p_n}}{p_n}$$

If $x_1^{p_1} = x_2^{p_2} = \cdots x_n^{p_n}$ then:

$$\frac{1}{p_1}x_1^{p_1} + \frac{1}{p_2}x_2^{p_2} + \dots + \frac{1}{p_n}x_n^{p_n} = \frac{1}{p_1}x_1^{p_1} + \frac{1}{p_2}x_1^{p_1} + \dots + \frac{1}{p_n}x_1^{p_1} =$$

$$= \left(\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_n}\right)x_1^{p_1} = x_1^{p_1} = x_1^{p_1 \cdot 1} = x_1^{\frac{1}{p_1}+\frac{1}{p_2}+\dots+\frac{1}{p_n}} =$$

$$= x_1^1 \cdot x_1^{\frac{p_1}{p_2}} \cdot x_1^{\frac{p_1}{p_3}} \cdot \dots \cdot x_1^{\frac{p_1}{p_n}} = x_1 \cdot \left(x_1^{p_1}\right)^{\frac{1}{p_2}} \cdot \left(x_1^{p_1}\right)^{\frac{1}{p_3}} \cdot \dots \cdot \left(x_1^{p_1}\right)^{\frac{1}{p_n}}$$

$$= x_1 \cdot \left(x_2^{p_2}\right)^{\frac{1}{p_2}} \cdot \left(x_3^{p_3}\right)^{\frac{1}{p_3}} \cdot \dots \cdot \left(x_n^{p_n}\right)^{\frac{1}{p_n}} = x_1 \cdot x_2 \cdot \dots \cdot x_n$$

Application 1: If $x_1, x_2 \ge 0$ then:

$$\boldsymbol{x_1 x_2 \le \frac{1}{2}\left(x_1^2 + x_2^2\right)}$$

Proof. We take in (1): $p = q = 2$. Equality holds for $x_1 = x_2$.

Application 2: If $x_1, x_2 \ge 0$ then: $3x_1x_2 \le x_1^3 + 2x_2\sqrt{x_2}$

Proof: We take in (1): $p = 3, q = \frac{3}{2}$. Equality holds for $x_1^3 = x_2^{\frac{3}{2}}$.

Application 3: If $x_1, x_2, x_3 \ge 0$ then:

$$\boldsymbol{x_1x_2x_3 \le \frac{1}{3}\left(x_1^3 + x_2^3 + x_3^3\right)}$$

Proof. We take in (2): $p = q = r = 3$. Equality holds for $x_1x_2 = x_3$.

Application 4: If $x_1, x_2, x_3 \geq 0$ then:

$$x_1x_2x_3 \leq \frac{1}{2}x_1^2 + \frac{1}{3}x_2^3 + \frac{1}{6}x_3^6$$

Proof. We take in (2): $p = 2, q = 3, r = 6$. Equality holds for $x_1^2 = x_2^3 = x_3^6$.

Application 5: If $x_1, x_2, \dots, x_n \geq 0$ then:

$$x_1 \cdot x_2 \cdot \dots \cdot x_n \leq \frac{1}{n}(x_1^n + x_2^n + \dots + x_n^n); n \in \mathbb{N}, n \geq 2$$

Proof. We take in (3): $p_1 = p_2 = \dots = p_n = \frac{1}{n}$.

Application 6.

YOUNG'S INEQUALITY INTEGRAL FORM $(n = 2)$:

If $f, g: [a, b] \to [0, \infty); a < b; f, g$ –continuous, $p, q > 1; \frac{1}{p} + \frac{1}{q} = 1$ then:

$$\int_a^b f(x)g(x)\,dx \leq \frac{1}{p}\int_a^b f^p(x)dx + \frac{1}{q}\int_a^b g^q(x)dx$$

Proof. We take in (1): $x_1 = f(x); x_2 = g(x)$, then $f(x)g(x) \leq \frac{1}{p}f^p(x) + \frac{1}{q}g^q(x)$

By integrating:

$$\int_a^b f(x)g(x)\,dx \leq \frac{1}{p}\int_a^b f^p(x)dx + \frac{1}{q}\int_a^b g^q(x)dx$$

Equality holds for $f^p(x) = g^q(x); \forall x \in [a, b]$

Application 7.

YOUNG'S INEQUALITY INTEGRAL FORM $(n = 3)$:

If $f, g, h: [a, b] \to [0, \infty); a < b; f, g, h$ –continuous, $p, q, r > 1; \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ then:

$$\int_a^b \boldsymbol{f(x)g(x)\,h(x)dx} \le \frac{\mathbf{1}}{\boldsymbol{p}}\int_a^b \boldsymbol{f^p(x)dx} + \frac{\mathbf{1}}{\boldsymbol{q}}\int_a^b \boldsymbol{g^q(x)dx} + \frac{\mathbf{1}}{\boldsymbol{r}}\int_a^b \boldsymbol{h^r(x)dx}$$

Proof. We take in (2): $x_1 = f(x); x_2 = g(x); x_3 = h(x)$, then

$$f(x)g(x)h(x) \le \frac{1}{p}f^p(x) + \frac{1}{q}g^q(x) + \frac{1}{r}h^r(x)$$

By integrating:

$$\int_a^b f(x)g(x)h(x)\,dx \le \frac{1}{p}\int_a^b f^p(x)dx + \frac{1}{q}\int_a^b g^q(x)dx + \frac{1}{r}\int_a^b h^r(x)dx$$

Equality holds for $f^p(x) = g^q(x) = h^r(x); \forall x \in [a,b]$

YOUNG'S INEQUALITY GENERAL FORM

If $\boldsymbol{f_1, f_2, \dots, f_n}: [\boldsymbol{a}, \boldsymbol{b}] \to [\mathbf{0}, \infty); \boldsymbol{a} < b; \boldsymbol{f_1, f_2, \dots, f_n}$ −continuous, $\boldsymbol{p_1, p_2, \dots, p_n} > 1$;

$\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_n} = 1$ then:

$$\int_a^b \boldsymbol{f_1(x)f_2(x)\cdot\ldots\cdot f_n(x)\,dx} \le \frac{\mathbf{1}}{\boldsymbol{p_1}}\int_a^b \boldsymbol{f^{p_1}(x)dx} + \frac{\mathbf{1}}{\boldsymbol{p_2}}\int_a^b \boldsymbol{f^{p_2}(x)dx} + \dots + \frac{\mathbf{1}}{\boldsymbol{p_n}}\int_a^b \boldsymbol{f_n^{p_n}(x)dx}$$

Proof. We take in (3): $x_1 = f_1(x); x_2 = f_2(x), \dots, x_n = f_n(x)$ then

$$f_1(x)f_2(x)\cdot\ldots\cdot f_n(x) \le \frac{1}{p_1}f_1^{p_1}(x) + \frac{1}{p_2}f_2^{p_2}(x) + \dots + \frac{1}{p_n}f_n^{p_n}(x)$$

By integrating:

$$\int_a^b f_1(x)f_2(x)\cdot\ldots\cdot f_n(x)\,dx \le \frac{1}{p_1}\int_a^b f^{p_1}(x)dx + \frac{1}{p_2}\int_a^b f^{p_2}(x)dx + \dots + \frac{1}{p_n}\int_a^b f_n^{p_n}(x)dx$$

Equality holds for $f_1^{p_1}(x) = f_2^{p_2}(x) = f_n^{p_n}(x); \forall x \in [a,b]$

Application 9: If $\boldsymbol{x_1, x_2} \ge \mathbf{0}; \boldsymbol{a} \in \left(\mathbf{0}, \frac{\pi}{2}\right)$ then:

$$\boldsymbol{x_1x_2} \le \frac{\mathbf{1}}{\boldsymbol{\sin^2 x}} \cdot \boldsymbol{x_1^{\sin^2 a}} + \frac{\mathbf{1}}{\boldsymbol{\cos^2 a}} \cdot \boldsymbol{x_2^{\cos^2 a}}$$

Proof. We take in (1): $p = \frac{1}{\sin^2 a}; q = \frac{1}{\cos^2 a}$. Obviously $p, q > 1; \frac{1}{p} + \frac{1}{q} = 1$.

Equality holds for $x_1^{\sin^2 a} = x_2^{\cos^2 a}$.

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A USEFUL LEMMA FOR SOME EXPONENTIAL INEQUALITIES

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Abstract: *A lemma regarding an inequality between an exponential and a rational expression is highlighted and studied. Several applications of this lemma are presented.*

Keywords and phrases **:** *Bernoulli's inequality , exponential inequality , fractional inequality,*

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Although the following lemma is a result of ours - over 40 years old - it was highlighted only in problems published by us after 2006 (see for example [3] - [9]).

Initially it was used strictly to solve exponential inequalities ; it was noticed by the regretted mathematician *Alexander Bogomolny*, who presented it as a stand-alone result in his prestigious website, ***Cut.the.knot ,*** [1]. We will present it below (with a slight modification). This lemma has a simple and very short expression:

Lemma. (See [1])

If $m > 0$ and $n \in (0, 1)$, then the following relationship holds:

$$\boldsymbol{m^n > \frac{m}{m+n}}; \quad (\boldsymbol{L})$$

Proof 1. Indeed, by logaritmic in $(\boldsymbol{L})$, we obtain equvalently $(n-1)\log m + \log(m+n) > 0; (1)$

To prove (1), we consider the function $f: (0,1) \to \mathbb{R}, f(x) = (n-1)\log x + \log(x+n)$, for which we have

$$f'(x) = n \cdot \frac{x+n-1}{x(x+n)}$$

So, the function has a minimum in $x_0 = 1 - n > 0$.Therefore, for any $x \in (0, \infty)$, we have:

$$f(x) \geq f(x_0) = f(1-n) = (n-1)\log(1-n)\,, \text{hence } (1).$$

Proof 2. Starting from Bernoulli's inequality for subunit powers:

If $x > -1$ and $a \in (0, a)$, then $(1+x)^a < 1 + ax$; $(\boldsymbol{B})$. In which choose $x = m - 1 > -1$ and

$a = 1 - n \in (0,1)$, and we get:

$$m^{1-n} = [(1 + m - 1)]^{1-n} < 1 + (m - 1)(1 - n) = m + n - mn < m + n$$

$$\Rightarrow \frac{m}{m^n} < m + n \Rightarrow m^n > \frac{m}{m + n}$$

2. Remark. The inequality in the Lemma can obviously extend to $n \in (0,1]$, because for $n = 1$ inequality $(\boldsymbol{L})$**,** in the following form

$$m^n > \frac{m}{m + n - mn} > \frac{m}{m + n}; (2)$$

(obtained in fact in the 2nd proof of the *lemma ,* too) . Independently, the same refinement it was reported to us (in a private communication) by the mathematician *Dan-Stefan Marinescu* .

Exponential inequalities are generally difficult inequalities . Inequality (L) has the great advantage that it manages to compare an exponential with a fraction. In this way *exponential inequalities* are reduced to *rational inequalities* - which are easier to manipulate .

In the following we will see during several applications how important this *lemma* is and how much it simplifies some inequalities with exponentials.

3. Application

Let $\boldsymbol{x}, \boldsymbol{y}$ be two strictly positive real numbers. Prove that $\boldsymbol{x}^{\boldsymbol{y}} + \boldsymbol{y}^{\boldsymbol{x}} > 1; (\boldsymbol{3})$

(France, Concours General, 1996, Exercise IV.2)

Solution. If at least one of the numbers x and y is greater than or equal to one, inequality occurs. Take for example, $x \geq 1$ and $f(x) = x^y$. How $f'(x) = x^y \log x \geq x^y \log 1 \geq 0$, it follows that f is increasing, therefore $f(x) \geq f(1) = 1$. Meaning $x^h \geq 1$ and how $y^x > 0$, result $x^y + y^x > 1$. Due to the symmetry of the inequality (3), the same happens if $y \geq 1$.

It remains to prove as for $x, y \in (0,1)$, inequality (3) occurs. We are now in Lemma's conditions, so we will have $x^y + y^x > \frac{x}{x+y} + \frac{y}{x+y} = 1$.

4. Consequence.

If $\boldsymbol{x} \in \left(\boldsymbol{0}, \frac{\boldsymbol{\pi}}{\boldsymbol{2}}\right)$, then $\sin \boldsymbol{x}^{\cos \boldsymbol{x}} + \cos \boldsymbol{y}^{\sin \boldsymbol{x}} > 1;\ \ (\boldsymbol{4})$.

It results from the last part of the above solution, or directly with Lemma.

5. Application

If $\boldsymbol{a_1}, \boldsymbol{a_2}, \dots, \boldsymbol{a_n} \in (\boldsymbol{0}, \boldsymbol{1})$ and $\boldsymbol{S} = \boldsymbol{a_1} + \boldsymbol{a_2} + \cdots + \boldsymbol{a_n}$, then:

$$(\boldsymbol{S} - \boldsymbol{a_1})^{a_1} + (\boldsymbol{S} - \boldsymbol{a_2})^{a_2} + \cdots + (\boldsymbol{S} - \boldsymbol{a_n})^{a_n} > n - 1; (\boldsymbol{5})$$

Solution. We apply Lemma n −times, we obtain:

$$(S-a_1)^{a_1}+(S-a_2)^{a_2}+\cdots+(S-a_n)^{a_n}\overset{(L)}{>}\frac{S-a_1}{(S-a_1)+a_1}+\frac{S-a_2}{(S-a_2)+a_2}+\cdots$$

$$+\frac{S-a_n}{(S-a_n)+a_n}=\frac{1}{S}\sum_{k=1}^{n}(S-a_k)=\frac{1}{S}(n-1)S=n-1$$

Inequality (5) can also be considered a generalization of inequality (3) - for the case of subunit variables. It can also generate the following inequality.

6. Consequence (See [5])

Prove that in any triangle ABC holds the inequality:

$$(\sin A+\sin B)^{\sin C}+(\sin B+\sin C)^{\sin A}+(\sin C+\sin A)^{\sin B}>2;\ (\mathbf{6})$$

Solution. Indeed for $n=3$ and $a_1=\sin A$, $a_2=\sin B$, $a_3=\sin C$, in (5), resulting the inequality (6). It is necessary here the extension of Remark 2, in order to be able to include here also the right triangles ($\sin 90°=1$).

7. Consequence: Let $a_1,a_2,\dots,a_n\in(0,1)$. If A_n,G_n,H_n,Q_n are the means arithmetic, geometric, harmonic, respectively quadratic of numbers $a_1,a_2,\dots,a_n$, then occurs the inequality,

$$(A_n+G_n+H_n)^{Q_n}+(G_n+H_n+Q_n)^{A_n}+(H_n+Q_n+A_n)^{G_n}+(Q_n+A_n+G_n)^{H_n}>3;\ (\mathbf{7})$$

Solution. Evidently, $A_n,G_n,H_n,Q_n\in(0,1)$, because they are means. The rest results from the relationship (5), for $n=4$.

8. Application: If $a,b,c\in\left(0,\frac{1}{2}\right)$, then holds the inequality

$$a^{b+c}+b^{c+a}+c^{a+b}>1;\ (\mathbf{8})$$

Solution. We have the obvious $a+b, b+c, c+a\in(0,1)$, so with Lemma, we have:

$$a^{b+c}+b^{c+a}+c^{a+b}>\frac{a}{a+(b+c)}+\frac{b}{b+(c+a)}+\frac{c}{c+(a+b)}=1.$$

9. Application (See [4]): If $a>0, b,c,d\in(0,1)$, then we have inequality

$$a^b(a+b)^c(a+b+c)^d>\frac{a}{a+b+c+d};\ (\mathbf{9})$$

Solution. In the conditions of the statement, the conditions in Lemma are also fulfilled, so, applying three times the inequality (L), we obtain:

$$a^b(a+b)^c(a+b+c)^d>\frac{a}{a+b}\cdot\frac{a+b}{a+b+c}\cdot\frac{a+b+c}{a+b+c+d}=\frac{a}{a+b+c+d}$$

10. Remark. It follows, therefore, that the following inequality also occurs.

$$\sum_{cyc} \boldsymbol{a^b(a+b)^c(a+b+c)^d} > \sum_{cyc} \frac{\boldsymbol{a}}{\boldsymbol{a+b+c+d}} = \mathbf{1}; (\mathbf{10})$$

11. Application (See [6]): If $\boldsymbol{a, b, c} \in (\mathbf{0}, \mathbf{1})$, then prove that

$$\boldsymbol{a^b(a+b)^c + b^c(b+c)^a + c^a(c+a)^b} > 1; \quad (\mathbf{11})$$

Solution. Using (L) three times (2 times each), we obtain:

$$a^b(a+b)^c \overset{Lemma}{>} \frac{a}{a+b} \cdot \frac{a+b}{(a+b)+c} = \frac{a}{a+b+c}$$

$$b^c(b+c) \overset{Lemma}{>} \frac{b}{b+c} \cdot \frac{b+c}{(b+c)+a} = \frac{b}{a+b+c}$$

$$c^a(c+a)^b \overset{Lemma}{>} \frac{c}{c+a} \cdot \frac{c+a}{(c+a)+b} = \frac{c}{a+b+c}$$

By adding there three inequalities, the relation in the statement is obtained.

12. Application (See [9]) MISSOURI JOURNAL OF MATHEMATICAL SCIENCES.

If $\boldsymbol{a, b, c} \in (\mathbf{0}, \mathbf{1})$, prove that

$$\boldsymbol{2^a(b+c)^{1-a} + 2^b(c+a)^{1-b} + 2^c(a+b)^{1-c}} < 4\boldsymbol{(a+b+c)}; (\mathbf{12})$$

Solution. In conditions of enonce we have $\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2} \in (0,1)$ and applying the inequality from Lemma, we obtain:

$$\left(\frac{a+b}{2}\right)^c > \frac{\frac{a+b}{2}}{\frac{a+b}{2}+c} = \frac{a+b}{a+b+2c}, \text{hence } a+b+2c > 2^c(a+b)^{1-c}; \quad (13)$$

Analogously,

$$a+2b+c > 2^b(a+c)^{1-b}; \ (14)$$

$$2a+b+c > 2^a(b+c)^{1-a}; \ \ (15)$$

By adding the inequalities (11)-(13), we obtain

$$4(a+b+c) > 2^a(b+c)^{1-a} + 2^b(c+a)^{1-b} + 2^c(a+b)^{1-c}$$

hence, the inequality from (12).

13. Application (See [8]): If $\boldsymbol{a, b, c} \in (\mathbf{0}, \mathbf{1})$, prove that:

$$\boldsymbol{(a+b)(b+c)(c+a)(a^b+c)(b^c+a)(c^a+b)} > \boldsymbol{a^{1-bc}b^{1-ca}c^{1-ab}}; \quad (\mathbf{16})$$

Solution. Using inequality (L), two times, we obtain:

$$a^{bc} = (a^b)^c > \frac{a^b}{a^b + c} > \frac{\frac{a}{a+b}}{a^b + c} = \frac{a}{(a+b)(a^b + c)}$$

and the analogous:

$$b^{ca} > \frac{b}{(b+c)(b^c + a)} \text{ and } c^{ab} > \frac{c}{(c+a)(c^a + b)}$$

which by multiplying gives:

$$a^{bc} \cdot b^{ca} \cdot c^{ab} > \frac{abc}{(a+b)(b+c)(c+a)(a^b + c)(b^c + a)(c^a + b)}$$

equivalent with the inequality from enonce.

14. Application (See [7]): If $\boldsymbol{a, b, c, d} \in (\mathbf{0}, \mathbf{1})$, prove that:

$$\left(\frac{\boldsymbol{a+b}}{\mathbf{2}}\right)^{\frac{c+d}{2}} + \left(\frac{\boldsymbol{b+c}}{\mathbf{2}}\right)^{\frac{d+a}{2}} + \left(\frac{\boldsymbol{c+d}}{\mathbf{2}}\right)^{\frac{a+b}{2}} + \left(\frac{\boldsymbol{d+a}}{\mathbf{2}}\right)^{\frac{b+c}{2}} > 2; \;\; (17)$$

Solution. In conditions of enonce we have: $\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+d}{2}, \frac{d+a}{2} \in (0,1)$ and applying the inequality from Lemma, on obtain:

$$\left(\frac{a+b}{2}\right)^{\frac{c+d}{2}} > \frac{\frac{a+b}{2}}{\frac{a+b}{2} + \frac{c+d}{2}} = \frac{a+b}{a+b+c+d}; (18)$$

and three other inequalities similar to (16). By summing we obtain:

$$\sum_{cyc} \left(\frac{a+b}{2}\right)^{\frac{c+d}{2}} > \sum_{cyc} \frac{a+b}{a+b+c+d} = 2$$

Obviously, many other exponential inequalities can be conceived and solved using Lemma $(\boldsymbol{L})$.

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ABOUT A FEW SPECIAL LIMITS AND SUMS (III)

By Florică Anastase-Romania

Abstract: *In this paper are presented few special limits with great integer function.*

App.1) Find:

$$\boldsymbol{\Omega = \lim_{p\to\infty}\frac{1}{p}\cdot\sum_{m=0}^{p-1}\sum_{n=1}^{m}\left(\sum_{k=1}^{n}\left[\sqrt{k^2+k+1}+\sqrt{k^2-k+1}\right]\right)^{-1}}$$

,where $[\cdot]$ great integer function.

Solution: Let be the notations:

$$S_1=\sum_{k=1}^{n}\left[\sqrt{k^2+k+1}+\sqrt{k^2-k+1}\right]$$

$$S_2=\sum_{n=1}^{m}\left(\sum_{k=1}^{n}\left[\sqrt{k^2+k+1}+\sqrt{k^2-k+1}\right]\right)^{-1}$$

$$S_3=\sum_{m=0}^{p-1}\sum_{n=1}^{m}\left(\sum_{k=1}^{n}\left[\sqrt{k^2+k+1}+\sqrt{k^2-k+1}\right]\right)^{-1}$$

For $n=1$ we have: $\left[\sqrt{1}+\sqrt{3}\right]=2$ and for $n=2$ we have: $\left[\sqrt{3}+\sqrt{7}\right]=4$

But, $2k\le\sqrt{k^2-k+1}+\sqrt{k^2+k+1}\le 2k+1$, then

$$\left[\sqrt{k^2-k+1}+\sqrt{k^2+k+1}\right]=2k$$

$$S_n=\sum_{k=1}^{n}\left[\sqrt{k^2+k+1}+\sqrt{k^2-k+1}\right]=\sum_{k=1}^{n}2k=n(n+1)$$

$$S_m=\sum_{n=1}^{m}\left(\sum_{k=1}^{n}\left[\sqrt{k^2+k+1}+\sqrt{k^2-k+1}\right]\right)^{-1}=$$

$$=\sum_{n=1}^{m}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\frac{m}{m+1}=1-\frac{1}{m+1}$$

$$S_p=\sum_{m=0}^{p-1}\sum_{n=1}^{m}\left(\sum_{k=1}^{n}\left[\sqrt{k^2+k+1}+\sqrt{k^2-k+1}\right]\right)^{-1}=\sum_{m=0}^{p-1}\left(1-\frac{1}{m+1}\right)=$$

$$=p-\sum_{m=0}^{p}\frac{1}{m+1}=p-\left(1+\frac{1}{2}+\cdots+\frac{1}{p}\right)=p-H_p$$

Therefore,

$$\Omega=\lim_{p\to 0}\frac{p-H_p}{p}=\lim_{p\to\infty}\left(1-\frac{H_p}{p}\right)\overset{LC-S}{=}1$$

App. 2) Find:

$$\boldsymbol{\Omega=\lim_{n\to\infty}\frac{\left[\frac{1}{\sqrt{1-\frac{1}{2}}}+\frac{1}{\sqrt{1-\frac{1}{2^2}}}+\cdots+\frac{1}{\sqrt{1-\frac{1}{2^n}}}\right]^{\alpha}}{\left[\sqrt[3]{1}\right]+\left[\sqrt[3]{2}\right]+\left[\sqrt[3]{3}\right]+\cdots+\left[\sqrt[3]{n^3-1}\right]}}, [*]-\mathbf{GIF}, \boldsymbol{\alpha}\in\mathbb{R}$$

Solution: We have: $1<\frac{1}{\sqrt{1-\frac{1}{2^k}}}<1+\frac{1}{2^k}, k=\overline{1,n}$ and summing, we get:

$$n<\sum_{k=1}^{n}\frac{1}{\sqrt{1-\frac{1}{2^k}}}<n+\sum_{k=1}^{n}\frac{1}{2^k}=n+1-\frac{1}{2^k}<n+1$$

Hence,

$$\left[\sum_{k=1}^{n}\frac{1}{\sqrt{1-\frac{1}{2^k}}}\right]=n;\ (1)$$

Now, we have: $\left[\sqrt[3]{k}\right]=\left[\sqrt[3]{k^3+1}\right]=\cdots=\left[\sqrt[3]{(k+1)^3-1}\right]=k$

$$\sum_{k=1}^{n-1}\left(\sum_{i=0}^{3k(k+1)}\left[\sqrt[3]{k^3+i}\right]\right)=\sum_{k=1}^{n-1}(3k^3+3k^2+k)=\frac{(n-1)n^2(3n+1)}{4};\ (2)$$

From (1) and (2) it follows that:

$$\Omega=\lim_{n\to\infty}\frac{\left[\frac{1}{\sqrt{1-\frac{1}{2}}}+\frac{1}{\sqrt{1-\frac{1}{2^2}}}+\cdots+\frac{1}{\sqrt{1-\frac{1}{2^n}}}\right]^\alpha}{\left[\sqrt[3]{1}\right]+\left[\sqrt[3]{2}\right]+\left[\sqrt[3]{3}\right]+\cdots+\left[\sqrt[3]{n^3-1}\right]}=\lim_{n\to\infty}\frac{4n^\alpha}{n^2(n-1)(3n+1)}=$$

$$=\begin{cases}0,\text{ if }\alpha<4\\ \frac{4}{3},\text{ if }\alpha=4\\ \infty,\text{ if }\alpha>4\end{cases}$$

App. 3) If $\boldsymbol{x_m=\lim_{n\to\infty}\sum_{k=0}^m\left\{\sqrt{n^2+(2k+1)n+k^2+k}\right\},m,n\in\mathbb{N}^*}$, then:

$$\boldsymbol{m\sqrt[m]{m+1}<\sum_{k=1}^m\frac{1}{x_k}<2\sqrt{m}}$$

$$\boldsymbol{\{x\}=x-[x],[\cdot]-GIF.}$$

Solution: $n^2+(2k+1)n+k^2+k=n^2+2kn+k^2+k=(n+k)^2+k>(n+k)^2$

$$n^2+(2k+1)n+k^2+k=n^2+2kn+k^2+k=(n+k)^2+k<(n+k+1)^2$$

Hence,

$$(n+k)^2<n^2+(2k+1)n+k^2+k<(n+k+1)^2$$

$$n+k<\sqrt{n^2+(2k+1)n+k^2+k}<n+k+1\Rightarrow$$

$$\left[\sqrt{n^2+(2k+1)n+k^2+k}\right]=n+k,$$

$$\because\{t\}=t-\{t\},(\forall)t\in\mathbb{R}$$

$$\left\{\sqrt{n^2+(2k+1)n+k^2+k}\right\}=\sqrt{n^2+(2k+1)n+k^2+k}-(n+k)=$$

$$=\frac{n^2+(2k+1)n+k^2+k-(n+k)^2}{\sqrt{n^2+(2k+1)n+k^2+k}+n+k}=\frac{n^2+(2k+1)n+k^2+k-n^2-2kn-k^2}{\sqrt{n^2+(2k+1)n+k^2+k}+n+k}=$$

$$=\frac{n+k}{\sqrt{n^2+(2k+1)n+k^2+k}+n+k}$$

$$x_m=\lim_{n\to\infty}\sum_{k=0}^m\left\{\sqrt{n^2+(2k+1)n+k^2+k}\right\}=\lim_{m\to\infty}\sum_{k=0}^m\frac{n+k}{\sqrt{n^2+(2k+1)n+k^2+k}+n+k}=$$

$$=\frac{1}{2}+\frac{1}{2}+\overset{m-terms}{\cdots}+\frac{1}{2}=\frac{m+1}{2}\in(\sqrt{m},m),m\in\mathbb{N}^*$$

$$x_k\in(\sqrt{k},k),k\in\mathbb{N}^*\Rightarrow\frac{1}{x_k}\in\left(\frac{1}{k},\frac{1}{\sqrt{k}}\right)$$

$$\frac{1}{k} \le \frac{1}{x_k} \le \frac{1}{\sqrt{k}}, k \in \mathbb{N}^*$$

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} \le \sum_{k=1}^{m} \frac{1}{x_k} \le 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{m}}; (1)$$

$$2 + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{m+1}{m} \ge m\sqrt[m]{m+1} \Leftrightarrow m + \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right) \ge m\sqrt[m]{m+1}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} \ge m\left(\sqrt[m]{m+1} - 1\right); (2)$$

$$2\left(\sqrt{k+1} - \sqrt{k}\right) < \frac{1}{\sqrt{k}} < 2\left(\sqrt{k} - \sqrt{k-1}\right) \Rightarrow 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{m}} < 2\sqrt{m}; (3)$$

From (1), (2) and (3), we get: $m\sqrt[m]{m+1} < \sum_{k=1}^{m} \frac{1}{x_k} < 2\sqrt{m}, m \in \mathbb{N}^*$

App. 4) Let $(x_n)_{n\ge1}$ and $(y_n)_{n\ge1}$ be sequences of real numbers such that

$$\boldsymbol{x_n = \sum_{k=3}^{n} \tan\left(\frac{\pi}{k}\right) - \pi \log n, y_n = \sum_{k=1}^{n} 2^{k-1} \cdot \left[\frac{k^2}{k+1}\right], [*] - \mathrm{GIF}.}$$

$$\textbf{Find: } \boldsymbol{\Omega = \lim_{n\to\infty} \frac{2^n \cdot x_n}{y_n}}$$

Solution:

$$x_n = \sum_{k=3}^{n} \tan\left(\frac{\pi}{k}\right) - \pi \log n = \pi\left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) + \sum_{k=3}^{n} \tan\left(\frac{\pi}{k}\right) - \pi \sum_{k=1}^{n} \frac{1}{k} =$$

$$= \pi\gamma_n + \sum_{k=3}^{n} \left(\tan\left(\frac{\pi}{k}\right) - \frac{\pi}{k}\right) - \frac{3\pi}{2} = \pi\gamma_n + a_n - \frac{3\pi}{2}, \text{where}$$

$$a_n = \sum_{k=3}^{n} \left(\tan\left(\frac{\pi}{k}\right) - \frac{\pi}{k}\right) < \sum_{k=3}^{n} \left(\frac{\pi}{k}\right)^3 = \pi^3 \sum_{k=3}^{n} \frac{1}{k^3} < \pi^3 \sum_{k=3}^{n} \frac{1}{k^2} <$$

$$< \pi^3 \sum_{k=3}^{n} \frac{1}{k(k-1)} = \pi^3 \left(\frac{1}{2} - \frac{1}{n}\right) < \frac{\pi^3}{2}; (1)$$

$$a_{n+1} - a_n = \tan\left(\frac{\pi}{n+1}\right) - \frac{\pi}{n+1} > 0; (2)$$

From (1) and (2) it follows that $(a_n)_{n\ge3}$ is convergent, so $x_n = \pi\gamma_n + a_n - \frac{3\pi}{2}$ is convergent; (3).

Now, we have: $k - 1 < \frac{k^2}{k+1} < k \Rightarrow \left[\frac{k^2}{k+1}\right] = k - 1$, then

$$y_n = \sum_{k=1}^{n} 2^{k-1} \cdot \left[\frac{k^2}{k+1}\right] = \sum_{k=1}^{n} 2^{k-1}(k-1) = \frac{1}{2}\sum_{k=1}^{n} k \cdot 2^k - \sum_{k=1}^{n} 2^{k-1} \overset{(*)}{=};$$

$$\left(\because \sum_{k=1}^{n} k \cdot a^k = \frac{n \cdot a^{n+2} - (n+1) \cdot a^{n+1} + a}{(a-1)^2}; (*)\right)$$

$$\overset{(*)}{=} n\cdot 2^{n+1}-(n+1)\cdot 2^n+1-2^n+1=n\cdot 2^n-2^{n+1}+2;(4)$$

From (3) and (4) it follows that:

$$\Omega=\lim_{n\to\infty}\frac{2^n\cdot x_n}{y_n}=\lim_{n\to\infty}\frac{2^n\cdot x_n}{n\cdot 2^n-2^{n+1}+2}=\lim_{n\to\infty}\frac{x_n}{n-2+2^{1-n}}=0$$

App. 5) Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be sequences of real numbers such that

$$\boldsymbol{x_n=\sum_{k=1}^{n}\sin\frac{1}{k}+\log\left(\sin\frac{1}{n}\right),y_n=\sum_{k=1}^{n^2+n}\left[\sqrt{k}+\frac{1}{2}\right],[*]-\text{GIF}.}$$

$$\textbf{Find}:\boldsymbol{\Omega=\lim_{n\to\infty}\frac{x_n}{y_n}}$$

Solution:

$$x_n=\sum_{k=1}^{n}\sin\frac{1}{k}+\log\left(\sin\frac{1}{n}\right)=\sum_{k=1}^{n}\left(\frac{1}{k}+\sin\frac{1}{k}\right)-\sum_{k=1}^{n}\frac{1}{k}-\log n+\log\left(\sin\frac{1}{n}\right)+\log n$$

$$=\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)-\sum_{k=1}^{n}\left(\frac{1}{k}-\sin\frac{1}{k}\right)+\log n\sin\frac{1}{n}=$$

$$=\gamma_n+\log\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right)-\sum_{k=1}^{n}\left(\frac{1}{k}-\sin\frac{1}{k}\right)=\gamma_n+\log\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right)-a_n,\text{where}$$

$$a_n=\sum_{k=1}^{n}\left(\frac{1}{k}-\sin\frac{1}{k}\right)<\sum_{k=1}^{n}\frac{1}{k^3}<\sum_{k=1}^{n}\frac{1}{k^2}<1+\sum_{k=2}^{n}\frac{1}{k(k-1)}=2-\frac{1}{n}<2;(1)$$

$$a_{n+1}-a_n=\frac{1}{n+1}-\sin\left(\frac{1}{n+1}\right)+\sin\left(\frac{1}{n}\right)=$$

$$=\frac{1}{n+1}-2\sin\left(\frac{1}{2n(n+1)}\right)\cos\left(\frac{1}{n+1}\right)\Rightarrow(a_n)_{n\geq 1}\text{ increasing; }(2)$$

From (1),(2) $(a_n)_{n\geq 1}$ are convergent, then $x_n=\gamma_n+\log\left(n\sin\frac{1}{n}\right)-a_n$ are convergent; (3)

Now, we have: $\left[\sqrt{k}+\frac{1}{2}\right]=q, q\in\mathbb{Z}\Leftrightarrow q\leq\frac{1}{2}+\sqrt{k}<q+1\Leftrightarrow$

$(q-1)q+1\leq q(q+1)$, so we can write:

$$y_n=\sum_{k=1}^{n^2+n}\left[\sqrt{k}+\frac{1}{2}\right]=\sum_{q=1}^{n}\sum_{k=(q-1)q+1}^{q(q+1)}\left[\sqrt{k}+\frac{1}{2}\right]=$$

$$=\sum_{q=1}^{n}\sum_{k=(q-1)q+1}^{q(q+1)}q=\sum_{q=1}^{n}[q(q+1)-(q-1)q]q=2\sum_{q=1}^{n}q^2=\frac{n(n+1)(2n+1)}{3};(4)$$

From (3) and (4) it follows that $\Omega=\lim_{n\to\infty}\frac{x_n}{y_n}=0.$

App. 6) Let $(a_n)_{n\geq 1},(b_n)_{n\geq 1}$ be sequences of real numbers such that:

$$\boldsymbol{a_1=2,na_n=b_{n+2}(a_n+b_{n+1}\cdot 2^{n-1}),b_n=\left[\sum_{k=1}^{n^2}\frac{1}{\sqrt{k}}\right],n\geq 1,[\cdot]-GIF.}$$

$$\textbf{Find}: \boldsymbol{\Omega = \lim_{n\to\infty} \sqrt[n^2]{a_1 \cdot a_2 \cdot \ldots \cdot a_n}}$$

Solution: For all $k \geq 2$, we have: $\sqrt{k-1}+\sqrt{k} < 2\sqrt{k} < \sqrt{k}+\sqrt{k+1}$

$$\sqrt{k} < \frac{\sqrt{k+1}+\sqrt{k}}{2}$$

$$2\left(\sqrt{k+1}-\sqrt{k}\right) < \frac{1}{\sqrt{k}} < 2\left(\sqrt{k}-\sqrt{k-1}\right); (1)$$

$$1+2\sum_{k=2}^{n^2}\left(\sqrt{k+1}-\sqrt{k}\right) \leq \sum_{k=1}^{n^2}\frac{1}{\sqrt{k}} \leq 1+2\sum_{k=2}^{n^2}\left(\sqrt{k}-\sqrt{k-1}\right)$$

$$1+2\left(\sqrt{n^2+1}-\sqrt{2}\right) \leq \sum_{k=1}^{n^2}\frac{1}{\sqrt{k}} < 1+2(n-1)$$

$$2n-2 \leq \sum_{k=1}^{n}\frac{1}{\sqrt{k}} < 2n-1 \Rightarrow \left[\sum_{k=1}^{n^2}\frac{1}{\sqrt{k}}\right] = 2n-2 \Rightarrow b_n = 2(n-1)$$

$$na_n = b_{n+2}(a_n + b_{n+1}\cdot 2^{n-1}) \Rightarrow na_n = 2(n+1)(a_n + n\cdot 2^n), n \geq 1 \Leftrightarrow$$

$$\frac{a_{n+1}}{n+1} = 2\left(\frac{a_n}{n}+2^n\right) \Leftrightarrow \frac{a_{n+1}}{(n+1)2^{n+1}} = \frac{a_n}{n\cdot 2^n}+1 \Leftrightarrow \frac{a_n}{n\cdot 2^n} = n \Leftrightarrow a_n = n^2\cdot 2^n$$

So, we get:

$$\sqrt[n^2]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} = 2^{\frac{n+1}{2n}} \cdot \sqrt[n^2]{(n!)^2}$$

From $1 \leq n! \leq n^n$, we obtain $1 \leq \sqrt[n^2]{(n!)^2} \leq \left(\sqrt[n]{n}\right)^2 \to 1$

$$\Rightarrow \lim_{n\to\infty}\sqrt[n^2]{(n!)^2} = 1 \Rightarrow \Omega = \lim_{n\to\infty}\sqrt[n^2]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} = \lim_{n\to\infty} 2^{\frac{n+1}{2n}} \cdot \sqrt[n^2]{(n!)^2} = \sqrt{2}.$$

App. 7) Let $(x_n)_{n\geq 1}$ be sequence of real numbers such that

$$\boldsymbol{x_{n-1} = \left[\sum_{k=1}^{n^3}\frac{1}{\sqrt[3]{k^2}}\right], n \geq 1, [\cdot] - GIF.}\ \textbf{Find}:$$

$$\boldsymbol{\Omega = \lim_{n\to\infty}\sqrt[n]{\sum_{i=0}^{n} x_i \binom{n}{i}^2}}$$

Solution: Using the double inequality:

$$\frac{1}{3\sqrt[3]{(k+1)^2}} < \sqrt[3]{k+1}-\sqrt[3]{k} < \frac{1}{3\sqrt[3]{k^2}}$$

$$\sum_{k=1}^{n^3-1}\frac{1}{\sqrt[3]{(k+1)^2}} < 3\left(\sum_{k=1}^{n^3-1}\left(\sqrt[3]{k+1}-\sqrt[3]{k}\right)\right) < \sum_{k=1}^{n^3-1}\frac{1}{\sqrt[3]{k^2}}$$

$$\sum_{k=1}^{n^3}\frac{1}{\sqrt[3]{k^2}} - 1 < 3(n-1) < \sum_{k=1}^{n^3}\frac{1}{\sqrt[3]{k^2}} - \frac{1}{n\sqrt[3]{n}}$$

$$3n-3 < 3n-3+\frac{1}{n\sqrt[3]{n}} < \sum_{k=1}^{n^3}\frac{1}{\sqrt[3]{k^2}} < 3n-2$$

Hence,

$$x_{n-1} = \left[\sum_{k=1}^{n^3}\frac{1}{\sqrt[3]{k^2}}\right] = 3n-3 \Rightarrow x_n = 3n, n \geq 0$$

$(x_n)_{n\geq 0}$ –is a arithmetic progression with ratio $r = 3$

$$S_n = \sum_{i=0}^{n} x_i\binom{n}{i}^2 = x_0\binom{n}{0}^2 + x_1\binom{n}{1}^2 + \cdots + x_n\binom{n}{n}^2$$

$$2S_n = (x_0+x_n)\binom{n}{0}^2 + (x_1+x_{n-1})\binom{n}{1}^2 + \cdots + (x_n+x_0)\binom{n}{n}^2 =$$

$$= (2x_0+nr)\left(\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2\right) =$$

$$= 3n\left(\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2\right) = 3n\binom{2n}{n}$$

Because

$$\lim_{n\to\infty}\frac{S_{n+1}}{S_n} = \lim_{n\to\infty}\frac{3(n+1)}{3n}\cdot\frac{2(2n+1)}{n+1} = 4 \Rightarrow \Omega = \lim_{n\to\infty}\sqrt[n]{\sum_{i=0}^{n} x_i\binom{n}{i}^2} = 4$$

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FAMOUS INEQUALITIES REDESIGNED IN THE TRIANGLE

WITH SIDES SUM BY SIDES

By Daniel Sitaru-Romania

Abstract: *In this paper its considered an initial triangle with sides a, b, c. We take the triangle with sides $a' = b+c, b' = c+a, c' = a+b$ and we find the form of a few famous inequalities in triangle obtained in this new triangle.*

Notations:

a, b, c, s, R, r, F –sides, semiperimeter, circumradii, inradii and area of the initial triangle.

$a', b', c', s', R', r', F'$ —sides, semiperimeter, circumradii, inradii and area of the new triangle.

r_a, r_b, r_c —exradii in the initial triangle.

r'_a, r'_b, r'_c —exradii in the new triangle.

h_a, h_b, h_c —altitudes in the initial triangle.

h'_a, h'_b, h'_c —altitudes in the new triangle.

A, b, c —angles in the initial triangle.

A', B', C' —angles in the new triangle.

Preliminaries:

$$s' = \frac{1}{2}(a' + b' + c') = \frac{1}{2}(2a + 2b + 2c) = a + b + c = 2s$$

$$F' = \sqrt{s')s' - a')(s' - b')(s' - c')} = \sqrt{(a + b + c)abc} =$$

$$= \sqrt{2s \cdot 4RF} = \sqrt{2s \cdot 4Rrs} = 2s\sqrt{2Rr}$$

$$R' = \frac{a'b'c'}{4F'} = \frac{(a + b)(b + c)(c + a)}{4 \cdot 2s\sqrt{2Rr}} = \frac{(a + b)(b + c)(c + a)}{8s\sqrt{2Rr}} =$$

$$= \frac{2s(s^2 + r^2 + 2Rr)}{4 \cdot 2s\sqrt{2Rr}} = \frac{s^2 + r^2 + 2Rr}{4\sqrt{2Rr}}$$

$$r' = \frac{F'}{s'} = \frac{2s\sqrt{2Rr}}{2s} = \sqrt{2Rr}$$

$$r'_a = \frac{F'}{s' - a'} = \frac{\sqrt{(a + b + c)abc}}{a + b + c - b - c} = \sqrt{\frac{(a + b + c)abc}{a^2}} = \sqrt{\frac{(a + b + c)bc}{a}}$$

$$r'_b = \frac{F'}{s' - b'} = \sqrt{\frac{(a + b + c)ca}{b}} \text{ and } r'_c = \frac{F'}{s' - c'} = \sqrt{\frac{(a + b + c)ab}{c}}$$

$$r'_a + r'_b + r'_c = \frac{F'}{s' - a'} + \frac{F'}{s' - b'} + \frac{F'}{s' - c'} = F'\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) =$$

$$=F'\cdot\frac{ab+bc+ca}{abc}=F'\cdot\frac{ab+bc+ca}{4R'F'}=\frac{ab+bc+ca}{4R'}=$$

$$=\frac{ab+bc+ca}{4\cdot\frac{s^2+r^2+2Rr}{4\sqrt{2Rr}}}=\frac{(ab+bc+ca)\sqrt{2Rr}}{s^2+r^2+2Rr}$$

$$\frac{1}{r_a'}+\frac{1}{r_b'}+\frac{1}{r_c'}=\frac{s'-a'}{F'}+\frac{s'-b'}{F'}+\frac{s'-c'}{F'}=\frac{3s'-2s'}{F'}=\frac{s'}{F'}=\frac{1}{r'}=\frac{1}{\sqrt{2Rr}}$$

$$\sum_{cyc}(m_a')^2=\frac{3}{4}\sum_{cyc}(a')^2=\frac{3}{4}\sum_{cyc}(b+c)^2=\frac{3}{4}\sum_{cyc}(b^2+c^2+2bc)=$$

$$\frac{3}{4}\cdot 2\sum_{cyc}a^2+\frac{3}{2}\sum_{cyc}bc=\frac{3}{2}\cdot\frac{4}{3}\sum_{cyc}m_a^2+\frac{3}{2}\sum_{cyc}bc=\sum_{cyc}m_a^2+\sum_{cyc}m_a^2+\frac{3}{2}\sum_{cyc}bc=$$

$$=\sum_{cyc}m_a^2+\frac{3}{4}\cdot 2(s^2-r^2-4Rr)+\frac{3}{2}(s^2+r^2+4Rr)=$$

$$=\sum_{cyc}m_a^2+\frac{3}{2}(s^2-r^2-4Rr+s^2+r^2+4Rr)=4s^2+\sum_{cyc}m_a^2$$

$$\sum_{cyc}(m_a')^2=3s^2+\sum_{cyc}m_a^2=\frac{3}{4}(a+b+c)^2+\frac{3}{4}(a^2+b^2+c^2)$$

$$h_a'=\frac{2F'}{b+c}=\frac{2\sqrt{(a+b+c)abc}}{b+c}$$

$$h_b'=\frac{2F'}{c+A}=\frac{2\sqrt{(a+b+c)abc}}{c+a}$$

$$h_c'=\frac{2F'}{a+b}=\frac{2\sqrt{(a+b+c)abc}}{a+b}$$

$$\sin A'=\frac{a'}{2R'}=\frac{b+c}{2\cdot\frac{(a+b)(b+c)(c+a)}{8s\sqrt{2Rr}}}=\frac{4s\sqrt{2Rr}}{(a+b)(a+c)}$$

$$\sin B'=\frac{4s\sqrt{2Rr}}{(b+c)(b+a)}\text{ and }\sin C'=\frac{4s\sqrt{2Rr}}{(c+a)(c+b)}$$

$$\cos A' = \frac{(b')^2+(c')^2-(a')^2}{2b'c'} = \frac{(a+c)^2+(a+b)^2-(b+c)^2}{2(a+b)(a+c)} =$$

$$= \frac{a^2+ac+ab-bc}{(a+b)(a+c)} = \frac{2sa-bc}{(a+b)(a+c)}$$

$$\cos B' = \frac{2sb-ca}{(b+a)(b+c)} \text{ and } \cos C' = \frac{2sc-ab}{(c+a)(c+b)}$$

$$\sin\frac{A'}{2} = \sqrt{\frac{(s'b')(s'-c')}{b'c'}} = \sqrt{\frac{bc}{(a+b)(a+c)}}$$

$$\sin\frac{B'}{2} = \sqrt{\frac{ca}{(b+c)(b+a)}} \text{ and } \sin\frac{C'}{2} = \sqrt{\frac{ab}{(c+a)(c+b)}}$$

$$\cos\frac{A'}{2} = \sqrt{\frac{s'(s'-a')}{b'c'}} = \sqrt{\frac{(a+b+c)a}{(a+b)(a+c)}}$$

$$\cos\frac{B'}{2} = \sqrt{\frac{(a+b+c)b}{(b+a)(c+a)}} \text{ and } \cos\frac{C'}{2} = \sqrt{\frac{(a+b+c)c}{(c+a)(c+b)}}$$

1. MITRINOVIC'S INEQUALITY: In $\Delta A'B'C'$ the following inequality holds:

$$\mathbf{3\sqrt{3}r' \le s' \le \frac{3\sqrt{3}}{2}R'}$$

Redesigned: $s' \ge 3\sqrt{3}r' \Leftrightarrow a+b+c \ge 3\sqrt{3}\cdot\sqrt{2Rr} \Leftrightarrow a+b+c \ge 3\sqrt{6Rr}$

$$s' \le \frac{3\sqrt{3}}{2}R' \Leftrightarrow a+b+c \le \frac{3\sqrt{3}}{2}\cdot\frac{(a+b)(b+c)(c+a)}{8s\sqrt{2Rr}} = \frac{3\sqrt{6}(a+b)(b+c)(c+a)}{16s\sqrt{Rr}}$$

$$\boldsymbol{a+b+c \le \frac{3\sqrt{6}(a+b)(b+c)(c+a)}{16s\sqrt{Rr}}}$$

2. IONESCU-WEITZENBOCK'S INEQUALITY: In $\Delta A'B'C'$ the following relationship holds:

$$\boldsymbol{(a')^2+(b')^2+(c')^2 \ge 4\sqrt{3}F'}$$

Redesigned: $(b+c)^2+(c+a)^2+(a+b)^2 \ge 4\sqrt{3}\cdot\sqrt{(a+b+c)abc}$

$$(\boldsymbol{b}+\boldsymbol{c})^2+(\boldsymbol{c}+\boldsymbol{a})^2+(\boldsymbol{a}+\boldsymbol{b})^2 \geq \mathbf{4}\sqrt{\mathbf{3}\boldsymbol{abc}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})}$$

3. ZETEL'S INEQUALITY: In $\Delta A'B'C'$ the following relationship holds:

$$\boldsymbol{h'_a} \cdot \boldsymbol{h'_b} \cdot \boldsymbol{h'_c} \geq \mathbf{27}(\boldsymbol{r'})^3$$

Redesigned: In ΔABC holds:

$$\frac{8(a+b+c)abc\sqrt{(a+b+c)abc}}{(a+b)(b+c)(c+a)} \geq 27 \cdot 2Rr \cdot \sqrt{2Rr}$$

$$\frac{(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})\boldsymbol{abc}\sqrt{(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})\boldsymbol{abc}}}{(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{b}+\boldsymbol{c})(\boldsymbol{c}+\boldsymbol{a})} \geq \frac{\mathbf{27}\boldsymbol{Rr}\sqrt{\mathbf{2}\boldsymbol{Rr}}}{\mathbf{4}}$$

4. SANTALO'S INEQUALITY: In $\Delta A'B'C'$ the following relationship holds:

$$(\boldsymbol{m'_a})^2+(\boldsymbol{m'_b})^2+(\boldsymbol{m'_c})^2 \geq \mathbf{3}\sqrt{\mathbf{3}}\boldsymbol{F'}$$

Redesigned: In ΔABC holds:

$$\sum_{cyc}(m'_a)^2=\frac{3}{4}(a+b+c)^2+\frac{3}{4}(a^2+b^2+c^2)$$

$$\frac{3}{4}(a+b+c)^2+\frac{3}{4}(a^2+b^2+c^2) \geq 3\sqrt{3} \cdot \sqrt{(a+b+c)abc}$$

$$(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})^2+\boldsymbol{a}^2+\boldsymbol{b}^2+\boldsymbol{c}^2 \geq \mathbf{4}\sqrt{\mathbf{3}\boldsymbol{abc}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})}$$

5. GORDON'S INEQUALITY: In $\Delta A'B'C'$ the following relationship holds:

$$\boldsymbol{a'b'}+\boldsymbol{b'c'}+\boldsymbol{c'a'} \geq \mathbf{4}\sqrt{\mathbf{3}}\boldsymbol{F'}$$

Redesigned: In ΔABC holds:

$$(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{b}+\boldsymbol{c})+(\boldsymbol{b}+\boldsymbol{c})(\boldsymbol{c}+\boldsymbol{a})+(\boldsymbol{c}+\boldsymbol{a})(\boldsymbol{c}+\boldsymbol{b}) \geq \mathbf{4}\sqrt{\mathbf{3}\boldsymbol{abc}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})}$$

6. GOLDNER'S INEQUALITY-I: In $\Delta A'B'C'$ the following relationship holds:

$$(\boldsymbol{a'})^4+(\boldsymbol{b'})^4+(\boldsymbol{c'})^4 \geq \mathbf{16}(\boldsymbol{F'})^2$$

Redesigned: In ΔABC holds:

$$\boldsymbol{(a+b)^4+(b+c)^4+(c+a)^4 \geq 16abc(a+b+c)}$$

7. GOLDNER'S INEQUALITY-II: In $\boldsymbol{\Delta A'B'C'}$ the following relationship holds:

$$\boldsymbol{(a')^2(b')^2+(b')^2(c')^2+(c')^2(a')^2 \geq 16(F')^2}$$

Redesigned: In ΔABC holds:

$$\boldsymbol{(a+b)^2(b+c)^2+(b+c)^2(c+a)^2+(c+a)^2(a+b)^2 \geq 16abc(a+b+c)}$$

8. BĂNDILĂ'S INEQUALITY: In $\boldsymbol{\Delta A'B'C'}$ the following relationship holds:

$$\boldsymbol{\max\left\{\frac{a'}{b'}+\frac{b'}{a'},\frac{b'}{c'}+\frac{c'}{b'},\frac{c'}{a'}+\frac{a'}{c'}\right\} \leq \frac{R'}{r'}}$$

Redesigned: In ΔABC holds:

$$\max\left\{\frac{a+b}{a+c}+\frac{a+c}{a+b},\frac{b+c}{b+a}+\frac{b+a}{b+c},\frac{c+a}{c+b}+\frac{c+b}{c+a}\right\} \leq \frac{\frac{(a+b)(b+c)(c+a)}{8s\sqrt{2Rr}}}{\sqrt{2Rr}}=$$

$$=\frac{(a+b)(b+c)(c+a)}{16Rrs}=\frac{(a+b)(b+c)(c+a)}{16RF}$$

$$\boldsymbol{\max\left\{\frac{a+b}{a+c}+\frac{a+c}{a+b},\frac{b+c}{b+a}+\frac{b+a}{b+c},\frac{c+a}{c+b}+\frac{c+b}{c+a}\right\} \leq \frac{(a+b)(b+c)(c+a)}{16RF}}$$

9.LEIBNIZ'S INEQUALITY: In $\boldsymbol{\Delta A'B'C'}$ the following relationship holds:

$$\boldsymbol{(a')^2+(b')^2+(c')^2 \leq 9(R')^2}$$

Redesigned: In ΔABC holds:

$$(a+b)^2+(b+c)^2+(c+a)^2 \leq 9\left(\frac{(a+b)(b+c)(c+a)}{8s\sqrt{2Rr}}\right)^2=$$

$$=\frac{9(a+b)^2(b+c)^2(c+a)^2}{128s^2Rr}=\frac{9(a+b)^2(b+c)^2(c+a)^2}{128RFs}$$

$$(a+b)^2+(b+c)^2+(c+a)^2 \leq \frac{9(a+b)^2(b+c)^2(c+a)^2}{128RFs}$$

10. EULER'S INEQUALITY: In $\Delta A'B'C'$ the following relationship holds:

$$R' \geq 2r'$$

Redesigned: In ΔABC holds:

$$\frac{(a+b)(b+c)(c+a)}{8s\sqrt{2Rr}} \geq 2\sqrt{2Rr} \Leftrightarrow (a+b)(b+c)(c+a) \geq 32sRr$$

$$(a+b)(b+c)(c+a) \geq 32RF$$

11. STEINIG'SINEQUALITY: In $\Delta A'B'C'$ the following relationship holds:

$$\frac{1}{a'}+\frac{1}{b'}+\frac{1}{c'} \leq \frac{\sqrt{3}}{2r'}$$

Redesigned: In ΔABC holds:

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \leq \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2Rr}} = \frac{1}{2}\sqrt{\frac{3}{2Rr}} = \frac{1}{4}\sqrt{\frac{6}{Rr}}$$

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \leq \frac{1}{4}\sqrt{\frac{6}{Rr}}$$

12. LEUENBERGER'S INEQUALITY: In $\Delta A'B'C'$ the following relationship holds:

$$\frac{1}{(R')^2} \leq \frac{1}{a'b'}+\frac{1}{b'c'}+\frac{1}{c'a'} \leq \frac{1}{4(r')^2}$$

Redesigned: In ΔABC holds:

$$\frac{128s^2Rr}{(a+b)^2(b+c)^2(c+a)^2} \leq \frac{1}{(a+b)^2(a+c)^2}+\frac{1}{(b+c)^2(b+a)^2}+\frac{1}{(c+a)^2(c+b)^2} \leq \frac{1}{4\cdot 2Rr}$$

$$\frac{128RFs}{(a+b)^2(b+c)^2(c+a)^2} \leq \frac{1}{(a+b)^2(a+c)^2}+\frac{1}{(b+c)^2(b+a)^2}+\frac{1}{(c+a)^2(c+b)^2} \leq \frac{1}{8Rr}$$

13. KLAMKIN'S INEQUALITY-I: In ΔABC the following relationship holds:

$$\mathbf{4(r')^2 \le \frac{a'b'c'}{a'+b'+c'} \le (R')^2}$$

Redesigned: In ΔABC holds:

$$4\cdot 2Rr \le \frac{(a+b)(b+c)(c+a)}{a+b+b+c+c+a} \le \frac{(a+b)^2(b+c)^2(c+a)^2}{128Rrs^2}$$

$$8Rr \le \frac{(a+b)(b+c)(c+a)}{2(a+b+c)} \le \frac{(a+b)^2(b+c)^2(c+a)^2}{128RFs}$$

$$\mathbf{16Rr \le \frac{(a+b)(b+c)(c+a)}{a+b+c} \le \frac{(a+b)^2(b+c)^2(c+a)^2}{64RFs}}$$

14. KLAMKIN'S INEQUALITY-II: In $\Delta A'B'C'$ the following relationship holds:

$$\mathbf{9r' \le r'_a + r'_b + r'_c \le \frac{9R'}{2}}$$

Redesigned: In ΔABC holds:

$$9\sqrt{2Rr} \le \frac{(ab+bc+ca)\sqrt{2Rr}}{s^2+r^2+2Rr} \le \frac{9(a+b)(b+c)(c+a)}{16s\sqrt{2Rr}}$$

$$\mathbf{9 \le \frac{ab+bc+ca}{s^2+r^2+2Rr} \le \frac{9(a+b)(b+c)(c+a)}{32RF}}$$

15. BOKOV'S INEQUALITY: In $\Delta A'B'C'$ the following relationship holds:

$$\mathbf{(r'_a)^2 + (r'_b)^2 + (r'_c)^2 \ge (s')^2}$$

Redesigned: In ΔABC holds:

$$\frac{(F')^2}{(s'-a')^2} + \frac{(F')^2}{(s'-b')^2} + \frac{(F')^2}{(s'-c')^2} \ge (a+b+c)^2$$

$$(a+B+c)abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \ge (a+b+c)^2$$

$$abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \ge a+b+c$$

$$\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\geq a+b+c$$

16. JANIC'S INEQUALITY: In $\Delta A'B'C'$ the following relationship holds:

$$\frac{(a')^2}{r'_br'_c}+\frac{(b')^2}{r'_cr'_a}+\frac{(c')^2}{r'_ar'_b}\geq 4$$

Redesigned: In ΔABC holds:

$$\frac{(b+c)^2}{\frac{F'}{s'-a'}\cdot\frac{F'}{s'-c'}}+\frac{(c+a)^2}{\frac{F'}{s'-c'}\cdot\frac{F'}{s'-a'}}+\frac{(a+b)^2}{\frac{F'}{s'-a'}\cdot\frac{F'}{s'-b'}}\geq 4$$

$$\frac{(b+c)^2}{\frac{(a+b+c)abc}{bc}}+\frac{(c+a)^2}{\frac{(a+b+c)abc}{ca}}+\frac{(a+b)^2}{\frac{(a+b+c)abc}{ab}}\geq 4$$

$$\frac{(b+c)^2}{a}+\frac{(c+a)^2}{b}+\frac{(a+b)^2}{c}\geq 4(a+b+c)$$

17. TRIGONOMETRIC INEQUALITY-I: In $\Delta A'B'C'$ the following relationship holds:

$$\sin\frac{A'}{2}+\sin\frac{B'}{2}+\sin\frac{C'}{2}\leq\frac{3}{2}$$

Redesigned: In ΔABC holds:

$$\sqrt{\frac{bc}{(a+b)(a+c)}}+\sqrt{\frac{ca}{(b+c)(b+a)}}+\sqrt{\frac{ab}{(c+a)(c+b)}}\leq\frac{3}{2}$$

18. TRIGONOMETRIC INEQUALITY-II: In $\Delta A'B'C'$ the following relationship holds:

$$\cos\frac{A'}{2}+\cos\frac{B'}{2}+\cos\frac{C'}{2}\leq\frac{3\sqrt{3}}{2}$$

Redesigned: In ΔABC holds:

$$\sqrt{\frac{(a+b+c)a}{(a+b)(a+c)}}+\sqrt{\frac{(a+b+c)b}{(b+a)(b+c)}}+\sqrt{\frac{(a+b+c)c}{(c+a)(c+b)}}\leq\frac{3\sqrt{3}}{2\sqrt{a+b+c}}$$

19. TRIGONOMETRIC INEQUALITY-III: In $\Delta A'B'C'$ the following relationship holds:

$$\sin\frac{A'}{2}\sin\frac{B'}{2}+\sin\frac{B'}{2}\sin\frac{C'}{2}+\sin\frac{C'}{2}\sin\frac{A'}{2}\le\frac{3}{4}$$

Redesigned: In ΔABC holds:

$$\sqrt{\frac{bc}{(a+b)(a+c)}\cdot\frac{ca}{(b+a)(b+c)}}+\sqrt{\frac{ca}{(b+a)(b+c)}\cdot\frac{ab}{(c+a)(c+b)}}+$$

$$+\sqrt{\frac{ab}{(c+a)(c+b)}\cdot\frac{bc}{(a+b)(a+c)}}\le\frac{3}{4}$$

$$\frac{c}{a+b}\sqrt{\frac{abc}{(a+c)(b+c)}}+\frac{a}{b+c}\sqrt{\frac{abc}{(b+a)(c+a)}}+\frac{b}{c+a}\sqrt{\frac{abc}{(c+b)(a+b)}}\le\frac{3}{4}$$

$$\frac{c}{(a+b)\sqrt{(a+c)(b+c)}}+\frac{a}{(b+c)\sqrt{(b+a)(c+a)}}+\frac{b}{(c+a)\sqrt{(c+b)(c+a)}}\le\frac{3}{4\sqrt{abc}}$$

20. TRIGONOMETRIC INEQUALITY-IV: In $\Delta A'B'C'$ the following relationship holds:

$$\cos\frac{A'}{2}\cos\frac{B'}{2}+\cos\frac{B'}{2}\cos\frac{C'}{2}+\cos\frac{C'}{2}\cos\frac{A'}{2}\le\frac{9}{4}$$

Redesigned: In ΔABC holds:

$$\sqrt{\frac{(a+b+c)c}{(a+b)(a+c)}}\cdot\sqrt{\frac{(a+b+c)b}{(b+c)(b+a)}}+\sqrt{\frac{(a+b+c)b}{(b+a)(b+c)}}\cdot\sqrt{\frac{(a+b+c)c}{(c+a)(c+b)}}+$$

$$+\sqrt{\frac{(a+b+c)c}{(c+a)(c+b)}}\cdot\sqrt{\frac{(a+b+c)a}{(a+b)(a+c)}}\le\frac{9}{4}$$

$$\frac{1}{a+b}\sqrt{\frac{ab}{(c+a)(c+b)}}+\frac{1}{b+c}\sqrt{\frac{bc}{(a+b)(a+c)}}+\frac{1}{c+a}\sqrt{\frac{ca}{(b+c)(b+a)}}\le\frac{9}{4(a+b+c)}$$

21. TRIGONOMETRIC INEQUALITY-V: In $\Delta A'B'C'$ the following relationship holds:

$$\sec^2\frac{A'}{2}+\sec^2\frac{B'}{2}+\sec^2\frac{C'}{2}\ge 4$$

Redesigned: In ΔABC holds:

$$\frac{(a+b)(a+c)}{a(a+b+c)}+\frac{(b+a)(b+c)}{b(a+b+c)}+\frac{(c+a)(c+b)}{c(a+b+c)}\ge 4$$

$$\left(1+\frac{b}{a}\right)\left(1+\frac{c}{a}\right)+\left(1+\frac{a}{b}\right)\left(1+\frac{c}{b}\right)+\left(1+\frac{a}{c}\right)\left(1+\frac{b}{c}\right)\ge 4(a+b+c)$$

22. TRIGONOMETRIC INEQUALITY-VI: In $\Delta A'B'C'$ the following relationship holds:

$$\csc^2\frac{A'}{2}+\csc^2\frac{B'}{2}+\csc^2\frac{C'}{2}\ge 12$$

Redesigned: In ΔABC the following relationship holds:

$$\frac{(a+b)(a+c)}{bc}+\frac{(b+c)(b+a)}{ca}+\frac{(c+a)(c+b)}{ab}\ge 12$$

$$a(a+b)(a+c)+b(b+c)(b+a)+c(c+a)(c+b)\ge 12abc$$

REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT AN INEQUALITY BY JOSE LUIS DIAZ BARRERO-I

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\ge \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_ah_bh_c}}$$

Proposed by Jose Luis Diaz-Barrero - Spain

Proof: *We have*

$$2)\sum\frac{a^2+bc}{b+c}=\frac{\sum(a^2+bc)(a+b)(a+b)}{\prod(b+c)}=\frac{5s^4-10s^2r^2+r^2(4R+r)^2}{2s(s^2+r^2+2Rr)}$$

which follows from

$$\sum(a^2+bc)\,(a+b)(a+b)=5s^4-10s^2r^2+r^2(4R+r)^2 \text{ and}$$

$$\prod(b+c)=2s(s^2+r^2+2Rr)$$

Let's get back to the main problem.

Using the Lemma and the known inequalities in triangle $abc=4Rrs$ *and* $\prod h_a=\frac{2s^2r^2}{R}$

We prove the stronger inequality

3) In ΔABC the following inequality holds:

$$\boldsymbol{\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\geq a+b+c}$$

Solution: $\frac{5s^4-10s^2r^2+r^2(4R+r)^2}{2s(s^2+r^2+2Rr)}\geq 2s \Leftrightarrow s^4-s^2(8Rr+14r^2)+r^2(4R+r)^2\geq 0 \Leftrightarrow$

$$\Leftrightarrow s^2(s^2-8Rr-14r^2)+r^2(4R+r)^2\geq 0$$

We distinguish the following cases:

Case 1). If $(s^2-8Rr-14r^2)\geq 0$, *the inequality is obvious.*

Case 2). If $(s^2-8Rr-14r^2)<0$, *we rewrite the inequality:*

$r^2(4R+r)^2\geq s^2(8Rr+14r^2-s^2)$, *which follows from Gerretsen's inequality*

$$16Rr-5r^2\leq s^2\leq \frac{R(4R+r)^2}{2(2R-r)}\leq 4R^2+4Rr+3r^2.$$

It remains to prove that:

$$r^2(4R+r)^2\geq \frac{R(4R+r)^2}{2(2R-r)}(8Rr+14r^2-16Rr+5r^2)\Leftrightarrow$$

$$\Leftrightarrow 2r^2(2R-r)\leq R(-8Rr+19r^2)\Leftrightarrow 8R^2-15Rr-2r^2\geq 0\Leftrightarrow (R-2r)(8R+r)\geq 0,$$
obviously from Euler's inequality $R\geq 2r$.

4) In ΔABC the following inequality holds:

$$\boldsymbol{a+b+c\geq \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_ah_bh_c}}}$$

Solution *We prove that:*

$$a+b+c\geq \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_ah_bh_c}}\Leftrightarrow 2s\geq \frac{3\cdot 4Rrs}{2R}\sqrt[3]{\frac{R}{2s^2r^2}}\Leftrightarrow 1\geq 3r\cdot\sqrt[3]{\frac{R}{2s^2r^2}}\Leftrightarrow$$

$\Leftrightarrow 2s^2 \geq 27Rr$, *which follows from Gerretsen's inequality* $s^2 \geq 16Rr - 5r^2$. *It remains to prove that:*

$$2(16Rr - 5r^2) \geq 27Rr \Leftrightarrow R \geq 2r. \text{ (Euler's inequality).}$$

Equality holds if and only if the triangle is equilateral.

Remark. *We can write:*

5) In ΔABC the following relationship holds:

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b} \geq a+b+c \geq \frac{3abc}{2R}\sqrt[3]{\frac{1}{h_a h_b h_c}}$$

Solution *See 3) and 4).*

Equality holds if and only if the triangle is equilateral.

Remark. *Let's find an inequality having an opposite sense:*

6) In ΔABC the following relationship holds:

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b} \leq (a+b+c)\frac{R}{2r}$$

Proposed by Marin Chirciu – Romania

Solution *Using the Lemma, the inequality can be written:*

$$\frac{5s^4-10s^2r^2+r^2(4R+r)^2}{2s(s^2+r^2+2Rr)} \leq s\cdot\frac{R}{r} \Leftrightarrow s^2[s^2(2R-5r)+2r(2R^2+Rr+5r^2)] \geq r^3(4R+r)^2.$$

We distinguish the following cases:

Case 1). If $(2R-5r) \geq 0$, *we use Gerretsen's inequality* $s^2 \geq 16Rr - 5r^2 \geq \frac{r(4R+r)^2}{R+r}$. *It remains to prove that:*

$$\frac{r(4R+r)^2}{R+r}[(16Rr-5r^2)(2R-5r)+2r(2R^2+Rr+5r^2)] \geq r^3(4R+r)^2 \Leftrightarrow$$

$$\Leftrightarrow 36R^2 - 89Rr + 34r^2 \geq 0, \text{ obviously, because in this case } 2R \geq 5r.$$

Case 2). If $(2R-5r) < 0$, *we rewrite the inequality:*

$s^2[2r(2R^2+Rr+5r^2) - s^2(5r-2R)] \geq r^3(4R+r)^2$, *and we use Gerretsen's inequality*

$$\frac{r(4R+r)^2}{R+r} \leq 16Rr - 5r^2 \leq s^2 \leq \frac{R(4R+r)^2}{2(2R-r)} \leq 4R^2+4Rr+3r^2.$$

It remains to prove that:

$$\frac{r(4R+r)^2}{R+r}[2r(2R^2+Rr+5r^2)-(4R^2+4Rr+3r^2)(5r-2R)] \geq r^3(4R+r)^2 \Leftrightarrow$$

$\Leftrightarrow 8R^3 - 8R^2r - 13Rr^2 - 6r^3 \geq 0 \Leftrightarrow (R - 2r)(8R^2 + 8Rr + 3r^2) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

Remark. *We can write:*

7) In ΔABC the following relationship holds:

$$a + b + c \leq \frac{a^2 + bc}{b + c} + \frac{b^2 + ca}{c + a} + \frac{c^2 + ab}{a + b} \leq (a + b + c)\frac{R}{2r}$$

Proposed by Marin Chirciu – Romania

Solution *See 3) and 6). Equality holds if and only if the triangle is equilateral.*

8) In ΔABC the following relationship holds:

$$a + b + c \leq \frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \leq (a + b + c)\frac{R}{2r}.$$

Proposed by Marin Chirciu – Romania

Solution *We prove the following lemma:*

Lemma: In ΔABC the following relationship holds:

$$\frac{a^2 + bc}{b + c} + \frac{b^2 + ca}{c + a} + \frac{c^2 + ab}{a + b} = \frac{5s^4 - 10s^2r^2 + r^2(4R + r)^2}{2s(s^2 + r^2 + 2Rr)}$$

Proof:

We have $\sum \frac{a^2+bc}{b+c} = \frac{\sum(a^2+bc)(a+b)(a+b)}{\prod(b+c)} = \frac{5s^2-10s^2r^2+r^2(4R+r)^2}{2s(s^2+r^2+2Rr)}$, *which follows from*

$\sum(a^2 + bc)\,(a + b)(a + b) = 5s^4 - 10s^2r^2 + r^2(4R + r)^2$ *and*

$\prod(b + c) = 2s(s^2 + r^2 + 2Rr)$. *Let's get to the main problem.*

LHS inequality. Using the Lemma the inequality can be written:

$$\frac{5s^4 - 10s^2r^2 + r^2(4R + r)^2}{2s(s^2 + r^2 + 2Rr)} \geq 2s \Leftrightarrow s^4 - s^2(8Rr + 14r^2) + r^2(4R + r)^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow s^2(s^2 - 8Rr - 14r^2) + r^2(4R + r)^2 \geq 0$. *We distinguish the following cases:*

Case 1). If $(s^2 - 8Rr - 14r^2) \geq 0$, *the inequality is obvious.*

Case 2). If $(s^2 - 8Rr - 14r^2) < 0$, *the inequality rewrites itself:*

$r^2(4R + R)^2 \geq s^2(8R + 14r^2 - s^2)$, *which follows from Gerretsen's inequality:*

$$16Rr - 5r^2 \leq s^2 \leq \frac{R(4R + r)^2}{2(2R - r)} \leq 4R^2 + 4Rr + 3r^2$$

It remains to prove that:

$$r^2(4R+r)^2 \geq \frac{R(4R+r)^2}{2(2R-r)}(8Rr+14r^2-16Rr+5r^2) \Leftrightarrow 2r^2(2R-r) \leq R(-8Rr+19r^2) \Leftrightarrow$$

$\Leftrightarrow 8R^2-15Rr-2r^2 \geq 0 \Leftrightarrow (R-2r)(8R+r) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$. *Equality holds if and only if the triangle is equilateral.*

RHS inequality. Using the Lemma, the inequality rewrites itself:

$$\frac{5s^4-10s^2r^2+r^2(4R+r)^2}{2s(s^2+r^2+2Rr)} \leq s \cdot \frac{R}{r} \Leftrightarrow s^2[s^2(2R-5r)+2r(2R^2+Rr+5r^2)] \geq r^3(4R+r)^2.$$

We distinguish the following cases:

Case 1). If $(2R-5r) \geq 0$, *we use Gerretsen's inequality* $s^2 \geq 16Rr-5r^2 \geq \frac{r(4R+r)^2}{R+r}$

It remain to prove that:

$$\frac{r(4R+r)^2}{R+r}[(16Rr-5r^2)(2R-5r)+2r(2R^2+Rr+5r^2)] \geq r^3(4R+r)^2 \Leftrightarrow$$

$\Leftrightarrow 36R^2-89Rr+34r^2 \geq 0$, *obviously, because in this case* $2R \geq 5r$.

Case 2). If $(2R-5r) < 0$, *we rewrite the inequality*

$s^2[2r(2R^2+Rr+5r^2)-s^2(5r-2R)] \geq r^3(4R+r)^2$ *and we use Gerretsen's inequality*

$$\frac{r(4R+r)^2}{R+r} \leq 16Rr-5r^2 \leq s^2 \leq \frac{R(4R+r)^2}{2(2R-r)} \leq 4R^2+4Rr+3r^2$$

It remains to prove that:

$$\frac{r(4R+r)^2}{R+r}[2r(2R^2+Rr+5r^2)-(4R^2+4Rr+3r^2)(5r-2R)] \geq r^3(4R+r)^2 \Leftrightarrow$$

$\Leftrightarrow 8R^3-8R^2r-13Rr^2-6r^3 \geq 0 \Leftrightarrow (R-2r)(8R^2+8Rr+3r^2) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$. *Equality holds if and only if the triangle is equilateral.*

REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

A SIMPLE PROOF FOR KLAMKIN'S INEQUALITY

By Bogdan Fuștei-Romania

Abstract: *In this short math note is presented a new technique to proof Klamkin's inequality.*

In $\Delta ABC, P, P' \in (ABC)$ the following relationship holds:

$$\boldsymbol{a \cdot AP \cdot AP' + b \cdot BP \cdot BP' + c \cdot CP \cdot CP' \geq abc \ (Klamkin)}$$

Proof: For $x, y, z \in \mathbb{R}$ we have:

$$(x+y+z)(x\cdot AP^2+y\cdot BP^2+z\cdot CP^2)\ge yz\cdot a^2+zx\cdot b^2+xy\cdot c^2$$

Let $P'\in Int(ABC)$ and $x=\frac{a\cdot AP'}{AP}, y=\frac{b\cdot BP'}{BP}, z=\frac{c\cdot CP'}{CP}$. Hence,

$$\left(\sum_{cyc}\frac{a\cdot AP'}{AP}\right)\cdot\left(\sum_{cyc}\frac{a\cdot AP'}{AP}\cdot AP^2\right)\ge\sum_{cyc}\frac{bc\cdot BP'\cdot CP'}{BP\cdot CP}\cdot a^2$$

$$\frac{1}{AP\cdot BP\cdot CP}\sum_{cyc}a\cdot AP'\cdot BP\cdot CP\cdot\sum_{cyc}a\cdot AP'\cdot AP\ge\frac{abc}{AP\cdot BP\cdot CP}\sum_{cyc}a\cdot AP\cdot BP'\cdot CP'$$

$$\sum_{cyc}a\cdot AP'\cdot BP\cdot CP\cdot\sum_{cyc}a\cdot AP'\cdot AP\ge abc\sum_{cyc}a\cdot AP\cdot BP'\cdot CP';(1)$$

Analogous,

$$\sum_{cyc}a\cdot AP\cdot BP'\cdot CP'\cdot\sum_{cyc}a\cdot AP\cdot AP'\ge abc\sum_{cyc}a\cdot AP'\cdot BP\cdot CP;(2)$$

By adding, it follows that:

$$\sum_{cyc}a\cdot AP\cdot AP'\left(\sum_{cyc}a\cdot AP'\cdot BP\cdot CP+\sum_{cyc}a\cdot AP\cdot BP'\cdot CP'\right)\ge$$

$$\ge abc\left(\sum_{cyc}a\cdot AP\cdot BP'\cdot CP'+\sum_{cyc}a\cdot AP\cdot BP'\cdot CP'\right)$$

$$a\cdot AP\cdot AP'+b\cdot BP\cdot BP'+c\cdot CP\cdot CP'\ge abc\ (Klamkin)$$

REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT THE PROBLEM JP.460-R.M.M. WINTER EDITION 2023

By Marin Chirciu-Romania

JP.460. If $a,b,c,m,n>0$, then

$$\left(\sum_{cyc}\frac{a}{\sqrt[3]{mb+nc}}\right)^3\ge\frac{(a+b+c)^4}{(m+n)(a^2+b^2+c^2)}$$

Proposed by Daniel Sitaru-Romania

Solution. **Lemma 1. If $a,b,c,m,n>0$ then:**

$$\sum_{cyc}\frac{\boldsymbol{a}}{\sqrt[3]{\boldsymbol{mb+nc}}}\ge\sqrt[3]{\frac{(\sum \boldsymbol{a})^4}{(\boldsymbol{m+n})(\sum \boldsymbol{bc})}}$$

Proof. Using Radon's inequality, we get:

$$\sum_{cyc}\frac{a}{\sqrt[3]{mb+nc}}=\sum_{cyc}\frac{a\sqrt[3]{a}}{\sqrt[3]{a(mb+nc)}}=\sum_{cyc}\frac{a^{\frac{4}{3}}}{(mab+nac)^{\frac{1}{3}}}\overset{Radon}{\ge}\frac{(\sum a)^{\frac{4}{3}}}{[\sum(mab+nac)]^{\frac{1}{3}}}=\sqrt[3]{\frac{(\sum a)^4}{(m+n)\sum bc}}$$

Equality holds for $a=b=c$.

Let's get back to the main problem, using Lemma 1, we get:

$$LHS=\left(\sum_{cyc}\frac{a}{\sqrt[3]{mb+nc}}\right)^3\overset{Lemma\ 1}{\ge}\frac{(\sum a)^4}{(m+n)\sum bc}\overset{(1)}{\ge}\frac{(a+b+c)^4}{(m+n)(a^2+b^2+c^2)}=RHS$$

$(1)\Leftrightarrow\sum a^2\ge\sum bc$. Equality holds for $a=b=c$. **Remark.** The problem can be developed.

If $a,b,c,m,n>0$ then:

$$\left(\sum_{cyc}\frac{\boldsymbol{a}}{\sqrt[4]{\boldsymbol{mb+nc}}}\right)^4\ge\frac{(\boldsymbol{a+b+c})^5}{(\boldsymbol{m+n})(\boldsymbol{a^2+b^2+c^2})}$$

Marin Chirciu

Solution. **Lemma 2. If $a,b,c,m,n>0$ then:**

$$\sum_{cyc}\frac{\boldsymbol{a}}{\sqrt[4]{\boldsymbol{mb+nc}}}\ge\sqrt[4]{\frac{(\sum \boldsymbol{a})^5}{(\boldsymbol{m+n})\sum \boldsymbol{bc}}}$$

Proof. Using Radon's inequality, we have:

$$\sum_{cyc}\frac{a}{\sqrt[4]{mb+nc}}=\sum_{cyc}\frac{a\sqrt[4]{a}}{\sqrt[4]{a(mb+nc)}}=\sum_{cyc}\frac{a^{\frac{5}{4}}}{(mab+nac)^{\frac{1}{4}}}\overset{Radon}{\ge}$$

$$\ge\frac{(\sum a)^{\frac{5}{4}}}{[\sum(mab+nac)]^{\frac{1}{4}}}=\frac{(\sum a)^{\frac{5}{4}}}{[(m+n)\sum bc]^{\frac{1}{4}}}=\sqrt[4]{\frac{(\sum a)^5}{(m+n)\sum bc}}$$

Equality holds for $a = b = c$. Let's get back to the main problem.

Using Lemma 2, we have:

$$LHS = \left(\sum_{cyc} \frac{a}{\sqrt[4]{mb + nc}}\right)^5 \overset{Lemma\ 2}{\geq} \frac{(\sum a)^5}{(m+n)\sum bc} \overset{(1)}{\geq} \frac{(a+b+c)^5}{(m+n)(a^2+b^2+c^2)} = RHS$$

$(1) \Leftrightarrow \sum a^2 \geq \sum bc$. Equality holds for $a = b = c$.

Remark. The problem can be developed.

If $a, b, c, x, y > 0$ and $n \in \mathbb{N}, n \geq 2$ then:

$$\left(\sum_{cyc} \frac{a}{\sqrt[n]{xb + yc}}\right)^n \geq \frac{(a+b+c)^{n+1}}{(x+y)(a^2+b^2+c^2)}$$

Marin Chirciu

Solution. **Lemma 3. If $a, b, c, x, y > 0$ then:**

$$\sum_{cyc} \frac{a}{\sqrt[n]{xb + yc}} \geq \sqrt[n]{\frac{(\sum a)^{n+1}}{(x+y)\sum bc}}$$

Proof. Using Radon's inequality we get:

$$\sum_{cyc} \frac{a}{\sqrt[n]{xb + yc}} = \sum_{cyc} \frac{a\sqrt[n]{a}}{\sqrt[n]{a(xb + yc)}} = \sum_{cyc} \frac{a^{\frac{n+1}{n}}}{(xab + yac)^{\frac{1}{n}}} \overset{Radon}{\geq}$$

$$\geq \frac{(\sum a)^{\frac{n+1}{n}}}{[\sum(xab + yac)]^{\frac{1}{n}}} = \frac{(\sum a)^{\frac{n+1}{n}}}{[(x+y)\sum bc]^{\frac{1}{n}}} = \sqrt[n]{\frac{(\sum a)^{n+1}}{(x+y)\sum bc}}$$

Equality holds for $a = b = c$. Let's back to the main problem.

Using Lemma 3, we get:

$$LHS=\left(\sum_{cyc}\frac{a}{\sqrt[n]{xb+yc}}\right)^{n+1}\overset{Lemma\ 3}{\geq}\frac{(\sum a)^{n+1}}{(x+y)\sum bc}\overset{(1)}{\geq}\frac{(a+b+c)^{n+1}}{(x+y)(a^2+b^2+c^2)}=RHS$$

$(1)\Leftrightarrow\sum a^2\geq\sum bc$. Equality holds for $a=b=c$.

Note: For $n=3$ we get the Problem JP.460 from R.M.M. WINTER EDITION 2023 proposed by *Daniel Sitaru*-Romania.

JP.460. If $a,b,c,m,n>0$, then

$$\left(\sum_{cyc}\frac{\boldsymbol{a}}{\sqrt[3]{\boldsymbol{mb}+\boldsymbol{nc}}}\right)^{3}\geq\frac{(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})^{4}}{(\boldsymbol{m}+\boldsymbol{n})(\boldsymbol{a}^{2}+\boldsymbol{b}^{2}+\boldsymbol{c}^{2})}$$

REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

SOLVED PROBLEMS-(III)

By D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

1. **Show that if $m\in[0,\infty)$, $x,y,z,t\in(0,\infty)$, then in any triangle ABC, with usual notations holds:**

$$\sum_{cyc}\frac{(\boldsymbol{x}\boldsymbol{a}^{2}+\boldsymbol{y}\boldsymbol{b}^{2})^{\boldsymbol{m}+1}}{(\boldsymbol{z}\boldsymbol{m}_{\boldsymbol{c}}^{2}+\boldsymbol{t}\boldsymbol{h}_{\boldsymbol{a}}^{2})^{\boldsymbol{m}}}\geq\frac{4^{\boldsymbol{m}+1}(\boldsymbol{x}+\boldsymbol{y})^{\boldsymbol{m}+1}}{3^{\boldsymbol{m}-\frac{1}{2}}(\boldsymbol{z}+\boldsymbol{t})^{\boldsymbol{m}}}\boldsymbol{S}$$

Solution: We have $h_a\leq m_a\Rightarrow h_a^2\leq m_a^2$, and other two analogous. So,

$$h_a^2+h_b^2+h_c^2\leq m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)\qquad(1)$$

By J. Radon's inequality, (1) and $\sum_{cyc}m_a^2=\frac{3}{4}\sum_{cyc}a^2$, we deduce that

$$\sum_{cyc}\frac{(xa^2+yb^2)^{m+1}}{(zm_c^2+th_a^2)^m}\geq\frac{\left(\sum_{cyc}(xa^2+yb^2)\right)^{m+1}}{\left(\sum_{cyc}zm_c^2+th_a^2\right)^m}=\frac{\left(x\sum_{cyc}a^2+y\sum_{cyc}a^2\right)^{m+1}}{\left(z\sum_{cyc}m_a^2+t\sum_{cyc}h_a^2\right)^m}\geq$$

$$=\frac{(x+y)^{m+1}\left(\sum_{cyc}a^2\right)^{m+1}}{(z+t)^m\left(\sum_{cyc}m_a^2\right)^m}=\frac{(x+y)^{m+1}\left(\sum_{cyc}a^2\right)^{m+1}}{(z+t)^m\left(\frac{3}{4}\right)^m\left(\sum_{cyc}a^2\right)^m}=\left(\frac{4}{3}\right)^m\frac{(x+y)^{m+1}}{(z+t)^m}\sum_{cyc}a^2\quad(2)$$

By Ion Ionescu – Weitzenböck inequality we have: $a^2+b^2+c^2\geq4\sqrt{3}S\quad(3)$

By (2) and (3) we obtain: $\sum_{cyc}\frac{(xa^2+yb^2)^{m+1}}{(zm_c^2+th_a^2)^m}\geq\frac{4^{m+1}(x+y)^{m+1}}{3^{m-\frac{1}{2}}(z+t)^m}S$, Q.E.D.

2. Prove that if $a,b,c,d,x,y,z,t\in\mathbb{R}_+^*$, then:

$$4(a^2+b^2+c^2+d^2+x^2+y^2+z^2+t^2)+8\sqrt{(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+t^2)}\geq(a+b+c+d+x+y+z+t)^2$$

Solution: By Bergström's inequality we have that:

$$a^2+b^2+c^2+d^2\geq\frac{(a+b+c+d)^2}{4}\quad(1)$$

$$x^2+y^2+z^2+t^2\geq\frac{(x+y+z+t)^2}{4}\quad(2)$$

Denoting LHS with E, then by (1) and (2) we obtain:

$$E\geq(a+b+c+d)^2+(x+y+z+t)^2+2(a+b+c+d)(x+y+z+t)=$$

$$=(a+b+c+d+x+y+z+t)^2,$$

and we are done.

3. If ABC is a triangle then prove the inequality:

$$\left(\sqrt[3]{\frac{b}{a}}+\sqrt[3]{\frac{c}{b}}+\sqrt[3]{\frac{a}{c}}\right)\cdot\sqrt[3]{6s\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\geq 9$$

Solution:

$$\left(\sqrt[3]{\frac{b}{a}}+\sqrt[3]{\frac{c}{b}}+\sqrt[3]{\frac{a}{c}}\right)\geq 3\cdot\sqrt[3]{\sqrt[3]{\frac{bca}{abc}}}=3\quad(1)$$

$$6p\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 27,\text{ so}$$

$$\sqrt[3]{6s\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\geq\sqrt[3]{27}=3\quad(2)$$

By multiplying (1) and (2) yields the inequality. The equality occurs iff ABC is equilateral.

4. Prove that in any triangle ABC occurs the inequality:

$$\frac{a^2}{(p-b)(p-c)}+\frac{b^2}{(p-c)(p-a)}+\frac{c^2}{(p-a)(p-b)}\geq\frac{4p^2}{(4R+r)r}$$

Solution: From Bergström's inequality:

$$U=\sum\frac{a^2}{(p-b)(p-c)}\geq\frac{(\sum a)^2}{\sum(p-b)(p-c)}=\frac{4p^2}{\sum(p-a)(p-b)}\text{ and since}$$

$$\sum(p-a)(p-b)=(4R+r)r,\ \text{ Q.E.D.}$$

5. If $x,y,z\in\mathbb{R}_+^*, m\in\mathbb{R}_+$, then in any triangle ABC holds:

$$\frac{r_a}{(xr_b+yr_c)^m}+\frac{r_b}{(xr_c+yr_a)^m}+\frac{r_c}{(xr_a+yr_b)^m}\geq\frac{(4R+r)^{m+1}}{(x+y)^m\cdot p^{2m}}$$

Solution: From J. Radon $\sum\frac{r_a}{(xr_b+yr_c)^m}=\sum\frac{r_a^{m+1}}{(xr_ar_b+yr_ar_c)^m}\geq\frac{(\sum r_a)^{m+1}}{(x+y)^m(\sum r_ar_b)^m}$

And by $\sum r_a=4R+r$ and $\sum r_ar_b=p^2$, Q.E.D.

6. **Prove that:**

If m and n are positive real numbers then in any triangle ABC (with usual notations) holds:

$$\frac{\cot\frac{A}{2}}{m+n\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}}+\frac{\cot\frac{B}{2}}{m+n\cdot\tan\frac{C}{2}\cdot\tan\frac{A}{2}}+\frac{\cot\frac{C}{2}}{m+n\cdot\tan\frac{A}{2}\cdot\tan\frac{B}{2}}\geq\frac{9s}{4mR+(m+3n)r}$$

Solution: We have;

$$U=\sum\frac{\cot\frac{A}{2}}{m+n\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}}=\sum\frac{1}{m\cdot\tan\frac{A}{2}+n\cdot\tan\frac{A}{2}\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}}$$

and by Bergström's inequality we deduce that:

$$U\geq\frac{9}{m\cdot\sum\tan\frac{A}{2}+3n\cdot\prod\tan\frac{A}{2}}$$

Because: $\sum\tan\frac{A}{2}=\frac{4R+r}{p}$ and $\prod\tan\frac{A}{2}=\frac{r}{p}$, we obtain the conclusion.

7. **If $x,y,z\in\mathbb{R}_+^*$, then in any triangle ABC prove that:**

$$\frac{\tan^4\frac{A}{2}}{x\sin^2\frac{A}{2}+y\sin^2\frac{B}{2}+z\cos^2\frac{C}{2}}+\frac{\tan^4\frac{B}{2}}{x\sin^2\frac{B}{2}+y\sin^2\frac{C}{2}+z\cos^2\frac{A}{2}}+$$

$$+\frac{\tan^4\frac{C}{2}}{x\sin^2\frac{C}{2}+y\sin^2\frac{A}{2}+z\cos^2\frac{B}{2}}\geq\frac{2R(16R^2+8Rr+r^2-p^2)^2}{(2(x+y+2z)R-(x+y-z)r)p^4}$$

Solution: By Bergström:

$$\sum\frac{\tan^4\frac{A}{2}}{x\sin^2\frac{A}{2}+y\sin^2\frac{B}{2}+z\cos^2\frac{C}{2}}=\sum\frac{\left(\tan^2\frac{A}{2}\right)^2}{x\sin^2\frac{A}{2}+y\sin^2\frac{B}{2}+z\cos^2\frac{C}{2}}\geq$$

$$\geq\frac{\left(\sum\tan^2\frac{A}{2}\right)^2}{(x+y)\sum\sin^2\frac{A}{2}+z\sum\cos^2\frac{A}{2}}$$

and

$$\sum\sin^2\frac{A}{2}=\frac{2R-r}{2R},\sum\cos^2\frac{A}{2}=\frac{4R+r}{2R},\sum\tan^2\frac{A}{2}=\frac{(4R+r)^2-2p^2}{p^2}$$

8. If $x,y,z\in\mathbb{R}_+^*$, then prove that in any triangle ABC the following inequality is true:

$$\frac{\cot^3\frac{A}{2}}{x\cot\frac{A}{2}+y\tan\frac{B}{2}+z\tan\frac{C}{2}}+\frac{\cot^3\frac{B}{2}}{x\cot\frac{B}{2}+y\tan\frac{C}{2}+z\tan\frac{A}{2}}+$$

$$+\frac{\cot^3\frac{C}{2}}{x\cot\frac{C}{2}+y\tan\frac{A}{2}+z\tan\frac{B}{2}}\ge\frac{p^2}{(3x+y+z)r^2}$$

Solution: By Bergström:

$$\sum\frac{\cot^3\frac{A}{2}}{x\cot\frac{A}{2}+y\tan\frac{B}{2}+z\tan\frac{C}{2}}=\sum\frac{\cot^2\frac{A}{2}}{x+y\tan\frac{B}{2}\tan\frac{A}{2}+z\tan\frac{C}{2}\tan\frac{A}{2}}\ge$$

$$\ge\frac{\left(\sum\cot\frac{A}{2}\right)^2}{3x+(y+z)\sum\tan\frac{A}{2}\tan\frac{B}{2}}$$

Since $\sum\sin^2\frac{A}{2}=\frac{2R-r}{2R}$ and $\sum\tan\frac{A}{2}\tan\frac{B}{2}=1$

9. If $x,y\in\mathbb{R}_+^*, m\in\mathbb{R}_+$, the prove that in any triangle ABC holds

$$\frac{\tan^{m+1}\frac{A}{2}}{\left(x\cdot\tan\frac{B}{2}+y\cdot\tan\frac{C}{2}\right)^m}+\frac{\tan^{m+1}\frac{B}{2}}{\left(x\cdot\tan\frac{C}{2}+y\cdot\tan\frac{A}{2}\right)^m}+\frac{\tan^{m+1}\frac{C}{2}}{\left(x\cdot\tan\frac{A}{2}+y\cdot\tan\frac{B}{2}\right)^m}$$

$$\ge\frac{4R+r}{(x+y)^m p}$$

Solution: $U=\sum\frac{\tan^{m+1}\frac{A}{2}}{\left(x\tan\frac{B}{2}+y\tan\frac{C}{2}\right)^m}$, by Radon $U\ge\frac{\left(\sum\tan\frac{A}{2}\right)^{m+1}}{(x+y)^m\left(\sum\tan\frac{A}{2}\right)^m}=\frac{\sum\tan\frac{A}{2}}{(x+y)^m}$

Using $\sum\tan\frac{A}{2}=\frac{4R+r}{p}$, Q.E.D.

10. If $x,y,z\in\mathbb{R}_+^*, m\in\mathbb{R}_+$, then in any triangle ABC occurs:

$$\frac{r_a}{(x+yr_b+zr_c)^m}+\frac{r_b}{(x+yr_c+zr_a)^m}+\frac{r_c}{(x+yr_a+zr_b)^m}\ge\frac{(4R+r)^{m+1}}{\left((4R+r)x+(y+z)p^2\right)^m}$$

Solution: From J. Radon inequality: $\sum\frac{r_a}{(x+yr_b+zr_c)^m}=\sum\frac{r_a^{m+1}}{(xr_a+yr_ar_b+zr_ar_c)^m}\ge\frac{(\sum r_a)^{m+1}}{(x\sum r_a+(y+z)\sum r_ar_b)^m}$

And from, $\sum r_a=4R+r$ and $\sum r_ar_b=p^2$, Q.E.D.

11. If $m,n\in\mathbb{R}_+^*$, then in any triangle ABC holds:

$$\frac{\tan\frac{A}{2}}{m\cdot(4R+r)+n\cdot p\cdot\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{m\cdot(4R+r)+n\cdot p\cdot\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{m\cdot(4R+r)+n\cdot p\cdot\tan\frac{A}{2}}$$
$$\geq\frac{(4R+r)^2}{p(m(4R+r)^2+np^2)}$$

Solution: $U=\sum\frac{\tan\frac{A}{2}}{m(4R+r)+np\cdot\tan\frac{B}{2}}=\sum\frac{\tan^2\frac{A}{2}}{m(4R+r)\tan\frac{A}{2}+np\tan\frac{A}{2}\tan\frac{B}{2}}$, and from Bergström

$U\geq\frac{\left(\tan\frac{A}{2}\right)^2}{m(4R+r)\sum\tan\frac{A}{2}+np\sum\tan\frac{A}{2}\tan\frac{B}{2}}$, since $\sum\tan\frac{A}{2}=\frac{4R+r}{p}$ and $\sum\tan\frac{A}{2}\tan\frac{B}{2}=1$, Q.E.D.

12. If $x,y\in\mathbb{R}_+^*,m\in\mathbb{R}_+$, the in any triangle ABC holds:

$$\frac{\tan\frac{A}{2}}{\left(x\cdot\tan\frac{B}{2}+y\cdot\tan\frac{C}{2}\right)^m}+\frac{\tan\frac{B}{2}}{\left(x\cdot\tan\frac{C}{2}+y\cdot\tan\frac{A}{2}\right)^m}+\frac{\tan\frac{C}{2}}{\left(x\cdot\tan\frac{A}{2}+y\cdot\tan\frac{B}{2}\right)^m}\geq\frac{(4R+r)^{m+1}}{(x+y)^m p^{m+1}}$$

Solution: $U=\sum\frac{\tan\frac{A}{2}}{\left(x\tan\frac{B}{2}+y\tan\frac{C}{2}\right)^m}=\sum\frac{\tan^{m+1}\frac{A}{2}}{\left(x\tan\frac{A}{2}\tan\frac{B}{2}+y\tan\frac{C}{2}\tan\frac{C}{2}\right)^m}$, by Radon

$$U\geq\frac{\left(\sum\tan\frac{A}{2}\right)^{m+1}}{(x+y)^m\left(\sum\tan\frac{A}{2}\tan\frac{B}{2}\right)^m}$$

Applying $\sum\tan\frac{A}{2}=\frac{4R+r}{p}$ and $\sum\tan\frac{A}{2}\tan\frac{B}{2}=1$, Q.E.D.

13. If $x,y\in\mathbb{R}_+^*$, then prove that in any triangle ABC holds:

$$\frac{\tan^2\frac{A}{2}}{x\cdot\tan\frac{A}{2}+y\cdot\tan\frac{B}{2}}+\frac{\tan^2\frac{B}{2}}{x\cdot\tan\frac{B}{2}+y\cdot\tan\frac{C}{2}}+\frac{\tan^2\frac{C}{2}}{x\cdot\tan\frac{C}{2}+y\cdot\tan\frac{A}{2}}\geq\frac{4R+r}{(x+y)p}$$

Solution: From Bergström: $\sum\frac{\tan^2\frac{A}{2}}{x\tan\frac{A}{2}+y\tan\frac{B}{2}}\geq\frac{\left(\sum\tan\frac{A}{2}\right)^2}{(x+y)\sum\tan\frac{A}{2}}=\frac{\sum\tan\frac{A}{2}}{x+y}$, and

$$\sum\tan\frac{A}{2}=\frac{4R+r}{p}\text{, Q.E.D.}$$

14. Prove that if $x,y\in\mathbb{R}_+^*,m\in\mathbb{R}_+$, the in any triangle ABC holds:

$$\frac{\tan^{2m+1}\frac{A}{2}}{\left(x\cdot\cot\frac{B}{2}+y\cdot\cot\frac{C}{2}\right)^m}+\frac{\tan^{2m+1}\frac{B}{2}}{\left(x\cdot\cot\frac{C}{2}+y\cdot\cot\frac{A}{2}\right)^m}+\frac{\tan^{2m+1}\frac{C}{2}}{\left(x\cdot\cot\frac{A}{2}+y\cdot\cot\frac{B}{2}\right)^m}\geq\frac{(4R+r)r^m}{(x+y)^m p^{m+1}}$$

Solution: $U=\sum\frac{\tan^{2m+1}\frac{A}{2}}{\left(x\cot\frac{B}{2}+y\cot\frac{C}{2}\right)^m}=\sum\frac{\tan^{m+1}\frac{A}{2}}{\left(x\cot\frac{A}{2}\cot\frac{B}{2}+y\cot\frac{A}{2}\cot\frac{C}{2}\right)^m}$, by Radon

$$U \geq \frac{\left(\sum \tan\frac{A}{2}\right)^{m+1}}{(x+y)^m\left(\sum \cot\frac{A}{2}\cot\frac{B}{2}\right)^m}$$

Taking account by $\sum \tan\frac{A}{2} = \frac{4R+r}{p}$ and $\sum \cot\frac{A}{2}\cot\frac{B}{2} = \frac{4R+r}{r}$, Q.E.D.

15. Prove that in all triangles ABC the following inequalities holds:

$$\boldsymbol{\sum \frac{\tan\frac{A}{2}\tan^2\frac{B}{2}}{m\tan\frac{A}{2}+n\tan\frac{B}{2}} \geq \frac{s^2}{(4R+r)^2\cdot m + s^2\cdot(n-2m)}\ (a)}$$

$$\boldsymbol{\sum \frac{\tan^2\frac{A}{2}\tan^2\frac{B}{2}}{m+n\tan^2\frac{B}{2}\tan^2\frac{C}{2}} \geq \frac{s^2}{(3m+n)\cdot s^2 - 2\cdot n\cdot r\cdot(4R+r)}\ (b)}$$

$$\boldsymbol{\sum \frac{\tan^2\frac{A}{2}\tan^2\frac{B}{2}}{m+n\tan\frac{B}{2}\tan\frac{C}{2}} \geq \frac{1}{3m+n}\ (c)}$$

$$\boldsymbol{\sum \frac{\tan\frac{A}{2}\tan\frac{B}{2}}{m+n\tan^2\frac{C}{2}} \geq \frac{s^2}{m\cdot s^2 + n\cdot(4R+r)\cdot r}\ (d)}$$

$$\boldsymbol{\sum \frac{\tan^2\frac{A}{2}\tan\frac{B}{2}}{m\tan\frac{A}{2}+n\tan\frac{B}{2}} \geq \frac{s^2}{(m-2n)\cdot s^2 + n\cdot(4R+r)^2}\ (e)}$$

for any positive real numbers m and n (the notations are usual and the sums are cyclic).

Solution Solution for (a). By Bergström's inequality we deduce that:

$$\sum \frac{\tan\frac{A}{2}\tan^2\frac{B}{2}}{m\tan\frac{A}{2}+n\tan\frac{B}{2}} = \sum \frac{\tan^2\frac{A}{2}\tan^2\frac{B}{2}}{m\tan^2\frac{A}{2}+n\tan\frac{A}{2}\tan\frac{B}{2}} \geq \frac{\left(\sum \tan\frac{A}{2}\tan\frac{B}{2}\right)^2}{m\sum\tan^2\frac{A}{2}+n\sum\tan\frac{A}{2}\tan\frac{B}{2}}$$

Since, $\sum \tan^2\frac{A}{2} = \frac{(4R+r)^2-2s^2}{s^2}$ and $\sum \tan\frac{A}{2}\tan\frac{B}{2} = 1$, (a) is proved.

Solution for (b). By Bergström's inequality we deduce that:

$$\sum \frac{\tan^2\frac{A}{2}\tan^2\frac{B}{2}}{m+n\tan^2\frac{B}{2}\tan^2\frac{C}{2}} \geq \frac{\left(\sum \tan\frac{A}{2}\tan\frac{B}{2}\right)^2}{3m+n\sum\tan^2\frac{A}{2}\tan^2\frac{B}{2}}$$

Because, $\sum \tan^2\frac{A}{2}\tan^2\frac{B}{2} = \frac{s^2-2r^2-8Rr}{s^2}$ and $\sum \tan\frac{A}{2}\tan\frac{B}{2} = 1$, (b) is proved.

Solution for (c). By Bergström's inequality we deduce that:

$$\sum\frac{\tan^2\frac{A}{2}\tan^2\frac{B}{2}}{m+n\tan\frac{B}{2}\tan\frac{C}{2}}\ge\frac{\left(\sum\tan\frac{A}{2}\tan\frac{B}{2}\right)^2}{3m+n\sum\tan\frac{A}{2}\tan\frac{B}{2}},\text{ since }\sum\tan\frac{A}{2}\tan\frac{B}{2}=1,\text{ (c) is proved.}$$

Solution for (d). By Bergström's inequality we deduce that:

$$\sum\frac{\tan\frac{A}{2}\tan\frac{B}{2}}{m+n\tan^2\frac{C}{2}}=\sum\frac{\tan^2\frac{A}{2}\tan^2\frac{B}{2}}{m\tan\frac{A}{2}\tan\frac{B}{2}+n\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\tan\frac{C}{2}}\ge$$

$$\ge\frac{\left(\sum\tan\frac{A}{2}\tan\frac{B}{2}\right)^2}{m\sum\tan\frac{A}{2}\tan\frac{B}{2}+n\prod\tan\frac{A}{2}\sum\tan\frac{C}{2}}$$

Because, $\sum\tan\frac{A}{2}\tan\frac{B}{2}=1,\prod\tan\frac{A}{2}=\frac{r}{s}$ and $\sum\tan\frac{A}{2}=\frac{4R+r}{s}$, (d) is proved.

Solution for (e). By Bergström's inequality we deduce that:

$$\sum\frac{\tan^2\frac{A}{2}\tan\frac{B}{2}}{m\tan\frac{A}{2}+n\tan\frac{B}{2}}=\sum\frac{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}{m\tan\frac{A}{2}\tan\frac{B}{2}+n\tan^2\frac{B}{2}}\ge\frac{\left(\sum\tan\frac{A}{2}\tan\frac{B}{2}\right)^2}{m\sum\tan\frac{A}{2}\tan\frac{B}{2}+n\sum\tan^2\frac{B}{2}}$$

Since, $\sum\tan\frac{A}{2}\tan\frac{B}{2}=1$ and $\sum\tan^2\frac{A}{2}=\frac{(4R+r)^2}{s^2}-2$, (e) is proved.

The proof is complete and we are done.

16. If $a,b,c>0$, then prove the inequality:

$$2\sqrt{\frac{a^2+b^2+c^2}{3}\cdot\frac{\sin x}{x}+\frac{a+b+c}{3}\cdot\frac{\tan x}{x}}>a+b=c,\forall x\in\left(0,\frac{\pi}{2}\right)$$

Solution

$$2\sqrt{\frac{a^2+b^2+c^2}{3}}\cdot\frac{\sin x}{x}+\frac{a+b+c}{3}\cdot\frac{\tan x}{x}\ge 2\cdot\frac{a+b+c}{3}\cdot\frac{\sin x}{x}+\frac{a+b+c}{3}\cdot\frac{\tan x}{x}=$$

$$=\frac{a+b+c}{3}\cdot\left(2\cdot\frac{\sin x}{x}+\frac{\tan x}{x}\right),\forall x\in\left(0,\frac{\pi}{2}\right)\quad(1)$$

and

$$\sin x>x-\frac{x^3}{6}\text{ and }\tan x>x+\frac{x^3}{3},\forall x\in\left(0,\frac{\pi}{2}\right)\Leftrightarrow$$

$$\Leftrightarrow\frac{\sin x}{x}>1-\frac{x^2}{6}\text{ and }\frac{\tan x}{x}>1+\frac{x^2}{3},\forall x\in\left(0,\frac{\pi}{2}\right).$$

Hence

$$2\cdot\frac{\sin x}{x}+\frac{\tan x}{x}>2-\frac{x^2}{3}+1+\frac{x^2}{3}=3,\forall x\in\left(0,\frac{\pi}{2}\right)\quad(2)$$

From (1) and (2), Q.E.D.

17. Prove that in all triangles ABC the following inequalities holds:

$$\sum \frac{\cot^{2m+1}\frac{A}{2}}{\left(x\cot\frac{A}{2}+y\tan\frac{B}{2}+z\tan\frac{B}{2}\tan\frac{C}{2}\right)^m} \geq \frac{s^{2m+1}}{\left((3x+y)s+3zr\right)^m r^{m+1}} \quad (a)$$

$$\sum \frac{\cot^{2m+1}\frac{A}{2}}{\left(x\cdot\tan\frac{B}{2}+y\cdot\tan\frac{C}{2}\right)^m} \geq \frac{s^{m+1}}{(x+y)^m r^{m+1}} \quad (b)$$

$$\sum \frac{\tan\frac{A}{2}\tan^{m+1}\frac{B}{2}}{\left(x\tan\frac{A}{2}+y\tan\frac{B}{2}\right)^m} \geq \frac{s^{2m}}{(x\cdot(4R+r)^2+(y-2x)\cdot s^2)^m} \quad (c)$$

for any $x,y,z>0$ and $m\geq 0$ (the notations are usual and the sums are cyclic).

Solution: Solution for (a). We apply the inequality of J. Radon and we deduce that:

$$\sum \frac{\cot^{2m+1}\frac{A}{2}}{\left(x\cot\frac{A}{2}+y\tan\frac{B}{2}+z\tan\frac{B}{2}\tan\frac{C}{2}\right)^m} = \sum \frac{\cot^{m+1}\frac{A}{2}}{\left(x+y\tan\frac{A}{2}\tan\frac{B}{2}+z\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\right)^m} \geq$$

$$\geq \frac{\left(\sum\cot\frac{A}{2}\right)^{m+1}}{\left(3x+y\sum\tan\frac{A}{2}\tan\frac{B}{2}+3z\prod\tan\frac{A}{2}\right)}$$

Using the well-known formulas, $\sum\cot\frac{A}{2}=\frac{s}{r}, \sum\tan\frac{A}{2}\tan\frac{B}{2}=1$ and $\prod\tan\frac{A}{2}=\frac{r}{s}$ and (a) is proved.

Solution for (b). We have:

$$V=\sum \frac{\cot^{2m+1}\frac{A}{2}}{\left(\sum\cdot\tan\frac{B}{2}+y\cdot\tan\frac{C}{2}\right)^m} = \sum \frac{\cot^{m+1}\frac{A}{2}}{\left(x\cdot\tan\frac{A}{2}\cdot\tan\frac{B}{2}+y\cdot\tan\frac{A}{2}\tan\frac{C}{2}\right)^m}$$

and by Radon's inequality we deduce that:

$$V\geq \frac{\left(\cot\frac{A}{2}\right)^{m+1}}{(x+y)^m\left(\sum\tan\frac{A}{2}\tan\frac{B}{2}\right)^m}$$

Using the well-known formulas, $\sum\cot\frac{A}{2}=\frac{s}{r}$ and $\sum\tan\frac{A}{2}\tan\frac{B}{2}=1$, (b) is proved.

Solution for (c) By Radon's inequality we obtain that:

$$\sum \frac{\tan\frac{A}{2}\tan^{m+1}\frac{B}{2}}{\left(x\tan\frac{A}{2}+y\tan\frac{B}{2}\right)^m} = \frac{\left(\tan\frac{A}{2}\tan\frac{B}{2}\right)^{m+1}}{\left(x\tan^2\frac{A}{2}+y\tan\frac{A}{2}\tan\frac{B}{2}\right)^m} \geq$$

$$\geq \frac{\left(\sum \tan\frac{A}{2}\tan\frac{B}{2}\right)^2}{\left(x\sum \tan^2\frac{A}{2} + y\sum \tan\frac{A}{2}\tan\frac{B}{2}\right)^m}$$

Using the well-known formulas, $\sum \tan\frac{A}{2}\tan\frac{B}{2} = 1$ and $\sum \tan^2\frac{A}{2} = \frac{(4R+r)^2}{s^2} - 2$, (c) is proved and we are done.

18. Show that in any triangle ABC (with usual notations) the following inequalities holds:

a) $(\boldsymbol{s^2 + r^2 + 4Rr})^2 \geq \boldsymbol{8r(4R+r)(s^2 - r^2 - 4Rr)}$

b) $(\boldsymbol{s^2 + r^2 - 8Rr})^2 \geq \boldsymbol{8(8R^2 + r^2 - s^2)\left(s^2 - 4R(R+r)\right)}$

Solution: For any $w_1, w_2, w_3 \in \mathbb{R}$ we have:

$$w_1^2 + w_2^2 + w_3^2 \geq w_1 \cdot w_2 + w_2 \cdot w_3 + w_2 \cdot w_1 \quad (1)$$

With equality if $w_1 = w_2 = w_3$.

If $\boldsymbol{m, n, p} \in \mathbb{R}_+^*$, then:

$$\boldsymbol{\frac{m\cdot n}{p} + \frac{n\cdot p}{m} + \frac{p\cdot m}{n} \geq m + n + p} \quad \textbf{(2)} \text{ With equality if } m = n = p.$$

Indeed if in (1) we take $w_1 = \sqrt{\frac{m\cdot n}{p}}, w_2 = \sqrt{\frac{n\cdot p}{m}}, w_3 = \sqrt{\frac{p\cdot m}{n}}$ yields (2)

If $t, u, v, x, y, z \in \mathbb{R}_+^*$, then:

$$\frac{(tx+uy+vz)(ty+uz+vx)}{tz+ux+vy} + \frac{(ty+uz+vx)(tz+ux+vy)}{tx+uy+vz} +$$

$$+\frac{(tz+ux+vy)(tx+uy+vz)}{ty+uz+vx} \geq (t+u+v)(x+y+z) \quad (3)$$

Indeed if in (2) we take $m = tx+uy+vz, n = ty+uz+vx, p = tz+ux+vy$ yields (3).

Lemma. For any $\boldsymbol{x, y, z} \in \mathbb{R}_+^*$, holds:

$$\boldsymbol{(xy+yz+zx)^2 + 2(x^2+y^2+z^2)^2 \geq (x^2+y^2+z^2)(x+y+z)^2} \quad \textbf{(4)}$$

Proof. In (3) we take $t = x, u = y, v = z$ and we obtain:

$$\frac{(x^2+y^2+z^2)(xy+yz+zx)}{xz+xy+zy} + \frac{(xy+yz+zx)(xz+xy+yz)}{x^2+y^2+z^2} +$$

$$+\frac{(xz+xy+yz)(x^2+y^2+z^2)}{xy+yz+xz} \geq (x+y+z)^2$$

$$\Leftrightarrow \frac{(xy+yz+zx)^2}{x^2+y^2+z^2} + 2(x^2+y^2+z^2) \geq (x+y+z)^2$$

$$\Leftrightarrow (xy+yz+zx)^2+2(x^2+y^2+z^2)^2 \geq (x+y+z)^2(x^2+y^2+z^2)$$

Proof of a) If in (4) we take $x=\sin A$, $y=\sin B$, $z=\sin C$, then:

$$(\sin A\sin B+\sin B\sin C+\sin C\sin A)^2+2(\sin^2 A+\sin^2 B+\sin^2 C)^2 \geq$$

$$\geq (\sin A+\sin B+\sin C)^2(\sin^2 A+\sin^2 B+\sin^2 C) \quad (5)$$

We use: $\sin^2 A+\sin^2 B+\sin^2 C=\frac{s^2-r^2-4Rr}{2R^2}$ (6)

$$\sin A\sin B+\sin B\sin C+\sin C\sin A=\frac{s^2+4Rr+r^2}{4R^2} \quad (7)$$

$$\sin A+\sin B+\sin C=\frac{s}{R} \quad (8)$$

By (6)-(8) and (5) we deduce:

$$(s^2+r^2+4Rr)^2+8(s^2-r^2-4Rr)^2 \geq 8s^2(s^2-r^2-4Rr)$$

$$\Leftrightarrow (s^2+r^2+4Rr)^2 \geq 8(s^2-r^2-4Rr)(s^2-s^2+r^2+4Rr)=$$

$$=8r(4R+r)(s^2-r^2-4Rr)$$

Proof of b). If in (4) we take $x=\sin^2\frac{A}{2}$, $y=\sin^2\frac{B}{2}$, $z=\sin^2\frac{C}{2}$, then:

$$\left(\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^2\frac{B}{2}\sin^2\frac{C}{2}+\sin^2\frac{C}{2}\sin^2\frac{A}{2}\right)^2+2\left(\sin^4\frac{A}{2}+\sin^4\frac{B}{2}+\sin^4\frac{C}{2}\right)^2 \geq$$

$$\geq \left(\sin^4\frac{A}{2}+\sin^4\frac{B}{2}+\sin^4\frac{C}{2}\right)\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right)^2 \quad (9)$$

Using: $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=\frac{2R-r}{2R}$ (10)

$$\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^2\frac{B}{2}\sin^2\frac{C}{2}+\sin^2\frac{C}{2}\sin^2\frac{A}{2}=\frac{s^2+r^2-8Rr}{16R^2} \quad (11)$$

$$\sin^4\frac{A}{2}+\sin^4\frac{B}{2}+\sin^4\frac{C}{2}=\frac{8R^2+r^2-s^2}{8R^2}$$

and (9) follows:

$$\frac{(s^2+r^2-8Rr)^2}{256R^4}+2\cdot\frac{(8R^2+r^2-s^2)^2}{64R^4} \geq \frac{8R^2+r^2-s^2}{8R^2}\cdot\frac{(2R-r)^2}{4R^2}$$

$$\Leftrightarrow (s^2+r^2-8Rr)^2 \geq 8(8R^2+r^2-s^2)((2R-r)^2-8R^2-r^2+s^2)=$$

$$=8(8R^2+r^2-s^2)\big(s^2-4R(R+r)\big)$$

19. Show that in any triangle ABC (with usual notations) holds the following inequalities:

a) $(16R^2+8Rr+r^2+s^2)^2 \geq 8s^2(16R^2+8Rr+r^2-s^2)$

b) $s^4 \geq (4R^2+4Rr+r^2-2s^2)(8R^2-r^2+4s^2)$

Solution: For any $w_1, w_2, w_3 \in \mathbb{R}$ we have:

$w_1^2 + w_2^2 + w_3^2 \geq w_1 \cdot w_2 + w_2 \cdot w_3 + w_3 \cdot w_1$ (1). With equality iff $w_1 = w_2 = w_3$.

If $m, n, p \in \mathbb{R}_+^*$, then: $\frac{m\cdot n}{p} + \frac{n\cdot p}{m} + \frac{p\cdot m}{n} \geq m + n + p$ (2). With equality if $m = n = p$.

Indeed if in (1) we take $w_1 = \sqrt{\frac{m\cdot n}{p}}, w_2 = \sqrt{\frac{n\cdot p}{m}}, w_3 = \sqrt{\frac{p\cdot m}{n}}$ yields (2).

If $t, u, v, x, y, z \in \mathbb{R}_+^*$, then:

$$\frac{(tx+uy+vz)(ty+uz+vx)}{tz+ux+vy} + \frac{(ty+uz+vx)(tz+ux+vy)}{tx+uy+vz} +$$

$$+\frac{(tz+ux+vy)(tx+uy+vz)}{ty+uz+vx} \geq (t+u+v)(x+y+z) \quad (3)$$

Indeed if in (2) we take $m = tx+uy+vz, n = ty+uz+vx, p = tz+ux+vy$ yields (3).

Lemma. For any $x, y, z \in \mathbb{R}_+^*$, holds:

$$(xy+yz+zx)^2 + 2(x^2+y^2+z^2)^2 \geq (x^2+y^2+z^2)(x+y+z)^2 \quad \textbf{(4)}$$

Proof. In (3) we take $t = x, u = y, v = z$ and we obtain:

$$\frac{(x^2+y^2+z^2)(xy+yz+zx)}{xz+xy+zy} + \frac{(xy+yz+zx)(xz+xy+yz)}{x^2+y^2+z^2} +$$

$$+\frac{(xz+xy+yz)(x^2+y^2+z^2)}{xy+yz+xz} \geq (x+y+z)^2$$

$$\Leftrightarrow \frac{(xy+yz+zx)^2}{x^2+y^2+z^2} + 2(x^2+y^2+z^2) \geq (x+y+z)^2$$

$$\Leftrightarrow (xy+yz+zx)^2 + 2(x^2+y^2+z^2)^2 \geq (x+y+z)^2(x^2+y^2+z^2)$$

Proof of a). If in (4) we take $x = \cos^2\frac{A}{2}, y = \cos^2\frac{B}{2}, z = \cos^2\frac{C}{2}$, yields:

$$\left(\cos^2\frac{A}{2}\cos^2\frac{B}{2} + \cos^2\frac{B}{2}\cos^2\frac{C}{2} + \cos^2\frac{C}{2}\cos^2\frac{A}{2}\right)^2 + 2\left(\cos^4\frac{A}{2} + \cos^4\frac{B}{2} + \cos^4\frac{C}{2}\right)^2 \geq$$

$$\geq \left(\cos^4\frac{A}{2} + \cos^4\frac{B}{2} + \cos^4\frac{C}{2}\right)\left(\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}\right)^2 \quad (5)$$

By:

$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = \frac{4R+r}{2R} \quad (6)$$

$$\cos^2\frac{A}{2}\cos^2\frac{B}{2} + \cos^2\frac{B}{2}\cos^2\frac{C}{2} + \cos^2\frac{C}{2}\cos^2\frac{A}{2} = \frac{(4R+r)^2+s^2}{16R^2} \quad (7)$$

$$\cos^4\frac{A}{2} + \cos^4\frac{B}{2} + \cos^4\frac{C}{2} = \frac{(4R+r)^2-s^2}{8R^2} \quad (8)$$

and (5) we obtain:

$$\frac{((4R+r)^2+s^2)^2}{256R^4}+2\cdot\frac{((4R+r)^2-s^2)^2}{64R^4}\ge\frac{(4R+r)^2-s^2}{8R^2}\cdot\frac{(4R+r)^2}{4R^2}$$

$$\Leftrightarrow (16R^2+8Rr+r^2+s^2)^2\ge$$

$$\ge 8(16R^2+8Rr+r^2-s^2)(16R^2+8Rr+r^2-16R^2-8Rr-r^2+s^2)=$$

$$=8s^2\,(16R^2+8Rr+r^2-s^2)$$

Proof of b). If in (4) we take $x=\tan\frac{A}{2}, y=\tan\frac{B}{2}, z=\tan\frac{C}{2}$, yields:

$$\left(\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}\right)^2+\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\right)^2\ge$$

$$\ge\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\right)\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2\quad(9)$$

By:

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{4R+r}{s}\quad(10),\quad \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1\quad(11)$$

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}=\frac{(2R+r)^2-2s^2}{s^2}\quad(12)$$

and (9) we deduce:

$$1+2\cdot\frac{((2R+r)^2-2s^2)^2}{s^4}\ge\frac{(2R+r)^2-2s^2}{s^2}\cdot\frac{(4R+r)^2}{s^2}$$

$$\Leftrightarrow s^4\ge((2R+r)^2-2s^2)\big(16R^2+8Rr+r^2-2(4R^2+4Rr+r^2-2s^2)\big)=$$

$$=(4R^2+4Rr+r^2-2s^2)(8R^2-r^2+4s^2)$$

REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

A FEW LOGARITHMIC INEQUALITIES

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Abstract: *In this paper are presented a few logarithmic inequalities.*

Application 1. If $[a,b]\subset\mathbb{R}$, then prove that:

$$\frac{b^2-x^2}{2x}+\frac{y^2-a^2}{2y}+\log\left[\left(\frac{y}{a}\right)^y\cdot\left(\frac{b}{x}\right)^x\right]\le\frac{(b-a+y-x)(a^2+b^2)}{ab};\forall x,y\in[a,b]$$

Solution 1: If $[a,b]\subset\mathbb{R}, x_k\in[a,b], k=\overline{1,n}$, then

$$\left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}\frac{1}{x_k}\right)\le\frac{(a+b)^2}{4ab}\cdot n^2,\forall n\in\mathbb{N};(Schweitzer)$$

For $n=2$ and $x_1=x, x_2=y$ we have:

$$(x+y)\left(\frac{1}{x}+\frac{1}{y}\right)\le\frac{(a+b)^2}{ab};\forall x,y\in[a,b]\Leftrightarrow 2+\frac{x}{y}+\frac{y}{x}\le\frac{(a+b)^2}{ab};\forall x,y\in[a,b];(*)$$

Integrating $(*)$ w.r.t. x from a to y, we have:

$$\int_a^y\left(2+\frac{x}{y}+\frac{y}{x}\right)dx\le\int_a^y\frac{(a+b)^2}{ab}dx\Leftrightarrow$$

$$2(y-a)+\frac{y^2-a^2}{2y}+y(\log y-\log a)\le\frac{(a+b)^2}{ab}\cdot(y-a)\Leftrightarrow$$

$$\frac{y^2-a^2}{2y}+\log\left(\frac{y}{a}\right)^y\le(y-a)\left[\frac{(a+b)^2}{ab}-2\right];\forall y\ge a;\ \ (1)$$

Now, integrating $(*)$ w.r.t. y from x to b, we have:

$$\int_x^b\left(2+\frac{x}{y}+\frac{y}{x}\right)dy\le\int_x^b\frac{(a+b)^2}{ab}dy\Leftrightarrow$$

$$2(b-x)+x(\log b-\log x)+\frac{b^2-x^2}{2x}\le\frac{(a+b)^2}{ab}\cdot(b-x)\Leftrightarrow$$

$$\frac{b^2-x^2}{2x}+\log\left(\frac{b}{x}\right)^x\le(b-x)\left[\frac{(a+b)^2}{ab}-2\right];\forall x\le b;(2)$$

By adding (1) and (2), we obtain:

$$\frac{b^2-x^2}{2x}+\frac{y^2-a^2}{2y}+\log\left(\frac{b}{x}\right)^x+\log\left(\frac{y}{a}\right)^y\le\frac{(b-a+y-x)(a^2+b^2)}{ab};\forall x,y\in[a,b]\Leftrightarrow$$

$$\frac{b^2-x^2}{2x}+\frac{y^2-a^2}{2y}+\log\left(\frac{b^xy^y}{x^xa^y}\right)\le\frac{(b-a+y-x)(a^2+b^2)}{ab};\forall x,y\in[a,b]$$

Solution 2: Let $f(t)=(t-1)\left(t+\frac{1}{t}\right)-\frac{t^2-1}{2}-\log t, t>0$

We have :

$$f'(t)=t+\frac{1}{t}+(t-1)\left(1-\frac{1}{t^2}\right)-t-\frac{1}{t}=\frac{(t-1)^2(t+1)}{t^2}\ge 0,\forall t>0.$$

Then f is increasing on $(0,\infty)$

Since $f(1)=0$ we have : $f(t)\ge 0,\forall t\ge 1$ and $f(t)\le 0,\forall t\in(0,1]$

$$\text{Since } \frac{b}{x} \ge 1 \to f\left(\frac{b}{x}\right) = \left(\frac{b}{x} - 1\right)\left(\frac{b}{x} + \frac{x}{b}\right) - \frac{b^2 - x^2}{2x^2} - \log\left(\frac{b}{x}\right) \ge 0 \overset{\times x}{\Leftrightarrow}$$

$$\frac{b^2 - x^2}{2x} + \log\left(\frac{b}{x}\right)^x \le (b - x)\left(\frac{b}{x} + \frac{x}{b}\right) \ (1)$$

$$\text{And : } \frac{a}{y} \le 1 \ \to \ f\left(\frac{a}{y}\right) = \left(\frac{a}{y} - 1\right)\left(\frac{a}{y} + \frac{y}{a}\right) - \frac{a^2 - y^2}{2y^2} - \log\left(\frac{a}{y}\right) \le 0 \overset{\times(-y)}{\Leftrightarrow}$$

$$\frac{y^2 - a^2}{2y} + \log\left(\frac{y}{a}\right)^y \le (y - a)\left(\frac{a}{y} + \frac{y}{a}\right) \ (2)$$

Now, let $g(t) = t + \frac{1}{t}, t \ge 1$. We have : $g'(t) = 1 - \frac{1}{t^2} \ge 0, \forall t \ge 1$ then

g is increasing on $[1, \infty)$

$$\text{Since } \frac{b}{a} \ge max\left\{\frac{b}{x}, \frac{y}{a}\right\} \ge 1, \quad \text{then : } g\left(\frac{b}{a}\right) \ge g\left(\frac{b}{x}\right) \text{ and } g\left(\frac{b}{a}\right) \ge g\left(\frac{y}{a}\right)$$

$$\text{Or : } \frac{b}{x} + \frac{x}{b} \le \frac{b}{a} + \frac{a}{b} = \frac{a^2 + b^2}{ab} \ (3) \text{ and } \frac{a}{y} + \frac{y}{a} \le \frac{b}{a} + \frac{a}{b} = \frac{a^2 + b^2}{ab} \ (4)$$

From $(1), (2), (3)$ and (4) we obtain:

$$\frac{b^2 - x^2}{2x} + \frac{y^2 - a^2}{2y} + \log\left(\left(\frac{b}{x}\right)^x \left(\frac{y}{a}\right)^y\right) \le (b - x) \cdot \frac{a^2 + b^2}{ab} + (y - a) \cdot \frac{a^2 + b^2}{ab}$$

Therefore,

$$\frac{b^2 - x^2}{2x} + \frac{y^2 - a^2}{2y} + \log\left(\frac{b^x y^y}{x^x a^y}\right) \le \frac{(b - a + y - x)(a^2 + b^2)}{ab}, \forall x, y \in [a, b].$$

Observation: Putting $y = x$ in

$$\frac{y^2 - a^2}{2y} + \log\left(\frac{y}{a}\right)^y \le (y - a)\left[\frac{(a + b)^2}{ab} - 2\right]; \forall y \ge a; \ \ (1)$$

we get:

$$2(x - a) + \frac{x^2 - a^2}{2x} + x(\log x - \log a) \le \frac{(a + b)^2}{ab}(x - a); (2)$$

By adding (1) and (2) it follows the proposed problem by Daniel Culea:

$$\boldsymbol{\frac{(x + y)(xy - a^2)}{2xy} + \log\left[\left(\frac{y}{a}\right)^y \cdot \left(\frac{x}{a}\right)^x\right] \le (x + y - 2a)\frac{a^2 + b^2}{ab}}$$

Application 2. If $a, b \in (1, \infty)$ then:

$$\log\left[\left(\frac{a+b}{2a}\right)^{2a}\cdot\left(\frac{b^2+2ab}{a^2+2ab}\right)^{3ab}\right]<b-a+\log\left[\left(\frac{b}{a}\right)^{2(a+b)}\right]$$

Proof. In Kurliancik's inequality

$$\sum_{i=1}^{n}\frac{i}{\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_i}}<2\sum_{i=1}^{n}a_i\,,a_i>0$$

we take $a_1=x;a_2=y;a_3=z$, then

$$x+\frac{2xy}{x+y}+\frac{3xyz}{xy+yz+zx}<2(x+y+z)$$

$$1+\frac{2y}{x+y}+\frac{3yz}{x(y+z)+yz}<2\left(1+\frac{y}{x}+\frac{z}{x}\right)$$

$$\int_y^z dx+\int_y^z\frac{2y}{x+y}dx+\int_y^z\frac{3yz}{x(y+z)+yz}dx<2\int_y^z\left(1+\frac{y}{x}+\frac{z}{x}\right)dx$$

$$z-y+2y\log\left(\frac{z+y}{2y}\right)+3yz\log\left[\frac{z(y+z)+yz}{y(y+z)+yz}\right]<2(z-y)+2y\log\left(\frac{z}{y}\right)+2z\log\left(\frac{z}{y}\right)$$

$$\log\left[\left(\frac{z+y}{2y}\right)^{2y}\cdot\left(\frac{z^2+2yz}{y^2+2yz}\right)^{3yz}\right]<z-y+\log\left[\left(\frac{z}{y}\right)^{2y}\cdot\left(\frac{z}{y}\right)^{2z}\right]$$

and for $y=a,z=b$ it follows the desired inequality.

Application 3. If $a_i>1\,,i=\overline{1,n+1}\,,n\in\mathbb{N}^*$ then prove:

$$\sum_{cyc}\log_{a_1}^{n}(a_2a_3\cdot\ldots\cdot a_{n+1})\geq n^{n+1}$$

Solution: $log_{a_1}^{n}(a_2a_3\cdot\ldots\cdot a_{n+1})=\left(log_{a_1}a_2+log_{a_1}a_3+\dots+log_{a_1}a_{n+1}\right)^n\overset{AM-GM}{\geq}$

$$\geq n^n\cdot log_{a_1}a_2\,log_{a_1}a_3\cdot\ldots\cdot log_{a_1}a_{n+1}$$

$$\sum_{cyc}log_{a_1}^{n}(a_2a_3\cdot\ldots\cdot a_{n+1})\geq n^n\sum_{cyc}log_{a_1}a_2\,log_{a_1}a_3\cdot\ldots\cdot log_{a_1}a_{n+1}\overset{AM-GM}{\geq}$$

$$\geq n^n\cdot n\cdot\sqrt[n]{log_{a_1}a_2\,log_{a_2}a_1\cdot\ldots\cdot log_{a_n}a_{n+1}\,log_{a_{n+1}}a_n}=n^{n+1}$$

Application 4. If $a_1,a_2,\dots,a_n>0,n\in\mathbb{N},n>1$ then:

$$\sum_{cyc}\log_{1+a_1a_2}\left(1+a_2^{1+a_2}\right)\left(1+a_3^{1+a_3}\right)\geq 2n$$

Solution: $1+a_i^{1+a_i}=1+(1+a_i-1)^{1+a_i}\overset{Bernoulli}{\overbrace{\geq}}1+a_i^2\Rightarrow$

$$\left(1+a_i^{1+a_i}\right)\left(1+a_j^{1+a_j}\right)\geq(1+a_i^2)(1+a_j^2)\geq(1+a_ia_j)^2\Rightarrow$$

$$\sum_{cyc}log_{1+a_1a_2}(1+a_2^{1+a_2})(1+a_3^{1+a_3})\geq 2\sum_{cyc}log_{1+a_1a_2}(1+a_2a_3)\overset{Am-Gm}{\overbrace{\geq}}$$

$$\geq 2n\sqrt[n]{\prod_{cyc}log_{1+a_1a_2}(1+a_2a_3)}\geq 2n$$

Application 5. If $\mathbf{0}<a\leq b<\pi$ then:

$$\boldsymbol{log\left(\frac{sinb}{sina}\right)\geq\left(1+\sqrt{\frac{a}{b}}\right)log\left(\frac{sinb}{sin(\sqrt{ab})}\right)}$$

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Solution:

$$log\left(\frac{sinb}{sina}\right)\geq\left(1+\sqrt{\frac{a}{b}}\right)log\left(\frac{sinb}{sin(\sqrt{ab})}\right)$$

$$\Leftrightarrow log\left(\frac{sinb}{sina}\right)-log\left(\frac{sinb}{sin(\sqrt{ab})}\right)\geq\sqrt{\frac{a}{b}}log\left(\frac{sinb}{sin(\sqrt{ab})}\right)$$

$$\Leftrightarrow log\left(\frac{sin\sqrt{ab}}{sina}\right)\geq\sqrt{\frac{a}{b}}log\left(\frac{sinb}{sin(\sqrt{ab})}\right)$$

$$\Leftrightarrow\sqrt{b}log\left(sin\sqrt{ab}\right)-\sqrt{b}log(sina)\geq\sqrt{a}log(sinb)-\sqrt{a}log\left(sin\sqrt{ab}\right)$$

$$\Leftrightarrow\left(\sqrt{a}+\sqrt{b}\right)log\left(sin\sqrt{ab}\right)\geq\sqrt{a}log(sinb)+\sqrt{b}log(sina)$$

$$\Leftrightarrow log\left(sin\sqrt{ab}\right)\geq\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}log(sinb)+\frac{\sqrt{b}}{\sqrt{a}+\sqrt{b}}log(sina);\ \ (1)$$

We have:

$$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}log(sinb)+\frac{\sqrt{b}}{\sqrt{a}+\sqrt{b}}log(sina)\overset{logt-concave}{\leq}$$

$$\leq log\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}sinb+\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}sina\right)\overset{sint-concave\ (0,\pi)}{\leq}$$

$$\leq log\left(sin\left(\frac{b\sqrt{a}+a\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)\right)=log\left(sin(\sqrt{ab})\right);(2)$$

From (1),(2) it follows that:

$$log\left(\frac{sinb}{sina}\right) \geq \left(1+\sqrt{\frac{a}{b}}\right) log\left(\frac{sinb}{sin(\sqrt{ab})}\right)$$

Application 6. If $a_i > 1$, $i = \overline{1, n+1}$, $n \in \mathbb{N}$ then:

$$\sum_{cyc} \boldsymbol{log_{a_1}^n(a_2a_3\cdot\ldots\cdot a_{n+1})\cdot log_{a_1a_2^2a_3^2\cdot\ldots\cdot a_{n*1}^2} a_1 \geq \frac{2n^n}{3}}$$

Solution.

$$\sum_{cyc} log_{a_1}^n(a_2a_3\cdot\ldots\cdot a_{n+1})\cdot log_{a_1a_2^2a_3^2\cdot\ldots\cdot a_{n*1}^2} a_1 \overset{Cebâşev}{\geq}$$

$$\geq \frac{1}{n}\left(\sum_{czc} log_{a_1}^n(a_2a_3\cdot\ldots\cdot a_{n+1})\right)\left(\sum_{cyc} log_{a_1a_2^2a_3^2\cdot\ldots\cdot a_{n*1}^2} a_1\right) \quad (1)$$

$$log_{a_1}^n(a_2a_3\cdot\ldots\cdot a_{n+1}) = \left(log_{a_1} a_2 + log_{a_1} a_3 + \cdots + log_{a_1} a_{n+1}\right)^n \overset{AM-GM}{\geq}$$

$$\geq n^n\cdot log_{a_1} a_2\, log_{a_1} a_3\cdot\ldots\cdot log_{a_1} a_{n+1}$$

$$\sum_{cyc} log_{a_1}^n(a_2a_3\cdot\ldots\cdot a_{n+1}) \geq n^n \sum_{cyc} log_{a_1} a_2\, log_{a_1} a_3\cdot\ldots\cdot log_{a_1} a_{n+1} \overset{AM-GM}{\geq}$$

$$\geq n^n\cdot n\cdot \sqrt[n]{log_{a_1} a_2\, log_{a_2} a_1\cdot\ldots\cdot log_{a_n} a_{n+1}\, log_{a_{n+1}} a_n} = n^{n+1} \quad (2)$$

$$\sum_{cyc} log_{a_1a_2^2a_3^2\cdot\ldots\cdot a_{n*1}^2} a_1 = \sum_{cyc} \frac{lna_1}{lna_1 + 2(lna_2 + lna_3 + \cdots + lna_{n+1}} =$$

$$= \sum_{cyc} \frac{ln^2a_1}{ln^2a_1 + 2(lna_2lna_3 + \cdots + lna_nlna_{n+1})} \geq$$

$$\geq \frac{(lna_1 + lna_2 + \cdots + lna_{n+1})^2}{ln^2a_1 + ln^2a_2 + \cdots + ln^2a_{n+1} + 4(lna_1lna_2 + lna_2lna_3 + \cdots + lna_{n+1}lna_1)} \geq \frac{2}{3} \Leftrightarrow$$

$$ln^2a_1 + ln^2a_2 + \cdots + ln^2a_{n+1} \geq lna_1lna_2 + lna_2lna_3 + \cdots + lna_nlna_{n+1} \quad (3)$$

From (1,2,3) we get the problem.

Application 7. If $x_i > 1, \forall i = \overline{1,n}; n \in \mathbb{N}, n \geq 3$ then prove:

$$\boldsymbol{\frac{logx_2}{log^2(x_1^2x_2)} + \frac{logx_3}{log^2(x_1^2x_2^2x_3)} + \cdots + \frac{logx_n}{log^2(x_1^2x_2^2\ldots x_{n-1}^2x_n)} \leq \frac{log\sqrt[4]{x_2x_3\ldots x_n}}{logx_1\cdot log(x_1x_2x_3\ldots x_n)}}$$

Solution:

$$\frac{a_2}{a_1(a_1+a_2)}+\frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)}=\frac{a_2+a_3}{a_1(a_1+a_2+a_3)}$$

WLOG suppose:

$$\frac{a_2}{a_1(a_1+a_2)}+\frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)}+\cdots+\frac{a_n}{(a_1+a_2+\cdots+a_{n-1})(a_1+a_2+\cdots+a_n)}=\frac{a_2+a_3+\cdots+a_n}{a_1(a_1+a_2+\cdots+a_n)}\Rightarrow$$

$$\frac{a_2}{a_1(a_1+a_2)}+\frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)}+\cdots+\frac{a_n}{(a_1+a_2+\cdots+a_{n-1})(a_1+a_2+\cdots+a_n)}+\frac{a_{n+1}}{(a_1+a_2+\cdots+a_n)(a_1+a_2+\cdots+a_{n+1})}=$$

$$=\frac{a_2+a_3+\cdots+a_n}{a_1(a_1+a_2+\cdots+a_n)}+\frac{a_{n+1}}{(a_1+a_2+\cdots+a_n)(a_1+a_2+\cdots+a_{n+1})}=$$

$$=\frac{(a_2+a_3+\cdots+a_n)^2+a_1(a_2+a_3+\cdots+a_n)+a_{n+1}(a_2+a_3+\cdots+a_n)+a_1a_{n+1}}{(a_1+a_2+\cdots+a_n)a_1(a_1+a_2+\cdots+a_{n+1})}=$$

$$=\frac{a_2+a_3+\cdots+a_{n+1}}{a_1(a_1+a_2+\cdots+a_{n+1})}$$

From $(x+y)^2\geq 4xy\Rightarrow \frac{1}{xy}\geq\frac{4}{(x+y)^2}$ we have:

$$\frac{a_2}{a_1(a_1+a_2)}\geq\frac{4a_2}{(2a_1+a_2)^2}$$

$$\frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)}\geq\frac{4a_3}{(2a_1+2a_2+a_3)^2}$$

$$\frac{a_n}{(a_1+a_2+\cdots+a_{n-1})(a_1+a_2+\cdots+a_n)}\geq\frac{4a_n}{(2a_1+2a_2+\cdots+2a_{n-1}+a_n)^2}$$

Adding up relationships, we have:

$$\frac{4a_2}{(2a_1+a_2)^2}+\frac{4a_3}{(2a_1+2a_2+a_3)^2}+\cdots+\frac{4a_n}{(2a_1+2a_2+\cdots+2a_{n-1}+a_n)^2}\leq$$

$$\leq\frac{a_2}{a_1(a_1+a_2)}+\frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)}+\cdots+\frac{a_n}{(a_1+a_2+\cdots+a_{n-1})(a_1+a_2+\cdots+a_n)}=\frac{a_2+a_3+\cdots+a_n}{a_1(a_1+a_2+\cdots+a_n)}\Rightarrow$$

$$\frac{a_2}{(2a_1+a_2)^2}+\frac{a_3}{(2a_1+2a_2+a_3)^2}+\cdots+\frac{a_n}{(2a_1+2a_2+\cdots+2a_{n-1}+a_n)^2}\leq\frac{a_2+a_3+\cdots+a_n}{4a_1(a_1+a_2+\cdots+a_n)}$$

For: $a_1 = logx_1;\ a_2 = logx_2; \dots ; a_n = logx_n$ *we get:*

$$\frac{logx_2}{log^2(x_1^2x_2)} + \frac{logx_3}{log^2(x_1^2x_2^2x_3)} + \dots + \frac{logx_n}{log^2(x_1^2x_2^2 \dots x_{n-1}^1x_n)} \le \frac{log\sqrt[4]{x_2x_3 \dots x_n}}{logx_1 \cdot log(x_1x_2x_3 \dots x_n)}$$

Application 8. For $x, y, z > 2$ prove:

$$\log\left(\frac{(3+x)^2(3+y)^2(3+z)^2}{8}\right)^3 \le \sum_{cyc}(x^2+8)\sin\frac{\pi}{x}$$

Solution: If $x > 2$, then $\frac{\pi}{x} < \frac{\pi}{2}, \tan\frac{\pi}{x} > \frac{\pi}{x} > \frac{3}{x}$,

$$\cos^2\frac{\pi}{x} = \frac{1}{1+\tan^2\frac{\pi}{x}} < \frac{1}{1+\left(\frac{\pi}{x}\right)^2} < \frac{1}{1+\left(\frac{3}{x}\right)^2} = \frac{x^2}{x^2+9} \Leftrightarrow \sin\frac{\pi}{x} > \frac{3}{\sqrt{x^2+9}}; (1)$$

$$\log(1+t) \le \frac{t}{\sqrt{1+t}}, \forall t \ge 0; (2)$$

$$t = x^2+8; (2) \Rightarrow \log(9+x^2) \le \frac{8+x^2}{\sqrt{9+x^2}} \Leftrightarrow \frac{\log(9+x^2)}{8+x^2} \le \frac{1}{\sqrt{9+x^2}}; (3)$$

From (1), (2), (3) it follows that:

$$\frac{\log(9+x^2)}{8+x^2} \le \frac{1}{\sqrt{9+x^2}} \le \frac{1}{3}\sin\frac{\pi}{x} \Leftrightarrow \log(9+x^2) \le \frac{1}{3}(x^2+8)\sin\frac{\pi}{x}$$

$$\log\left(\frac{(3+x)^2}{2}\right) \le \frac{1}{3}(x^2+8)\sin\frac{\pi}{x}$$

Therefore,

$$\log\left(\frac{(3+x)^2(3+y)^2(3+z)^2}{8}\right)^3 \le \sum_{cyc}(x^2+8)\sin\frac{\pi}{x}$$

Application 9. If $a, b, c > 1$, then:

$$\sum_{cyc} log_{a+b}\left(1+b^{b+1}\right)\left(1+c^{c+1}\right) \ge 6(a+b)^{c-b}(b+c)^{a-c}(c+a)^{b-a}$$

Solution. From Bernoulli's inequality, we have: $\begin{cases} a^{a+1} = (1+a-1)^{a+1} \ge a^2 \\ b^{b+1} = (1+b-1)^{b+1} \ge b^2 \\ c^{c+1} = (1+c-1)^{c+1} \ge c^2 \end{cases}$

$$\begin{cases} (1+a^{a+1})\left(1+b^{b+1}\right) \ge (1+a^2)(1+b^2) \ge (a+b)^2 \\ \left(1+b^{b+1}\right)(1+c^{c+1}) \ge (1+b^2)(1+c^2) \ge (b+c)^2 \\ (1+c^{c+1})(1+a^{a+1}) \ge (1+c^2)(1+a^2) \ge (c+a)^2 \end{cases}$$

$$\sum_{cyc} log_{a+b}\left(1+b^{b+1}\right)(1+c^{c+1}) \geq \sum_{cyc} log_{a+b}(b+c)^2 =$$

$$= 2\sum_{cyc} log_{a+b}(b+c) \overset{Am-Gm}{\geq} 6\cdot\sqrt[3]{\prod_{cyc} log_{a+b}(b+c)} = 6 \quad (i)$$

$$\therefore x^x y^y z^z \geq x^z y^x z^y, \forall x,y,z > 1 \leftrightarrow (z-x)lnx + (x-y)lny + (y-z)lnz \leq 0$$

$$If\ 1 \leq x \leq y \leq z \to (lnx \leq lny \leq lnz, z-x \geq x-y) \overset{Chebyshev's}{\Rightarrow}$$

$$(z-x)lnx + (x-y)lny \leq \frac{1}{2}(z-y)\,ln(xy) = (z-y)ln\sqrt{xy}$$

$$\to (z-x)lnx + (x-y)lny + (y-z)lnz \leq (z-y)ln\sqrt{xy} + (y-z)lnz =$$

$$= (z-y)ln\frac{\sqrt{xy}}{z} \leq 0 \therefore$$

From: $x = b+c, y = c+a, z = a+b \to (a+b)^{c-b}(b+c)^{a-c}(c+a)^{b-a} \leq 1 \quad (ii)$

From$(i),(ii)$ it follows that:

$$\sum_{cyc} log_{a+b}\left(1+b^{b+1}\right)(1+c^{c+1}) \geq 6(a+b)^{c-b}(b+c)^{a-c}(c+a)^{b-a}$$

Application 10. Let $\boldsymbol{a_i \in (1,\infty), i = \overline{1,n}, n \in \mathbb{N}, a_1 + a_2 + \cdots + a_n = ne^4}$, then:

$$\boldsymbol{\sqrt{\log a_1^{a_2} + \log a_1^{a_3} + \cdots + \log a_1^{a_n}} + \cdots + \sqrt{\log a_n^{a_1} + \log a_n^{a_2} + \cdots + \log a_n^{a_{n-1}}} \leq 9n^2}$$

Solution:

$$\sqrt{log\, a_1^{a_2} + log\, a_1^{a_3} + \cdots + log\, a_1^{a_n}} + \cdots + \sqrt{log\, a_n^{a_1} + log\, a_n^{a_2} + \cdots + log\, a_n^{a_{n-1}}} =$$

$$= \sqrt{(a_2+a_3+\cdots+a_n)\,log\, a_1} + \cdots + \sqrt{(a_1+a_2+\cdots+a_{n-1})\,log\, a_n} \overset{CBS}{\leq}$$

$$\overset{CBS}{\leq} \sqrt{(a_2+a_3+\cdots+a_n)+\cdots+(a_1+a_2+\cdots+a_{n-1})}\cdot\sqrt{\sum_{k=1}^{n} log\, a_k} =$$

$$= \sqrt{(n-1)S_n}\cdot\sqrt{log(a_1a_2\cdot\ldots\cdot a_n)} \leq \sqrt{(n-1)S_n}\cdot\sqrt{log\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^n} =$$

$$= \sqrt{(n-1)S_n}\cdot\sqrt{n\cdot log\left(\frac{S_n}{n}\right)} = \sqrt{(n-1)ne^4}\cdot\sqrt{4n} = 2ne^2\cdot\sqrt{n-1} \leq$$

$$\leq 2ne^2\cdot\frac{n-1+1}{2} = n^2e^2 < 9n^2$$

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ABOUT AN INEQUALITY BY CHUYEN HA NAM-I

By Marin Chirciu-Romania

1) If $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}>0$ such that $\boldsymbol{abc}=\boldsymbol{8}$ then:

$$\frac{a}{ac+4}+\frac{b}{ab+4}+\frac{c}{bc+4}\le\frac{1}{16}(a^2+b^2+c^2)$$

Proposed by Chuyen Ha Nam-Vietnam

Solution. **Lemma.** **2) If $x,y>0$, then $\frac{1}{x+y}\le\frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}\right)$**

Proof. $\frac{1}{x+y}\le\frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}\right)\Leftrightarrow(x-y)^2\ge 0$, equality holds when $x=y$.

Let's get back to the main problem. Using Lemma, we get:

$$LHS=\sum_{cyc}\frac{a}{ac+4}\le\frac{1}{4}\sum_{cyc}a\left(\frac{1}{ac}+\frac{1}{4}\right)=\frac{1}{4}\sum_{cyc}\left(\frac{1}{c}+\frac{a}{4}\right)=\frac{1}{4}\sum_{cyc}\frac{1}{a}+\frac{1}{16}\sum_{cyc}a=$$

$$=\frac{1}{4}\cdot\frac{\sum bc}{abc}+\frac{1}{16}\sum_{cyc}a=\frac{1}{4}\cdot\frac{1}{8}\sum_{cyc}bc+\frac{1}{16}\sum_{cyc}a\overset{(1)}{\le}\frac{1}{32}\sum_{cyc}a^2+\frac{1}{16}\cdot\frac{1}{2}\sum_{cyc}a^2=\frac{1}{16}\sum_{cyc}a^2=$$

$=RHS$,where (1) it follows from $\sum bc\le\sum a^2$ and $\sum a\le\frac{1}{2}\sum a^2$, true from

$$\sum_{cyc}a^2\overset{Chebyshev}{\ge}\frac{1}{3}\sum_{cyc}a\cdot\sum_{cyc}a\overset{AM-GM}{\ge}\sqrt[3]{abc}\sum_{cyc}a=\sqrt[3]{8}\sum_{cyc}a=2\sum_{cyc}a.$$

Equality holds if and only id $a=b=c=2$.

3) If $a,b,c>0$ such that $abc=8^n, n\in\mathbb{N}$ then

$$\frac{a}{ac+4^n}+\frac{b}{ab+4^n}+\frac{c}{bc+4^n}\le\frac{1}{2^{3n+1}}(a^2+b^2+c^2)$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. 4) If , $y>0$, then $\frac{1}{x+y}\le\frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}\right)$**

Proof. $\frac{1}{x+y}\le\frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}\right)\Leftrightarrow(x-y)^2\ge 0$, equality holds when $x=y$.

Let's get back to the main problem. Using Lemma, we get:

$$LHS=\sum_{cyc}\frac{a}{ac+4^n}\le\frac{1}{4}\sum_{cyc}a\left(\frac{1}{ac}+\frac{1}{4^n}\right)=\frac{1}{4}\sum_{cyc}\left(\frac{1}{c}+\frac{a}{4^n}\right)=\frac{1}{4}\sum_{cyc}\frac{1}{a}+\frac{1}{4^{n+1}}\sum_{cyc}a=$$

$$=\frac{1}{4}\cdot\frac{\sum bc}{abc}+\frac{1}{4^{n+1}}\sum_{cyc}a=\frac{1}{4}\cdot\frac{1}{8^n}\sum_{cyc}bc+\frac{1}{4^{n+1}}\sum_{cyc}a\overset{(1)}{\le}\frac{1}{2^{3n+2}}\sum_{cyc}a^2+\frac{1}{2^{3n+2}}\sum_{cyc}a^2=\frac{1}{2^n}\sum_{cyc}a^2$$
$$=RHS$$

where (1) it follows from $\sum bc\le\sum a^2$ and $\sum a\le\frac{1}{2}\sum a^2$, true from

$$\sum_{cyc}a^2\overset{Chebyshev}{\ge}\frac{1}{3}\sum_{cyc}a\cdot\sum_{cyc}a\overset{AM-GM}{\ge}\sqrt[3]{abc}\sum_{cyc}a=\sqrt[3]{8}\sum_{cyc}a=2\sum_{cyc}a.$$

Equality holds if and only if $a=b=c=2^n$.

REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT AN INEQUALITY FROM *SAUDI ARABIA MATHEMATICAL COMPETITION*

By Marin Chirciu-Romania

1) If $a,b,c>0$ then

$$a\sqrt{3a^2+6b^2}+b\sqrt{3b^2+6c^2}+c\sqrt{3c^2+6a^2}\ge(a+b+c)^2$$

Saudi Arabia Mathematical Competition

Solution: $a\sqrt{3a^2+6b^2}+b\sqrt{3b^2+6c^2}+c\sqrt{3c^2+6a^2}\ge(a+b+c)^2$

$$\Leftrightarrow a\sqrt{3a^2+6b^2}+b\sqrt{3b^2+6c^2}+c\sqrt{3c^2+6a^2}\ge a^2+b^2+c^2+2ab+2bc+2ca$$

$$\left(a\sqrt{3a^2+6b^2}-a^2-2ab\right)+\left(b\sqrt{3b^2+6c^2}-b^2-2bc\right)+\left(c\sqrt{3c^2+6a^2}-c^2-2ca\right)\ge 0$$

$$a\left(\sqrt{3a^2+6b^2}-a-2b\right)+b\left(\sqrt{3b^2+6c^2}-b-2c\right)+c\left(\sqrt{3c^2+6a^2}-c-2a\right)\ge 0,(1)$$

which follows from: $\sqrt{3a^2+6b^2}-a-2b\ge 0\Leftrightarrow\sqrt{3a^2+6b^2}\ge a+2b\Leftrightarrow$

$3a^2+6b^2\ge(a+2b)^2\Leftrightarrow 2(a-b)^2\ge 0$. Equality holds for $a=b$.

Similarly, $\sqrt{3b^2+6c^2}-b-2c\ge 0$ and $\sqrt{3c^2+6a^2}-c-2a\ge 0$

Remark. The problem can be developed.

2) If $a,b,c\ge 0$ and $x\ge 1,y\ge 4,xy=4x+y$ then:

$$a\sqrt{xa^2+yb^2}+b\sqrt{xb^2+yc^2}+c\sqrt{xc^2+ya^2}\ge(a+b+c)^2$$

Marin Chirciu

Solution: $a\sqrt{xa^2+yb^2}+b\sqrt{xb^2+yc^2}+c\sqrt{xc^2+ya^2}\ge(a+b+c)^2\Leftrightarrow$

$$a\sqrt{xa^2+yb^2}+b\sqrt{xb^2+yc^2}+c\sqrt{xc^2+ya^2}\ge a^2+b^2+c^2+2ab+2bc+2ca\Leftrightarrow$$

$$\left(a\sqrt{xa^2+yb^2}-a^2-2ab\right)+\left(b\sqrt{xb^2+yc^2}-b^2-2bc\right)+\left(c\sqrt{xc^2+ya^2}-c^2-2ca\right)\ge 0$$

$$a\left(\sqrt{xa^2+yb^2}-a-2b\right)+b\left(\sqrt{xb^2+yc^2}-b-2c\right)+c\left(\sqrt{xc^2+ya^2}-c-2a\right)\ge 0;(1)$$

Inequality (1) it follows from: $\sqrt{xa^2+yb^2}-a-2b\ge 0\Leftrightarrow\sqrt{xa^2+yb^2}\ge a+2b\Leftrightarrow$

$xa^2+yb^2\ge(a+2b)^2\Leftrightarrow xa^2+yb^2\ge a^2+4ab+4b^2\Leftrightarrow$

$(x-1)a^2-4ab+(y-4)b^2\ge 0\Leftrightarrow\left(a\sqrt{x-1}-b\sqrt{y-4}\right)^2\ge 0$, which is true from conditions $x\ge 1,y\ge 4,xy=4x+y\Leftrightarrow(x-1)(y-4)=4$.

Equality holds for $a\sqrt{x-1}=b\sqrt{y-4}$.

Similarly, $\sqrt{xb^2+yc^2}-b-2c\ge 0$ and $\sqrt{xc^2+ya^2}-c-2a\ge 0\Rightarrow(1)$ −true.

Note.For $(x,y)=(3,6)$, we get problem Saudi Arabia Mathematical Competition, 2020.

3) If $a,b,c\ge 0$ then:

$$a\sqrt[3]{9(a^3+2b^3)}+b\sqrt[3]{9(b^3+2c^3)}+c\sqrt[3]{9(c^3+2a^3)}\ge(a+b+c)^2$$

Marin Chirciu

Solution: $a\sqrt[3]{9(a^3+2b^3)}+b\sqrt[3]{9(b^3+2c^3)}+c\sqrt[3]{9(c^3+2a^3)}\ge(a+b+c)^2\Leftrightarrow$

$$a\sqrt[3]{9(a^3+2b^3)}+b\sqrt[3]{9(b^3+2c^3)}+c\sqrt[3]{9(c^3+2a^3)}\ge a^2+b^2+c^2+2ab+2bc+2ca\Leftrightarrow$$

$$\left(a\sqrt[3]{9(a^3+2b^3)}-2a^2-2ab\right)+\left(b\sqrt[3]{9(b^3+2c^3)}-b^2-2bc\right)+\left(c\sqrt[3]{9(c^3+2a^3)}-2c^2-2ca\right)\ge 0$$

$$a\left(\sqrt[3]{9(a^3+2b^3)}-a-2b\right)+b\left(\sqrt[3]{9(b^3+2c^3)}-b-2c\right)+c\left(\sqrt[3]{9(c^3+2a^3)}-c-2a\right)\ge 0;(1)$$

which follows from: $\sqrt[3]{9(a^3+2b^3)}-a-2b\ge 0\Leftrightarrow$

$$\sqrt[3]{9(a^3+2b^3)}\ge a+2b\Leftrightarrow 9(a^3+2b^3)\ge(a+2b)^3\Leftrightarrow$$

$4a^3-3a^2b-6ab^2+5b^2\ge 0\Leftrightarrow(a-b)^2(4a+5b)\ge 0$. Equality holds for $a=b$.

Similarly, $\sqrt[3]{9(b^3+2c^3)}-b-2c\ge 0$ and $\sqrt[3]{9(c^3+2a^3)}-c-2a\ge 0$.

Equality hold if and only if $a=b=c$.

4) If $a,b,c\ge 0$ then:

$$\boldsymbol{a\sqrt[4]{27(a^4+2b^4)}+b\sqrt[4]{27(b^4+2c^4)}+c\sqrt[4]{27(c^4+2a^4)}\ge(a+b+c)^2}$$

Marin Chirciu

Solution: $a\sqrt[4]{27(a^4+2b^4)}+b\sqrt[4]{27(b^4+2c^4)}+c\sqrt[4]{27(c^4+2a^4)}\ge(a+b+c)^2\Leftrightarrow$

$$a\sqrt[4]{27(a^4+2b^4)}+b\sqrt[4]{27(b^4+2c^4)}+c\sqrt[4]{27(c^4+2a^4)}\ge a^2+b^2+c^2+2ab+2bc+2ca$$

$$\Leftrightarrow\left(a\sqrt[4]{27(a^4+2b^4)}-a^2-2ab\right)+\left(b\sqrt[4]{27(b^4+2c^4)}-2bc\right)+\left(c\sqrt[4]{27(c^4+2a^4)}-2ca\right)\ge 0$$

$$a\left(\sqrt[4]{27(a^4+2b^4)}-a-2b\right)+b\left(\sqrt[4]{27(b^4+2c^4)}-b-2c\right)+c\left(\sqrt[4]{27(c^4+2a^4)}-c-2a\right)\ge 0,$$

which is true from $\sqrt[4]{27(a^4+2b^4)}-a-2b\ge 0\Leftrightarrow$

$$\sqrt[4]{27(a^4+2b^4)}\ge a+2b\Leftrightarrow 27(a^4+2b^4)\ge(a+2b)^4\Leftrightarrow$$

$$13a^4-4a^3b-12a^2b^2-16ab^3+19b^4\ge 0\Leftrightarrow(a-b)^2(13a^2+22ab+19b^2)\ge 0$$

Equality holds for $a=b$

Similarly, $\sqrt[4]{27(b^4+2c^4)}-b-2c\ge 0$ and $\sqrt[4]{27(c^4+2a^4)}-c-2a\ge 0$.

Equality holds if and only if $a=b=c$.

5) If $a,b,c\ge 0$ and $n\in\mathbb{N},n\ge 2$ then:

$$\boldsymbol{a\sqrt[n]{3^{n-1}(a^n+2b^n)}+b\sqrt[n]{3^{n-1}(b^n+2c^n)}+c\sqrt[n]{3^{n-1}(c^n+2b^n)}\ge(a+b+c)^2}$$

Marin Chirciu

Solution: $a\sqrt[n]{3^{n-1}(a^n+2b^n)}+b\sqrt[n]{3^{n-1}(b^n+2c^n)}+c\sqrt[n]{3^{n-1}(c^n+2b^n)}\ge(a+b+c)^2\Leftrightarrow$

$$a\sqrt[n]{3^{n-1}(a^n+2b^n)}+b\sqrt[n]{3^{n-1}(b^n+2c^n)}+c\sqrt[n]{3^{n-1}(c^n+2b^n)}$$
$$\geq a^2+b^2+c^2+2ab+2bc+2ca \Leftrightarrow$$

$$\left(a\sqrt[n]{3^{n-1}(a^n+2b^n)}-a^2-2ab\right)+\left(b\sqrt[n]{3^{n-1}(b^n+2c^n)}-2b^2-2bc\right)+\left(c\sqrt[n]{3^{n-1}(c^n+2b^n)}-c^2-2ca\right)\geq 0,$$ (1) which follows from

$$a\sqrt[n]{3^{n-1}(a^n+2b^n)}-a^2-2ab\geq 0 \Leftrightarrow a\sqrt[n]{3^{n-1}(a^n+2b^n)}\geq a^2+2ab \Leftrightarrow$$

$$3^{n-1}(a^n+2b^n)\geq (a+2b)^n \Leftrightarrow \frac{a^n+2b^n}{3}\geq\left(\frac{a+2b}{3}\right)^n \ (Jensen).$$

Equality holds for $a=b$.

Similarly, $b\sqrt[n]{3^{n-1}(b^n+2c^n)}-2b^2-2bc\geq 0$ and $c\sqrt[n]{3^{n-1}(c^n+2b^n)}-c^2-2ca\geq 0$.

Equality holds if $a=b=c$.

Note: For $n=2$, we get Problem 2, Saudi Arabia Mathematical Competition, 2020.

ABOUT RECURRENCE RELATIONSHIPS FOR REAL NUMBER SEQUENCES-(IV)

By Marian Ursărescu and Florică Anastase-Romania

Abstract: *In this paper are presented few special techniques for recurrent limits.*

Ap.1) Find:

$$\boldsymbol{\Omega_n=\lim_{x\to 0}\frac{1-\sqrt{1+x^2}\cdot\sqrt[3]{1+x^2}\cdot\ldots\cdot\sqrt[n]{1+x^2}}{x^2}, n\in\mathbb{N}}$$

Solution. Let be $f_k(x)=\sqrt[k]{1+x^2}, k\in\overline{2,n}, n\in\mathbb{N}$, then

$$f_k'(x)=\frac{2x}{k\sqrt[k]{(1+x^2)^{k-1}}}$$

$$\Omega_n=\lim_{x\to 0}\frac{1-\sqrt{1+x^2}\cdot\sqrt[3]{1+x^2}\cdot\ldots\cdot\sqrt[n]{1+x^2}}{x^2}=$$

$$=\lim_{x\to 0}\frac{1-\prod_{k=2}^n f_k(x)}{x^2}=\lim_{x\to 0}\frac{(1-\prod_{k=2}^n f_k(x))'}{2x}=$$

$$=-\lim_{x\to 0}\frac{(\prod_{k=2}^n f_k(x))'}{2x}=-\frac{1}{2}\lim_{x\to 0}\left[\prod_{k=2}^n f_k(x)\cdot\sum_{k=2}^n\left(\frac{f_k'(x)}{x}\cdot\frac{1}{f_k(x)}\right)\right]=$$

$$= -\frac{1}{2}\sum_{k=2}^{n}\lim_{x\to 0}\frac{f_k'(x)}{x} = -\frac{1}{2}\sum_{k=2}^{n}\frac{2}{k\sqrt[k]{(1+x)^{k-1}}} = -\sum_{k=1}^{n}\frac{1}{k}$$

Ap.2) Find:

$$\boldsymbol{\Omega_n = \lim_{x\to\infty}\frac{1-\cos x\cdot\cos 2x\cdot\ldots\cdot\cos nx}{x^2}, n} \in \mathbb{N}^*$$

Solution. For $n = 1$, we have:

$$\Omega_1 = \lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2} = \frac{1}{2}\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}$$

Now, we have:

$$\Omega_n - \Omega_{n-1} = \lim_{x\to 0}\frac{(1-\cos nx)\cos x\cdot\cos 2x\cdot\ldots\cdot\cos(n-1)x}{x^2} =$$

$$= \lim_{x\to 0}\frac{1-\cos nx}{x^2} = \lim_{x\to 0}\frac{2\sin^2\frac{nx}{2}}{x^2} = \frac{n^2}{2}\lim_{x\to 0}\left(\frac{\sin\frac{nx}{2}}{\frac{nx}{2}}\right)^2 = \frac{n^2}{2}$$

Hence, $\Omega_n - \Omega_{n-1} = \frac{n^2}{2}$. Therefore,

$$\begin{cases} \Omega_2 - \Omega_1 = \frac{2^2}{2} \\ \Omega_3 - \Omega_2 = \frac{3^2}{2} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \Omega_n - \Omega_{n-1} = \frac{n^2}{2} \end{cases}$$

By adding, we get:

$$\Omega_n = \Omega_1 + \frac{2^2+3^2+\cdots+n^2}{2} = \frac{n(n+1)(2n+1)}{12}$$

We'll prove using the mathematical induction.

$$P(1)\colon \Omega_1 = \frac{1\cdot 2\cdot 3}{12} = \frac{1}{2}$$

We consider $P(n-1)\colon \Omega_{n-1} = \frac{(n-1)n(2n-1)}{12}$ to be true. Thus, from $\Omega_n - \Omega_{n-1} = \frac{n^2}{2}$

we get: $\Omega_n = \Omega_{n-1} + \frac{n^2}{2} = \frac{(n-1)n(2n-1)}{12} + \frac{n^2}{2} = \frac{n(2n^2+3n+1)}{12} = \frac{n(n+1)(2n+1)}{12}$

Therefore,

$$\Omega_n = \frac{n(n+1)(2n+1)}{12}$$

Ap.3) Find:

$$\boldsymbol{\Omega_n = \lim_{x\to 0} \frac{n!\,x^n - \sin x\cdot \sin 2x\cdot \ldots \cdot \sin nx}{x^{n+2}}, n\in \mathbb{N}^*}$$

Solution. For $n = 1$ we have:

$$\Omega_1 = \lim_{x\to 0}\frac{x-\sin x}{x^3} = \frac{1}{6}$$

$$\Omega_n - n\Omega_{n-1} = \lim_{x\to 0}\frac{nx\cdot \sin x\cdot \ldots \cdot \sin(n-1)x - \sin x\cdot \sin 2x\cdot \ldots \cdot \sin nx}{x^{n+2}} =$$

$$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin 2x}{x}\cdot \ldots \cdot \frac{\sin(n-1)x}{x}\cdot \lim_{x\to 0}\frac{nx-\sin nx}{x^3} =$$

$$= (n-1)!\lim_{x\to 0}\frac{n-n\cos nx}{3n^2} = n!\lim_{x\to 0}\frac{n\sin nx}{6n} = n!\cdot\frac{n^2}{6}$$

Hence, $\Omega_n = n\Omega_{n-1} + n!\cdot\frac{n^2}{6}$

Let $P(n)$: $\Omega_n = \frac{n(2n+1)}{36}(n+1)!$

$$P(1): \Omega_1 = \frac{1\cdot 3\cdot 2!}{36} = \frac{1}{6}$$

Suppose that $P(n)$: $\Omega(n-1) = \frac{(n-1)(2n-1)n!}{36}$, then

$$\Omega_n = n\frac{(n-1)(2n-1)n!}{36} + \frac{n^2 n!}{6} = \frac{n!\,n}{36}[(n-1)(2n-1)+6n] = \frac{n!\,n(n+1)(2n+1)}{36}$$

Ap.4) Find:

$$\boldsymbol{\Omega_n = \lim_{x\to 0} \frac{n!\,x^n - \tan x\cdot \tan 2x\cdot \ldots \cdot \tan nx}{x^{n+2}}, n\in \mathbb{N}^*}$$

Solution. Let us denote:

$$f_k(x) = \frac{\tan kx}{kx}; \lim_{x\to 0} f_k(x) = 1 \Rightarrow \lim_{x\to 0}\prod_{k=1}^{n} f_k(x) = 1$$

$$\Omega_n = \lim_{x\to 0} n!\,x^n\cdot\frac{1-\prod_{k=1}^{n}\frac{\tan kx}{kx}}{x^{n+2}} = -n!\cdot\lim_{x\to 0}\frac{\prod_{k=1}^{n}\frac{\tan kx}{kx}-1}{x^2} =$$

$$= -n!\lim_{x\to 0}\frac{e^{\log\prod_{k=1}^{n} f_k(x)}-1}{\log\prod_{k=1}^{n} f_k(x)}\cdot\frac{\log\prod_{k=1}^{n} f_k(x)}{x^2}$$

$$\text{But: } \lim_{x\to 0}\frac{\log\prod_{k=1}^n f_k(x)}{x^2}=\lim_{x\to 0}\sum_{k=1}^n\frac{\log f_k(x)}{x^2}=\sum_{k=1}^n\frac{k^2}{3},\text{where}$$

$$\lim_{x\to 0}\frac{\log f_k(x)}{x^2}=\lim_{x\to 0}\frac{\log(1+f_k(x)-1)}{f_k(x)-1}\cdot\frac{f_k(x)-1}{x^2}=\lim_{x\to 0}\frac{\tan kx-kx}{kx^3}=$$

$$=k^2\cdot\lim_{x\to 0}\frac{\tan kx-kx}{(kx)^3}=\frac{k}{3},\text{where}$$

$$\lim_{x\to 0}\frac{\tan x-x}{x^3}=\frac{1}{3}$$

Therefore,

$$\Omega_n=\lim_{x\to 0}\frac{n!\,x^n-\tan x\cdot\tan 2x\cdot\ldots\cdot\tan nx}{x^{n+2}}=-\frac{n!}{3}\sum_{k=1}^n k^2=-n!\cdot\frac{n(n+1)(2n+1)}{18}$$

Ap.5) Find:

$$\boldsymbol{\Omega_n=\lim_{x\to 0}\frac{n!\,x^n-\sin^{-1}x\cdot\sin^{-1}2x\cdot\ldots\cdot\sin^{-1}nx}{x^{n+2}},n\in\mathbb{N}^*}$$

Ap.6) Find:

$$\boldsymbol{\Omega_n=\lim_{x\to 0}\frac{n!\,x^n-\tan^{-1}x\cdot\tan^{-1}2x\cdot\ldots\cdot\tan^{-1}nx}{x^{n+2}},n\in\mathbb{N}^*}$$

Ap.7) Find:

$$\boldsymbol{\Omega_n=\lim_{x\to 0}\frac{n!\,x^n-\log(1+x)\cdot\log(1+2x)\cdot\ldots\cdot\log(1+nx)}{x^{n+1}},n\in\mathbb{N}^*}$$

Solution. Let us denote:

$$f_k(x)=\frac{\log(1+kx)}{kx},\text{then }\lim_{x\to 0}f_k(x)=1\text{ and }\lim_{x\to 0}\prod_{k=1}^n f_k(x)=1$$

$$\Omega_n=\lim_{x\to 0}\frac{n!\,x^n}{x^{n+1}}\left(1-\prod_{k=1}^n\frac{\log(1+kx)}{kx}\right)=-n!\cdot\lim_{x\to 0}\frac{e^{\log\prod_{k=1}^n f_k(x)}-1}{x}=$$

$$=-n!\cdot\lim_{x\to 0}\frac{e^{\log\prod_{k=1}^n f_k(x)}-1}{\log\prod_{k=1}^n f_k(x)}\cdot\frac{\log\prod_{k=1}^n f_k(x)}{x}=-n!\cdot\lim_{x\to 0}\sum_{k=1}^n\frac{\log f_k(x)}{x}$$

$$\text{But: }\lim_{x\to 0}\frac{\log f_k(x)}{x}=\lim_{x\to 0}\frac{\log(1+f_k(x)-1)}{f_k(x)-1}\cdot\frac{f_k(x)-1}{x}=\lim_{x\to 0}\frac{\frac{\log(1+kx)}{kx}-1}{x}=$$

$$=k\cdot\lim_{x\to 0}\frac{\log kx-kx}{(kx)^2}=-\frac{k}{2}$$

Therefore,

$$\Omega_n = \lim_{x\to 0}\frac{n!\,x^n - \log(1+x)\cdot\log(1+2x)\cdot\ldots\cdot\log(1+nx)}{x^{n+1}} = (-n!)\sum_{k=1}^{n}\left(-\frac{k}{2}\right) = n!\frac{n(n+1)}{4}$$

Ap.8) Find:

$$\boldsymbol{\Omega_n = \lim_{x\to 0}\frac{n!\,x^n - (e^x-1)(e^{2x}-1)\cdot\ldots\cdot(e^{nx}-1)}{x^{n+1}}, n\in\mathbb{N}^*}$$

Ap.9) Find:

$$\boldsymbol{\Omega_n = \lim_{x\to 0}\frac{n!\tan x\cdot\tan\frac{x}{2}\cdot\ldots\cdot\tan\frac{x}{n} - x^n}{x^{n+2}}, n\in\mathbb{N}}$$

Ap.10) Find:

$$\boldsymbol{\Omega_n = \lim_{x\to 0}\frac{1-\cosh x\cdot\cosh^2 2x\cdot\ldots\cdot\cosh^n nx}{x^2}, n\in\mathbb{N}^*}$$

Solution. We have:

$$-\frac{\prod_{k=1}^{n}\cosh^k kx - 1}{x^2} = -\frac{e^{\log\prod_{k=1}^{n}\cosh^k x} - 1}{x^2} = -\frac{e^{\sum_{k=1}^{n} k\log\cosh kx} - 1}{\sum_{k=1}^{n} k\log\cosh kx}\cdot\frac{\sum_{k=1}^{n} k\log\cosh kx}{x^2}$$

$$\lim_{x\to 0}\frac{k\log\cosh kx}{x^2} = \lim_{x\to 0}\frac{k\log(1+\cosh kx - 1)}{\cosh kx - 1}\cdot\frac{\cosh kx - 1}{k^2x^2} = \frac{k}{2}\cdot k^2 = \frac{k^3}{2}$$

Because: $\lim\limits_{x\to 0}\frac{\cosh x - 1}{x^2} = \frac{1}{2}$

Therefore,

$$\Omega_n = \lim_{x\to 0}\frac{1-\cosh x\cdot\cosh^2 2x\cdot\ldots\cdot\cosh^n nx}{x^2} = -\frac{n^2(n+1)^2}{12}$$

Ap.11) Find:

$$\boldsymbol{\Omega_n = \lim_{x\to 0}\frac{n!\,x^n - \sinh x\cdot\sinh 2x\cdot\ldots\cdot\sinh nx}{x^{n+2}}}$$

Ap.12) Find:

$$\boldsymbol{\Omega_n = \lim_{x\to 0}\frac{\cosh x\cdot\cosh 2x\cdot\cosh 2^2x\cdot\ldots\cdot\cosh 2^n x - e^{x^2}}{x^2}, n\in\mathbb{N}}$$

Ap.13) Find:

$$\boldsymbol{\Omega_n = \lim_{x\to 0}\frac{a_1a_2\cdot\ldots\cdot a_n x^n - \mathrm{trig}\,a_1x\cdot\mathrm{trig}\,a_2x\cdot\ldots\cdot\mathrm{trig}\,a_nx}{x^{n+2}}, a_i\in\mathbb{R}^*,}$$

$$i = \overline{1, n}, \textbf{ where trig} \in \{\sin, \tan, \sinh, \tanh\}$$

Solution. Let us denote trig $a_k x = f(a_k x)$

$$\lim_{t\to 0} \frac{\text{trig } t}{t} = \lim_{t\to 0} \frac{f(t)}{t} = 1. \text{We have:}$$

$$\lim_{x\to 0} \frac{a_1 a_2 \cdot \ldots \cdot a_n x^n \left(1 - \frac{f(a_1 x)}{a_1 x} \cdot \ldots \cdot \frac{f(a_n x)}{a_n x}\right)}{x^{n+2}} = -a_1 a_2 \cdot \ldots \cdot a_n \lim_{x\to 0} \frac{e^{\log \prod_{k=1}^{n} \frac{f(a_k x)}{a_k x}} - 1}{x^2} =$$

$$= -a_1 a_2 \cdot \ldots \cdot a_n \lim_{x\to 0} \frac{e^{\log \prod_{k=1}^{n} \frac{f(a_k x)}{a_k x}} - 1}{\log \prod_{k=1}^{n} \frac{f(a_k x)}{a_k x}} \cdot \frac{\log \prod_{k=1}^{n} \frac{f(a_k x)}{a_k x}}{x^2}$$

$$\lim_{x\to 0} \frac{\log \prod_{k=1}^{n} \frac{f(a_k x)}{a_k x}}{x^2} = \lim_{x\to 0} \frac{\sum_{k=1}^{n} \log \frac{f(a_k x)}{a_k x}}{x^2} =$$

$$= \lim_{x\to 0} \sum_{k=1}^{n} \frac{\log\left(1 + \frac{f(a_k x)}{a_k x} - 1\right)}{\frac{f(a_k x)}{a_k x} - 1} \cdot \frac{f(a_k x) - a_k x}{(a_k x)^3} \cdot a_k^2 = \begin{cases} \frac{1}{6} \sum_{k=1}^{n} a_k^2, f \in \{\sinh, \sin\} \\ \frac{1}{3} \sum_{k=1}^{n} a_k^2, f \in \{\tan, \tanh\} \end{cases}$$

Ap.14) Find:

$$\boldsymbol{\Omega_n = \lim_{n\to\infty} \frac{\left(\frac{\pi}{2}\right)^n - \cos^{-1} x \cdot \cos^{-1} 2x \cdot \ldots \cdot \cos^{-1} nx}{x}, n \in \mathbb{N}^*}$$

Solution. We have:

$$\Omega_n = \frac{\pi}{2} \Omega_{n-1} + \lim_{x\to 0} \frac{\frac{\pi}{2} - \cos^{-1} nx}{x} \cdot \prod_{k=1}^{n-1} \cos^{-1}(n-1)x =$$

$$= \frac{\pi}{2} \Omega_{n-1} + \left(\frac{\pi}{2}\right)^{n-1} \cdot \lim_{y\to\frac{\pi}{2}} \frac{\frac{\pi}{2} - y}{\cos y} = \frac{\pi}{2} \Omega_{n-1} + \lim_{y\to\frac{\pi}{2}} \frac{\frac{\pi}{2} - t}{\sin\left(\frac{\pi}{2} - y\right)} \cdot \left(\frac{\pi}{2}\right)^{n-1} \cdot n = \frac{\pi}{2} \Omega_{n-1} + n \left(\frac{\pi}{2}\right)^{n-1}$$

Hence,

$$\left(\frac{2}{\pi}\right)^n \Omega_n = \Omega_{n-1} \left(\frac{2}{\pi}\right)^{n-1} + \frac{2n}{\pi}. \text{ Denote } \omega_n = \left(\frac{2}{\pi}\right)^n \Omega_n, \forall n \in \mathbb{N}^*, \text{then}$$

$$\omega_n = \omega_{n-1} + \frac{2n}{\pi}, \forall n \geq 2. \text{Hence,} \sum_{k=2}^{n} \omega_k = \sum_{k=1}^{n-1} \omega_k + \frac{2}{\pi} \sum_{k=2}^{n} k \Rightarrow \omega_n = \omega_1 + \frac{2}{\pi}(2 + 3 + \cdots + n); (1)$$

$$\omega_1 = \frac{2}{\pi} \cdot \lim_{x\to 0} \frac{\frac{\pi}{2} - \cos^{-1} x}{x} = \frac{2}{\pi} \cdot \lim_{y\to\frac{\pi}{2}} \frac{\frac{\pi}{2} - y}{\cos y} = \frac{2}{\pi}; (2)$$

$$(1)+(2) \Rightarrow \omega_n = \frac{2}{\pi} \cdot \frac{n(n+1)}{2} \Rightarrow \Omega_n = \left(\frac{\pi}{2}\right)^{n-1} - \frac{n(n+1)}{2}$$

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ABOUT AN INEQUALITY BY WALTER JANOUS-I

By *Marin Chirciu – Romania*

1) Show that:

$$\sum a \tan A \geq 10R - 2r$$

for any acute triangle ABC, where a, b, c are its sides, R its circumradius, and$r$ its inradius, and the sum is cyclic.

Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Solution *We prove the following lemma:*

Lemma.

2) In ΔABC the following relationship holds:

$$\sum a\tan A = 2r\cdot\frac{(4R+r)(2R+r)-s^2}{s^2-(2R+r)^2}$$

Proof. *We have* $\sum a\tan A = \sum 2R\sin A\cdot\frac{\sin A}{\cos A} = 2R\sum\frac{\sin^2 A}{\cos A} = 2R\sum\frac{1-\cos^2 A}{\cos A} =$

$2R\sum\left(\frac{1}{\cos A}-1\right) == 2R\sum\left(\frac{s^2+r^2-4R^2}{s^2-(2R+r)^2}-1\right) = 2r\cdot\frac{(4R+r)(2R+r)-s^2}{s^2-(2R+r)^2}$, *which follows from the known inequality in triangle* $\sum\frac{1}{\cos A} = \frac{s^2+r^2-4R^2}{s^2-(2R+r)^2}$. *Let's get back to the main problem.*

Using the lemma the inequality can be written:

$2r\cdot\frac{(4R+r)(2R+r)-s^2}{s^2-(2R+r)^2} \ge 10R-2r \Leftrightarrow r(4R+r)(2R+r)+(5R-r)(2R+r)^2 \ge 5Rs^2$, *which follows from Gerretsen's inequality* $s^2 \le 4R^2+4Rr+3r^2$ *and the observation that in the acute angled triangle* $s^2-(2R+r)^2 > 0$, *which assures the elimination of the denominator from the above inequality. It remains to prove that:*

$r(4R+r)(2R+r)+(5R-r)(2R+r)^2 \ge 5R(4R^2+4Rr+3r^2) \Leftrightarrow R \ge 2r$, *(Euler's inequality). Equality holds if and only if the triangle is equilateral.*

Remark *Let's strength the above inequality:*

3) In ΔABC the following inequality holds:

$$\sum a\tan A \ge xR+(18-2x)r,\ \text{where } x \le 18$$

Marin Chirciu

Solution *Using the Lemma the inequality can be rewritten:*

$$\sum a\tan A = 2r\cdot\frac{(4R+r)(2R+r)-s^2}{s^2-(2R+r)^2} \ge xR+(18-2x)r \Leftrightarrow$$

$$\Leftrightarrow 2r(4R+r)(2R+r)+(2R+r)^2[xR+(18-2x)r] \ge s^2[xR+(20-2x)r]$$

which follows from Gerretsen's inequality $s^2 \le 4R^2+4Rr+3r^2$ *and the observation that in the acute-angled triangle* $s^2-(2R+r)^2 > 0$, *which assures the elimination of the denominator in the above inequality. It remains to prove that:*

$$2r(4R+r)(2R+r)+(2R+r)^2[xR+(18-2x)r] \ge (4R^2+4Rr+3r^2)[xR+(20-2x)r] \Leftrightarrow$$

$\Leftrightarrow 4R^2+(2-x)Rr+(2x-20)r^2 \ge 0 \Leftrightarrow (R-2r)[4R+(10-x)r] \ge 0$, *obviously from Euler's inequality* $R \ge 2r$ *and the condition* $x \le 18$, *which assures* $[4R+(10-x)r] \ge 0$

Equality holds if and only if the triangle is equilateral.

Remark 1. *For $x=10$ we obtain Problem 1424 from Crux Mathematicorum Vol. 15, No.3 March 1989, Walther Janous, Innsbruck, Austria.*

Remark 2. *The best inequality having the form 2) it is obtained for $x=18$. In this case we obtain:*

4) In ΔABC the following inequality holds:

$$\sum \boldsymbol{a}\tan \boldsymbol{A} \geq \mathbf{18}(\boldsymbol{R}-\boldsymbol{r})$$

Marin Chirciu

Remark 3. *Inequality 4) is stronger then inequality 1)*

5) In ΔABC the following inequality holds:

$$\sum \boldsymbol{a}\tan \boldsymbol{A} \geq \mathbf{18}(\boldsymbol{R}-\boldsymbol{r}) \geq \mathbf{10}\boldsymbol{R}-\mathbf{2}\boldsymbol{r}$$

Solution *See inequality 4) and $18(R-r)\geq 10R-2r \Leftrightarrow R\geq 2r$ (Euler's inequality)*

Equality holds if and only if the triangle is equilateral.

REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT AN INEQUALITY BY TAN TEY TAN-I

By Marin Chirciu – Romania

1) If $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}>0$ then:

$$\frac{\boldsymbol{x}}{\mathbf{2}\boldsymbol{y}}+\frac{\boldsymbol{y}}{\mathbf{2}\boldsymbol{z}}+\frac{\boldsymbol{z}}{\mathbf{2}\boldsymbol{x}}+\frac{\mathbf{8}\boldsymbol{xyz}}{(\boldsymbol{x}+\boldsymbol{y})(\boldsymbol{y}+\boldsymbol{z})(\boldsymbol{z}+\boldsymbol{x})}\geq\frac{\mathbf{5}}{\mathbf{2}}$$

Proposed by Tan Tey Tan-China

Solution: Using the means inequality:

$$LHS=\frac{x}{2y}+\frac{y}{2z}+\frac{z}{2x}+\frac{8xyz}{(x+y)(y+z)(z+x)}=\frac{1}{2}\sum\frac{x}{y}+\frac{8xyz}{\prod(x+y)}=$$

$$=\frac{1}{2}\left(\sum\frac{x+y}{y}-3\right)+\frac{8xyz}{\prod(y+z)}=\frac{1}{2}\sum\frac{x+y}{y}-\frac{3}{2}+\frac{8xyz}{\prod(y+z)}=$$

$$=\sum\frac{x+y}{2y}+\frac{8xyz}{\prod(y+z)}-\frac{3}{2}\overset{AGM}{\geq}4\sqrt[4]{\frac{x+y}{2y}\cdot\frac{y+z}{2z}\cdot\frac{z+x}{2y}\cdot\frac{8xyz}{\prod(y+z)}}=4-\frac{3}{2}=\frac{5}{2}=RHS$$

Equality holds if and only if $x = y = z$. **Remark:**The problem can be developed.

2) If $x, y, z > 0$ then:

$$\frac{x}{2y}+\frac{y}{2z}+\frac{z}{2t}+\frac{t}{2x}+\frac{16xyzt}{(x+y)(y+z)(z+t)(t+x)} \geq 3$$

Marin Chirciu

Solution: Using the means inequality we obtain:

$$LHS = \frac{x}{2y}+\frac{y}{2z}+\frac{z}{2t}+\frac{t}{2x}+\frac{16xyzt}{(x+y)(y+z)(z+t)(t+x)} = \frac{1}{2}\sum\frac{x}{y}+\frac{16xyzt}{\prod(x+y)} =$$
$$= \frac{1}{2}\left(\sum\frac{x+y}{y}-4\right)+\frac{16xyzt}{\prod(x+y)} = \frac{1}{2}\sum\frac{x+y}{y}-2+\frac{16xyzt}{\prod(x+y)} =$$
$$= \sum\frac{x+y}{2y}+\frac{16xyz}{\prod(x+y)}-2 \overset{AGM}{\geq} 5\sqrt[5]{\frac{x+y}{2y}\cdot\frac{y+z}{2z}\cdot\frac{z+t}{2t}\cdot\frac{t+x}{2x}\cdot\frac{16xyzt}{\prod(x+y)}}-2 = 5-2 = 3$$
$$= RHS$$

Equality if and only if $x = y = z = t$. **Remark:** The problem can be developed.

3) If $x_1, x_2, \dots, x_n > 0$ then:

$$\frac{x_1}{2x_2}+\frac{x_2}{2x_3}+\cdots+\frac{x_n}{2x_1}+\frac{2^n x_1x_2\dots x_n}{(x_1+x_2)(x_2+x_3)\dots(x_n+x_1)} \geq 1+\frac{1}{2}n$$

Marin Chirciu

Solution: Using means inequality we obtain:

$$LHS = \frac{x_1}{2x_2}+\frac{x_2}{2x_3}+\cdots+\frac{x_n}{2x_1}+\frac{2^n x_1x_2\dots x_n}{(x_1+x_2)(x_2+x_3)\dots(x_n+x_1)} =$$
$$= \frac{1}{2}\sum\frac{x_1}{x_2}+\frac{2^n x_1x_2\dots x_n}{\prod(x_1+x_2)} =$$
$$= \frac{1}{2}\left(\sum\frac{x_1+x_2}{x_2}-n\right)+\frac{2^n x_1x_2\dots x_n}{\prod(x_1+x_2)} = \frac{1}{2}\sum\frac{x_1+x_2}{x_2}-\frac{1}{2}n+\frac{2^n x_1x_2\dots x_n}{\prod(x_1+x_2)} =$$
$$= \sum\frac{x_1+x_2}{2x_2}+\frac{2^n x_1x_2\dots x_n}{\prod(x_1+x_2)}-\frac{1}{2}n \overset{AGM}{\geq}$$
$$\geq (n+1)\sqrt[n+1]{\frac{x_1+x_2}{2x_2}\cdot\frac{x_2+x_3}{2x_3}\cdot\ldots\cdot\frac{x_n+x_1}{2x_1}\cdot\frac{2^n x_1x_2\dots x_n}{\prod(x_1+x_2)}} = n+1-\frac{1}{2}n = 1+\frac{1}{2}n = RHS$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Note: For $n = 3$ we obtain the proposed problem by Tan Tey Tan in RMM 10/2020.

REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-WWW.SSMRMH.RO

ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-(XIX)

By Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{a^2}\ge\frac{27}{4(4R+r)}$$

Proposed by Marian Ursărescu-Romania

Solution. **Lemma 2) In ΔABC the following relationship holds:**

$$\sum_{cyc}\frac{m_a}{a^2}\ge\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)$$

Proof. Using Tereshin inequality, we get:

$$\sum_{cyc}\frac{m_a}{a^2}\ge\sum_{cyc}\frac{\frac{b^2+c^2}{4R}}{a^2}=\frac{1}{4R}\sum_{cyc}\frac{b^2+c^2}{a^2}\ge\frac{1}{4R}\cdot\frac{3}{8}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)=$$

$$=\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right),\qquad \text{which follows from}$$

$$\sum_{cyc}\frac{b^2+c^2}{a^2}=\frac{s^6+s^4(r^2-12Rr)+s^2r^2(24R^2+8Rr-r^2)-r^3(4R+r)^3}{8R^2r^2s^2}=$$

$$=\frac{1}{8R^2r^2}\left[s^4+s^2(r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{s^2}\right]=$$

$$=\frac{1}{8R^2r^2}\left[s^2(s^2+r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{s^2}\right]\overset{Gerretsen}{\ge}$$

$$\ge\frac{1}{8R^2r^2}\left[(16Rr-5r^2)(16Rr-5r^2+r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{1}{8R^2r^2}[(16Rr-5r^2)(4Rr-4r^2)+r^2(24R^2+8Rr-r^2)-r^2(R+r)(4R+r)(4R+r)]=$$

$$=\frac{1}{8R^2}[(16R-5r)(4R-4r)+(24R^2+8R-r^2)-(R+r)(4R+r)]=$$

$$=\frac{1}{8R^2}[64R^2-64Rr-20Rr+20r^2+(24R^2+8Rr-r^2)-4R^2-5Rr-r^2]=$$

$$=\frac{84R^2-81Rr+18r^2}{8R^2}=\frac{3}{8}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)$$

Let's get back to the main problem. Using Lemma , it follows that:

$$E=\sum_{cyc}\frac{m_a}{a^2}\geq\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)\overset{(1)}{\geq}\frac{27}{4(4R+r)}=RHS$$

$$(1)\Leftrightarrow\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)\geq\frac{27}{4(4R+r)}\Leftrightarrow 40R^3-80R^2r-3Rr^2+6r^3\geq 0\Leftrightarrow$$

$(R-2r)(40R^2-3r^2)\geq 0$, which is obviously true from $R\geq 2r$ $(Euler)$.

Equality holds if and only if triangle is equilateral. **Remark.** Inequality can be much stronger.

3) In ΔABC the following relationship holds:

$$\boldsymbol{\sum_{cyc}\frac{m_a}{a^2}\geq\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)}$$

Marin Chirciu

Solution. Using Tereshin inequality, we get:

$$\sum_{cyc}\frac{m_a}{a^2}\geq\sum_{cyc}\frac{\frac{b^2+c^2}{4R}}{a^2}=\frac{1}{4R}\sum_{cyc}\frac{b^2+c^2}{a^2}\geq\frac{1}{4R}\cdot\frac{3}{8}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)=$$

$$=\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right),\qquad\text{which follows from}$$

$$\sum_{cyc}\frac{b^2+c^2}{a^2}=\frac{s^6+s^4(r^2-12Rr)+s^2r^2(24R^2+8Rr-r^2)-r^3(4R+r)^3}{8R^2r^2s^2}=$$

$$=\frac{1}{8R^2r^2}\left[s^4+s^2(r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{s^2}\right]=$$

$$=\frac{1}{8R^2r^2}\left[s^2(s^2+r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{s^2}\right]\overset{Gerretsen}{\geq}$$

$$\geq\frac{1}{8R^2r^2}\left[(16Rr-5r^2)(16Rr-5r^2+r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{1}{8R^2r^2}[(16Rr-5r^2)(4Rr-4r^2)+r^2(24R^2+8Rr-r^2)-r^2(R+r)(4R+r)(4R+r)]=$$

$$=\frac{1}{8R^2}[(16R-5r)(4R-4r)+(24R^2+8R-r^2)-(R+r)(4R+r)]=$$

$$=\frac{1}{8R^2}[64R^2-64Rr-20Rr+20r^2+(24R^2+8Rr-r^2)-4R^2-5Rr-r^2]=$$

$$=\frac{84R^2-81Rr+18r^2}{8R^2}=\frac{3}{8}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)$$

Equality holds if and only if triangle is equilateral. **Remark.** Inequality 3) is much stronger such inequality 1).

4) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{a^2}\ge\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)\ge\frac{27}{4(4R+r)}$$

Marin Chirciu

Solution. See inequality 3) and $\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)\ge\frac{27}{4(4R+r)}\Leftrightarrow$

$40R^3-80R^2r-3Rr^2+6r^3\ge 0\Leftrightarrow(R-2r)(40R^2-3r^2)\ge 0$ which is true from $R\ge 2r$ $(Euler)$. Equality holds if and only if triangle is equilateral.

Remark. Let's find an opposite inequality.

5) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{a^2}\le\frac{3R}{8r^2}$$

Marin Chirciu

Solution. Using Panaitopol's inequality $m_a\le\frac{Rs}{a}$ we get:

$$\sum_{cyc}\frac{m_a}{a^2}\le\sum_{cyc}\frac{\frac{Rs}{a}}{a^2}\le Rs\sum_{cyc}\frac{1}{a^3}\le Rs\cdot\frac{1}{64RF}\left(68-47\frac{r}{R}+14\frac{r^2}{R^2}\right)=$$

$$=\frac{1}{64r}\left(68-47\frac{r}{R}+14\frac{r^2}{R^2}\right)\overset{(1)}{\le}\frac{1}{64r}\cdot\frac{24R}{r}=\frac{3R}{8r^2},\text{ where}$$

$$(1)\Leftrightarrow 68-47\frac{r}{R}+14\frac{r^2}{R^2}\le\frac{24R}{r}\Leftrightarrow 24R^3-68R^2r+47Rr^2-14r^3\ge 0\Leftrightarrow$$

$(R-2r)(24R^2-20Rr+7r^2)\ge 0$, which is true from $R\ge 2r$ $(Euler)$.

Equality holds if and only if triangle is equilateral.

$$\sum_{cyc}\frac{1}{a^3}\le\frac{3}{8sr^2},\text{ which follows from:}$$

$$\sum_{cyc}\frac{1}{a^3}\le\frac{1}{64RF}\left(68-47\frac{r}{R}+14\frac{r^2}{R^2}\right)\overset{(1)}{\le}\frac{1}{64RF}\cdot\frac{24R}{r}=\frac{3}{8sr^2}\text{, which is true from:}$$

$$\sum_{cyc}\frac{1}{a^3}=\frac{\sum b^3c^3}{a^3b^3c^3}=\frac{s^6+s^4(3r^2-12Rr)+3s^2r^4+r^3(4R+r)^3}{64R^3r^3s^3}=$$

$$=\frac{1}{64R^3r^3s}\left[s^4+s^2(3r^2-12Rr)+3r^4+\frac{r^3(4R+r)^3}{s^2}\right]=$$

$$=\frac{1}{64R^3r^3s}\left[s^2(s^2+3r^2-12Rr)+3r^4+\frac{r^3(4R+r)^3}{s^2}\right]\overset{Gerretsen}{\le}$$

$$\le\frac{1}{64R^3r^3s}\left[(16Rr-5r^2)(16Rr-5r^2+3r^2-12Rr)+3r^4+\frac{r^3(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{1}{64R^3rs}[(16R-5r)(4R-2r)+3r^2+(R+r)(4R+r)]=$$

$$=\frac{1}{64R^3rs}[64r^2-32Rr-20Rr+10r^2+3r^2+4R^2+5Rr+r^2]=$$

$$=\frac{1}{64R^3F}[64R^2-32Rr-20Rr+10r^2+3r^2+4R^2+5Rr+r^2]=$$

$$=\frac{68R^2-47Rr+14r^2}{64R^3F}=\frac{1}{64RF}\left(68-47\frac{r}{R}+14\frac{r^2}{R^2}\right)$$

$$\sum_{cyc}b^3c^3=s^6+s^4(3r^2-12Rr)+3s^2r^4+r^3(4R+r)^3$$

6) In ΔABC the following relationship holds:

$$\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)\le\sum_{cyc}\frac{m_a}{a^2}\le\frac{3R}{8r^2}$$

Marin Chirciu

Solution. For LHS, using Tereshin inequality, we get:

$$\sum_{cyc}\frac{m_a}{a^2}\ge\sum_{cyc}\frac{\frac{b^2+c^2}{4R}}{a^2}=\frac{1}{4R}\sum_{cyc}\frac{b^2+c^2}{a^2}\ge\frac{1}{4R}\cdot\frac{3}{8}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)=$$

$$=\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)\text{, which follows from:}$$

Lemma 7) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{b^2+c^2}{a^2}\ge\frac{3}{8}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)$$

Proof.

$$\sum_{cyc}\frac{b^2+c^2}{a^2}=\frac{s^6+s^4(r^2-12Rr)+s^2r^2(24R^2+8Rr-r^2)-r^3(4R+r)^3}{8R^2r^2s^2}=$$

$$=\frac{1}{8R^2r^2}\left[s^4+s^2(r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{s^2}\right]=$$

$$=\frac{1}{8R^2r^2}\left[s^2(s^2+r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{s^2}\right]\overset{Gerretsen}{\ge}$$

$$\ge\frac{1}{8R^2r^2}\left[(16Rr-5r^2)(16Rr-5r^2+r^2-12Rr)+r^2(24R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{1}{8R^2r^2}[(16Rr-5r^2)(4Rr-4r^2)+r^2(24R^2+8Rr-r^2)-r^2(R+r)(4R+r)(4R+r)]=$$

$$=\frac{1}{8R^2}[(16R-5r)(4R-4r)+(24R^2+8R-r^2)-(R+r)(4R+r)]=$$

$$=\frac{1}{8R^2}[64R^2-64Rr-20Rr+20r^2+(24R^2+8Rr-r^2)-4R^2-5Rr-r^2]=$$

$$=\frac{84R^2-81Rr+18r^2}{8R^2}=\frac{3}{8}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)$$

Equality holds if and only if triangle is equilateral. For RHS, using Panaitopol's inequality $m_a\le\frac{Rs}{a}$ we get:

$$\sum_{cyc}\frac{m_a}{a^2}\le\sum_{cyc}\frac{\frac{Rs}{a}}{a^2}\le Rs\sum_{cyc}\frac{1}{a^3}\le Rs\cdot\frac{1}{64RF}\left(68-47\frac{r}{R}+14\frac{r^2}{R^2}\right)=$$

$$=\frac{1}{64r}\left(68-47\frac{r}{R}+14\frac{r^2}{R^2}\right)\overset{(1)}{\le}\frac{1}{64r}\cdot\frac{24R}{r}=\frac{3R}{8r^2},\text{where}$$

$$(1)\Leftrightarrow 68-47\frac{r}{R}+14\frac{r^2}{R^2}\le\frac{24R}{r}\Leftrightarrow 24R^3-68R^2r+47Rr^2-14r^3\ge 0\Leftrightarrow$$

$(R-2r)(24R^2-20Rr+7r^2)\ge 0$, which is true from $R\ge 2r$ $(Euler)$.

Equality holds if and only if triangle is equilateral.

$$\sum_{cyc}\frac{1}{a^3}\le\frac{3}{8sr^2},\text{which follows from:}$$

$$\sum_{cyc}\frac{1}{a^3}\le\frac{1}{64RF}\left(68-47\frac{r}{R}+14\frac{r^2}{R^2}\right)\overset{(1)}{\le}\frac{1}{64RF}\cdot\frac{24R}{r}=\frac{3}{8sr^2}$$, which is true from:

$$\sum_{cyc}\frac{1}{a^3}=\frac{\sum b^3c^3}{a^3b^3c^3}=\frac{s^6+s^4(3r^2-12Rr)+3s^2r^4+r^3(4R+r)^3}{64R^3r^3s^3}=$$

$$=\frac{1}{64R^3r^3s}\left[s^4+s^2(3r^2-12Rr)+3r^4+\frac{r^3(4R+r)^3}{s^2}\right]=$$

$$=\frac{1}{64R^3r^3s}\left[s^2(s^2+3r^2-12Rr)+3r^4+\frac{r^3(4R+r)^3}{s^2}\right]\overset{Gerretsen}{\le}$$

$$\le\frac{1}{64R^3r^3s}\left[(16Rr-5r^2)(16Rr-5r^2+3r^2-12Rr)+3r^4+\frac{r^3(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{1}{64R^3rs}[(16R-5r)(4R-2r)+3r^2+(R+r)(4R+r)]=$$

$$=\frac{1}{64R^3rs}[64r^2-32Rr-20Rr+10r^2+3r^2+4R^2+5Rr+r^2]=$$

$$=\frac{1}{64R^3F}[64R^2-32Rr-20Rr+10r^2+3r^2+4R^2+5Rr+r^2]=$$

$$=\frac{68R^2-47Rr+14r^2}{64R^3F}=\frac{1}{64RF}\left(68-47\frac{r}{R}+14\frac{r^2}{R^2}\right)$$

$$\sum_{cyc}b^3c^3=s^6+s^4(3r^2-12Rr)+3s^2r^4+r^3(4R+r)^3$$

Equality holds if and only if triangle is equilateral.

8) In ΔABC the following relationship holds:

$$\frac{27}{4(4R+r)}\le\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)\le\sum_{cyc}\frac{m_a}{a^2}\le\frac{1}{64r}\left(68-47\frac{r}{R}+14\frac{r^2}{R^2}\right)\le\frac{3R}{8r^2}$$

Marin Chirciu

Solution. See up these inequalities. Equality holds if and only if triangle is equilateral.

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SOLVED PROBLEMS-IV

By D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

1. Let $A_1A_2\ldots A_n, n\ge 3$ be a regular polygon, M a point on incircle and N a point on circumcircle of the polygon. Prove that:

$$\sum_{k=1}^{n}\frac{MA_k^4}{NA_k^2}\ge\frac{1}{16}\left(3+\cos\frac{2\pi}{n}\right)^2\sum_{k=1}^{n}NA_k^2$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution: Let O be the center of the polygon and xOy be the system of coordonates with

$[Ox)=[OA_n)$. WLOG we assume that the circumradius is $R=1$.

We have $A_k\left(\cos\frac{2k\pi}{n},\sin\frac{2k\pi}{n}\right), k=\overline{1,n}$. Let t and u be the arguments of the affixes of the points M and N. We have that in radius is $r=\cos\frac{\pi}{n}$, so $M\left(\cos\frac{\pi}{n}\cos t,\cos\frac{\pi}{n}\sin t\right)$. Therefore:

$$\sum_{k=1}^{n}MA_k^2=\sum_{k=1}^{n}\left(\cos\frac{\pi}{n}\cos t-\cos\frac{2k\pi}{n}\right)^2+\sum_{k=1}^{n}\left(\cos\frac{\pi}{n}\sin t-\sin\frac{2k\pi}{n}\right)^2=$$

$$=n\cos^2\frac{\pi}{n}+n-2\cos\frac{\pi}{n}\cos t\sum_{k=1}^{n}\cos\frac{2k\pi}{n}-2\cos\frac{\pi}{n}\sin t\sum_{k=1}^{n}\sin\frac{2k\pi}{n}=n\left(1+\cos^2\frac{\pi}{n}\right)\quad(1)$$

In (1) we use the fact that the affixes $\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}, k=\overline{1,n}$, of the points A_k are the roots of the equation $x^n-1=0$, so that $\sum_{k=1}^{n}\varepsilon_k=0$, from where we deduce that

$$\sum_{k=1}^{n}\cos\frac{2k\pi}{n}+i\sum_{k=1}^{n}\frac{2k\pi}{n}=0$$

Also, we have:

$$\sum_{k=1}^{n}NA_k^2=\sum_{k=1}^{n}\left(\cos u-\cos\frac{2k\pi}{n}\right)^2+\sum_{k=1}^{n}\left(\sin u-\sin\frac{2k\pi}{n}\right)^2=$$

$$=n+n-2\cos u\sum_{k=1}^{n}\cos\frac{2k\pi}{n}-2\sin u\sum_{k=1}^{n}\sin\frac{2k\pi}{n}=2n\quad(2)$$

By (1) and (2) we deduce that:

$$\sum_{k=1}^{n}MA_k^2=\frac{1}{2}\left(1+\cos^2\frac{\pi}{n}\right)\sum_{k=1}^{n}NA_k^2=\frac{1}{4}\left(3+\cos\frac{2\pi}{n}\right)\sum_{k=1}^{n}NA_k^2\quad(3)$$

By (3) and by Bergström's inequality we have:

$$\sum_{k=1}^{n}\frac{MA_k^4}{NA_k^2} \geq \frac{\left(\sum_{k=1}^{n} MA_k^2\right)^2}{\sum_{k=1}^{n} NA_k^2} = \frac{\frac{1}{16}\left(3+\cos\frac{2\pi}{n}\right)^2\left(\sum_{k=1}^{n} NA_k^2\right)^2}{\sum_{k=1}^{n} NA_k^2} = \frac{1}{16}\left(3+\cos\frac{2\pi}{n}\right)^2\sum_{k=1}^{n} NA_k^2$$

and we are done.

2. If $x, y \in \mathbb{R}_+^*, xy = 1$, then show in any ABC triangle with usual notations the following inequality holds: $(xr_a + r_b + yr_c)(yr_a + r_b + xr_c) \geq 81r^2$.

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution: By $xy = 1$ and AM-GM inequality, we get:

$$(xr_a + r_b + yr_c)(yr_a + r + b + xr_c) = r_a^2 + r_b^2 + r_c^2 + r_a r_b(x+y) + r_b r_c(x+y) +$$

$$+ r_a r_c(x^2 + y^2) \geq r_a^2 + r_b^2 + r_c^2 + 2r_a r_b\sqrt{xy} + 2r_b r_c\sqrt{xy} + 2r_a r_c xy =$$

$$= r_a^2 + r_b^2 + r_c^2 + 2r_a r_b + 2r_b r_c + 2r_a r_c = (r_a + r_b + r_c)^2$$

using $r_a + r_b + r_c = 4R + r$ and $R \geq 2r$, we obtain

$(xr_a + r_b + yr_c)(yr_a + r_b + xr_c) \geq (4R + r)^2 \geq 81r^2$, Q.E.D.

3. If a, b, c be positive real numbers, then show that

$$\boldsymbol{a + b + c + 3 \cdot \sqrt[3]{abc} \geq 2\left(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}\right)}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution: It is well-known Popoviciu's inequality:

"If $f: I \to \mathbb{R}, I \subset R$ is an interval and f is a continue and convex function on I, then

$$\frac{f(x)+f(y)+f(z)}{3} + f\left(\frac{x+y+z}{3}\right) \geq \frac{2}{3}\left(f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right), \forall x, y, z \in I \text{ "}$$

For $I = R$ and $f(x) = e^x$, then $f'(x) = f''(x) = e^x > 0, \forall x \in \mathbb{R}$, so we can apply Popoviciu's inequality. Then

$$\frac{e^x + e^y + e^z}{3} + e^{\frac{x+y+z}{3}} \geq \frac{2}{3}\left(e^{\frac{x+y}{2}} + e^{\frac{y+z}{2}} + e^{\frac{z+x}{2}}\right)$$

where we put $x = \ln a, y = \ln b, z = \ln c$, and we obtain the desired inequality.

4. If $f: \mathbb{R}_+^* \to \mathbb{R}_+^*$ is convex function on $\mathbb{R}_+^*$, then prove that:

$$\boldsymbol{3\left(f^2(x) + f^2(y) + f^2(z)\right) - 9f^2\left(\frac{x+y+z}{3}\right) \geq}$$

$$\boldsymbol{\geq \left(f(x) - f(y)\right)^2 + \left(f(y) - f(z)\right)^2 + \left(f(z) - f(x)\right)^2}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution: We have $f''(x) > 0, \forall x \in \mathbb{R}_+^*$. We consider the function $g: \mathbb{R}_+^* \to \mathbb{R}_+^*, g = f^2$. So,

$g'(x) = 2f(x)f'(x), \forall x \in \mathbb{R}_+^*$ and $g''(x) = 2\left(f'(x)\right)^2 + 2f(x)f''(x) > 0, \forall x \in \mathbb{R}_+^*$. So, g is convex on $\mathbb{R}_+^*$. By Jensen's inequality we have:

$$f(x) + f(y) + f(z) \geq 3f\left(\frac{x+y+z}{3}\right), \forall x, y, z \in \mathbb{R}_+^* \quad (1)$$

and

$$g(x) + g(y) + g(z) \geq 3g\left(\frac{x+y+z}{3}\right) \Leftrightarrow f^2(x) + f^2(y) + f^2(z) \geq 3f^2\left(\frac{x+y+z}{3}\right) \quad (2)$$

By (1) we deduce that:

$$\left(f(x) + f(y) + f(z)\right)^2 \geq 9f^2\left(\frac{x+y+z}{3}\right) \Leftrightarrow -3f^2\left(\frac{x+y+z}{3}\right) \geq -\frac{\left(f(x)+f(y)+f(z)\right)^2}{3} \quad (3)$$

By (2) and (3) we deduce that:

$$f^2(x) + f^2(y) + f^2(z) - 3f^2\left(\frac{x+y+z}{3}\right) \geq$$

$$\geq f^2(x) + f^2(y) + f^2(z) - \frac{\left(f(x) + f(y) + f(z)\right)^2}{3} =$$

$$= \frac{1}{3}\left[\left(f(x) - f(y)\right)^2 + \left(f(y) - f(z)\right)^2 + \left(f(z) - f(x)\right)^2\right]$$

and we are done.

5. **Prove that in all triangles ABC, with usual notations, holds:**

$$\frac{m_a^3}{R \cdot m_b + r \cdot m_c} + \frac{m_b^3}{R \cdot m_c + r \cdot m_a} + \frac{m_c^3}{R \cdot m_a + r \cdot m_b} \geq \frac{3\sqrt{3}}{R + r} S$$

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Solution: We have that:

$$U = \sum_{cyc} \frac{m_a^3}{R \cdot m_b + r \cdot m_c} = \sum_{cyc} \frac{(m_a^2)^2}{R \cdot m_a \cdot m_b + r \cdot m_a \cdot m_c} \geq 2 \cdot \sum_{cyc} \frac{(m_a^2)^2}{R(m_a^2 + m_b^2) + r(m_a^2 + m_c^2)}$$

where we apply Bergström's inequality and well-known formula: $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

We obtain that:

$$U \geq 2 \cdot \frac{\left(\sum_{cyc} m_a^2\right)^2}{R \cdot \sum_{cyc}(m_a^2 + m_b^2) + r \cdot \sum_{cyc}(m_a^2 + m_c^2)} = \frac{2 \cdot \left(\sum_{cyc} m_a^2\right)^2}{2R \cdot \sum_{cyc} m_a^2 + 2r \cdot \sum_{cyc} m_a^2} =$$

$$= \frac{\sum_{cyc} m_a^2}{R + r} = \frac{3}{4} \cdot \frac{a^2 + b^2 + c^2}{R + r}$$

where we use Ionescu – Weitzenböck inequality, i.e. $a^2+b^2+c^2 \geq 4S\sqrt{3}$, and we deduce that:

$$U \geq \frac{3}{4}\cdot\frac{1}{R+r}\cdot 4S\sqrt{3}=\frac{3\sqrt{3}}{R+r}S$$

6. If $\boldsymbol{n \in \mathbb{N}^* - \{1\}, a \in \mathbb{R}_+, b, c, d, m, p \in \mathbb{R}_+^*, x_k \in \mathbb{R}_+^*, k=\overline{1,n}, X_{n,m}=\sum_{k=1}^{n} x_k^m}$, $\boldsymbol{X_{n,p}=\sum_{k=1}^{n} x_k^p}$, with $\boldsymbol{c\cdot X_{n,p} > d\cdot \max_{1\leq k\leq n} x_k^p}$, then prove that:

$$\boldsymbol{\sum_{k=1}^{n}\frac{a\cdot X_{n,p}+b\cdot x_k^p}{c\cdot X_{n,p}-d\cdot x_k^p} \geq \frac{n\cdot(an+b)}{n-d}\cdot\frac{X_{n,m}}{X_{n,p}}}$$

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Solution: WLOG we can assume that: $x_1 \geq x_2 \geq \cdots \geq x_n$, then yields

$$x_1^m \geq x_2^m \ldots \geq x_n^m, x_1^p \geq x_2^p \ldots \geq x_n^p$$

$$a\cdot X_{n,m}+b\cdot x_1^m \geq a\cdot X_{n,m}+b\cdot x_2^m \geq \cdots \geq a\cdot X_{n,m}+b\cdot x_n^m$$

$$0 < c\cdot X_{n,p}-d\cdot x_1^p \leq c\cdot X_{n,p}-d\cdot x_2^p \leq \cdots \leq c\cdot X_{n,p}-d\cdot x_n^p$$

$$\frac{1}{c\cdot X_{n,p}-d\cdot x_1^p} \geq \frac{1}{c\cdot X_{n,p}-d\cdot x_2^p} \geq \cdots \geq \frac{1}{c\cdot X_{n,p}-d\cdot x_n^p}$$

Applying Chebyshev's inequality for sequences

$$\left(a\cdot X_{n,p}+d\cdot x_k^m\right)_{1\leq k\leq n},\left(\frac{1}{c\cdot X_{n,p}-d\cdot x_k^p}\right)_{1\leq k\leq n}$$

We obtain:

$$W=\sum_{k=1}^{n}\frac{a\cdot X_{n,m}+b\cdot b\cdot x_k^m}{c\cdot X_{n,p}-d\cdot x_k^p} \geq \frac{1}{n}\left(\sum_{k=1}^{n}\left(a\cdot X_{n,m}+d\cdot x_k^m\right)\right)\left(\sum_{k=1}^{n}\frac{1}{c\cdot X_{n,p}-d\cdot x_k^p}\right)=$$

$$=\frac{1}{n}\left(a\cdot n\cdot X_{n,m}+d\cdot\sum_{k=1}^{n}x_k^m\right)\left(\sum_{k=1}^{n}\frac{1}{c\cdot X_{n,p}-d\cdot x_k^p}\right)=$$

$$=\frac{(an+d)X_{n,m}}{n}\left(\sum_{k=1}^{n}\frac{1^2}{c\cdot X_{n,p}-d\cdot x_k^p}\right) \quad (1)$$

In (1) we apply the inequality of Harald Bergström and we obtain:

$$W \geq \frac{(an+d)X_{n,m}}{n}\cdot\frac{\left(\sum_{k=1}^{n}1\right)^2}{\sum_{k=1}^{n}\left(c\cdot X_{n,p}-d\cdot x_k^p\right)}=\frac{(an+d)X_{n,m}}{n}\cdot\frac{n^2}{c\cdot n\cdot X_{n,p}-d\cdot\sum_{k=1}^{n}x_k^p}=$$

$$=n(an+b)X_{n,m}\cdot\frac{1}{c\cdot n\cdot X_{n,p}-d\cdot X_{n,p}}=\frac{n(an+b)}{cn-d}\cdot\frac{X_{n,m}}{X_{n,p}}$$

7. If $a,b,c,m,n,p,x,y,z\in\mathbb{R}_+^*$, then prove:

$$\mathbf{5(a^3+b^3+c^3+m^3+n^3+p^3+x^3+y^3+z^3)+}$$

$$\mathbf{+3\sqrt[3]{(a^3+b^3+c^3)(m^3+n^3+p^3)(x^3+y^3+z^3)}\geq}$$

$$\mathbf{\geq 2(a+b+c)(m+n+p)(x+y+z)}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution: From J. Radon $a^3+b^3+c^3\geq\frac{(a+b+c)^3}{9},\forall a,b,c\in\mathbb{R}_+^*$, so

$$E\geq 5\cdot\frac{(a+b+c)^3+(m+n+p)^3+(x+y+z)^3}{9}+$$

$$+3\cdot\sqrt[3]{\frac{(a+b+c)^3(m+n+p)^3(x+y+z)^3}{9^3}}\geq 5\cdot\frac{3}{9}\sqrt[3]{(a+b+c)^3(m+n+p)^3(x+y+z)^3}+$$

$$+3\cdot\frac{1}{9}(a+b+c)(m+n+p)(x+y+z)=\left(\frac{5}{3}+\frac{1}{3}\right)(a+b+c)(m+n+p)(x+y+z)=$$

$$=2(a+b+c)(x+y+z)(m+n+p)$$

8. Show that in any triangle ABC with the usual notations, holds:

$$\mathbf{a^2r_a+b^2r_b+c^2r_c\geq 108r^3}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution: By Bergström's inequality and well-known $\sum\frac{1}{r_a}=\frac{1}{r}$ we have:

$$V=\sum a^2r_a=\sum\frac{a^2}{\frac{1}{r_a}}\overset{Bergström}{\geq}\frac{(a+b+c)^2}{\sum\frac{1}{r_a}}=\frac{4s^2}{\frac{1}{r}}=4rs^2$$

By Mitrinovic's inequality, i.e. $s\geq 3\sqrt{3}r\Leftrightarrow s^2\geq 27r^2$ we obtain: $V=4rs^2\geq 108r^3$, Q.E.D.

9. Prove that in any ABC triangle with the semiperimeter s, the inradius r and usual notations the following inequality is true $\frac{ab}{s-c}+\frac{bc}{s-a}+\frac{ca}{s-b}\geq 12\cdot\sqrt{3}\cdot r$.

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution: By AM-GM inequality, the fact that $abc=4\cdot R\cdot S$ and Heron's formula

$S=\sqrt{s(s-a)(s-b)(s-c)}$ we have

$$\sum_{cyc}\frac{ab}{s-c}\geq 3\cdot\left(\sqrt[3]{\frac{(abc)^2}{(s-a)(s-b)(s-c)}}\right)=3\cdot\sqrt[3]{\frac{16R^2S^2s}{S^2}}=6\cdot\sqrt[3]{2R^2s}\quad(1)$$

By Euler's inequality $(R\geq 2r)$ and Mitrinovic's inequality $\left(s\geq 3\sqrt{3}r\right)$ from (1) we get:

$$\sum_{cyc}\frac{ab}{s-c}\ge 6\cdot\sqrt[3]{2R^2s}\ge 6\cdot\sqrt[3]{4r^2\cdot 3\sqrt{3}\cdot r}=12\sqrt{3}r$$

10. If $m,n\in\mathbb{R}_+$ then in any triangle ABC the following inequality is true

$$mc^2+nab\ge 3\cdot\sqrt[3]{m(2nprR)^2}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution: We have $mc^2+nab=mc^2+\frac{nab}{2}+\frac{nab}{2}\ge 3\cdot\sqrt[3]{m\cdot\frac{n^2}{2^2}\cdot(abc)^2}$ and from $abc=4RS=4Rpr$, Q.E.D.

11. Prove that if $m,n\in\mathbb{R}_+$, then in any acute triangle ABC holds:

$$m\cdot\tan^2 A+n\cdot\tan B\cdot\tan C\ge 3\cdot\sqrt[3]{\left(\frac{prn}{p^2-(2R+r)^2}\right)^2\cdot m}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution: $m\cdot\tan^2 A+n\cdot\tan B\cdot\tan C=m\cdot\tan^2 A+\frac{n}{2}\cdot\tan B\cdot\tan C+\frac{n}{2}\cdot\tan B\cdot\tan C\ge$

$$\ge 3\cdot\sqrt[3]{\frac{mn^2}{4}\cdot(\tan A\cdot\tan B\cdot\tan C)^2}$$

$$\tan A\cdot\tan B\cdot\tan C=\frac{2pr}{p^2-(2R+r)^2}$$

12. If $m,n\in\mathbb{R}_+^*$, then prove that in any triangle ABC holds:

$$\frac{r_ar_b^2}{mr_a+nr_b}+\frac{r_br_c^2}{mr_b+nr_c}+\frac{r_cr_a^2}{mr_c+nr_a}\ge\frac{p^4}{(4R+r)^2m+(n-2m)p^2}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution: $U=\sum\frac{r_ar_b^2}{mr_a+nr_b}=\sum\frac{(r_ar_b)^2}{mr_a^2+nr_ar_b}$, and from H. Bergström inequality:

$$U\ge\frac{(r_ar_b+r_br_c+r_cr_a)^2}{m(r_a^2+r_b^2+r_c^2)+n(r_ar_b+r_br_c+r_cr_a)}$$

Since $r_ar_b+r_br_c+r_cr_a=p^2$ and $r_a^2+r_b^2+r_c^2=(4R+r)^2-2p^2$, we get

$$U\ge\frac{p^4}{m(4R+r)^2-2mp^2+np^2}=\frac{p^4}{(4R+r)^2m+(n-2m)p^2}$$

13. If $m,n\in\mathbb{R}_+^*$, then in any triangle ABC prove the following inequality is true

$$\frac{ab^2}{ma+nb}+\frac{bc^2}{mb+nc}+\frac{ca^2}{mc+na}\geq\frac{(p^2+r^2+4Rr)^2}{(2m+n)p^2+(n-2m)r^2+4(n-2m)Rr}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution: $V=\sum\frac{ab^2}{ma+nb}=\sum\frac{(ab)^2}{ma^2+nab}$, and by H. Bergström's inequality we deduce

$$V\geq\frac{(ab+bc+ca)^2}{m(a^2+b^2+c^2)+n(ab+bc+ca)}$$

Since, $ab+bc+ca=p^2+r^2+4Rr$ and $a^2+b^2+c^2=2(p^2-r^2-4Rr)$, we get

$$V\geq\frac{(p^2+r^2+4Rr)^2}{2m(p^2-r^2-4Rr)+n(p^2+r^2+4Rr)}=\frac{(p^2+r^2+4Rr)^2}{(2m+n)p^2+(n-2m)r^2+4(n-2m)Rr}$$

14. I**f $x,y\in\mathbb{R}_+$, then prove that in any triangle holds:**

$$(p\cdot x+R\cdot y)\cdot p^2+(R\cdot y-3\cdot p\cdot x)\cdot r^2+2\cdot(2\cdot R\cdot y-3\cdot p\cdot x)\cdot R\cdot r\geq$$

$$\geq 8\cdot R\cdot\sqrt[4]{6\cdot R^2\cdot(p\cdot r\cdot y)^3\cdot x}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution: In any triangle ABC holds:

$$x\sin^3 A+y\sin B\sin C\geq\frac{2}{R}\cdot\sqrt[4]{\frac{2x\cdot(pry)^3}{27R^2}}$$

Indeed, $x\sin^3 A+y\sin B\sin C=x\sin^3 A+\frac{y\sin B\sin C}{3}+\frac{y\sin B\sin C}{3}+\frac{y\sin B\sin C}{3}\geq$

$\geq 4\cdot\sqrt[4]{\frac{xy^3}{3^3}\cdot(\sin A\sin B\sin C)^3}$, and because $\sin A\sin B\sin C=\frac{pr}{2R^2}$, we get the above inequality. Writing other two inequalities and add up we deduce

$x\sum\sin^3 A+y\sum\sin B\sin C\geq\frac{6}{R}\cdot\sqrt[4]{\frac{2x(pry)^3}{27R^2}}$, where we used

$\sum\sin^3 A=\frac{p(p^2-3r^2-6Rr)}{4R^3}$, respectively $\sum\sin B\sin C=\frac{p^2+r^2+4Rr}{4R^2}$, and the conclusion follows.

15. Prove that if $x,y\in\mathbb{R}_+^*$, then in any triangle holds:

$$\frac{r_a^3}{xr_a+yr_b}+\frac{r_b^3}{xr_b+yr_c}+\frac{r_c^3}{xr_c+yr_a}\geq\frac{(4R+r)^2-2p^2}{x+y}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution: $U=\sum\frac{r_a^3}{xr_a+yr_b}=\sum\frac{r_a^3}{xr_a^2+yr_ar_b}=\sum\frac{(r_a^2)^2}{xr_a^2+yr_ar_b}$, and by H. Bergström and taking account by

$r_a^2+r_b^2+r_c^2\ge r_ar_b+r_br_c+r_cr_a$, we deduce

$$U\ge\frac{(\sum r_a^2)^2}{(x+y)\sum r_a^2}=\frac{\sum r_a^2}{x+y}$$

where we use $\sum r_a^2=(4R+r)^2-2p^2$, Q.E.D.

16. Prove that in all triangle holds:

$$(a^4+b^4+c^4)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\ge 12\sqrt{3}S$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution: By H. Bergström's inequality we deduce that:

$$\sum a^4=\sum(a^2)^2\ge\frac{\left(\sum a^2\right)^2}{3}\quad (1)$$

and also by H. Bergström's inequality we obtain that:

$$\sum\frac{1}{a^2}\ge\frac{3^2}{\sum a^2}\quad (2)$$

So, (1) and (2) yields that:

$$U=\left(\sum a^4\right)\left(\sum\frac{1}{a^2}\right)\ge 3\sum a^2\quad (3)$$

By Weitzenböck's inequality we have that:

$$\sum a^2\ge 4S\sqrt{3}\quad (4)$$

Therefore, from (3) and (4) we obtain that: $U\ge 3\cdot\left(4S\sqrt{3}\right)=12\sqrt{3}S$, and we are done.

17. Prove that if $m,n\in\mathbb{R}_+^*$, then in any triangle ABC holds:

$$\frac{a}{mb+nc}+\frac{b}{mc+na}+\frac{c}{ma+nb}\ge\frac{4p^2}{(m+n)(p^2+r^2+4Rr)}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution: $W=\sum\frac{a}{mb+nc}=\sum\frac{a^2}{mab+nac}$ and from H. Bergström's inequality

$W\ge\frac{(a+b+c)^2}{(m+n)(ab+bc+ca)}$, and because $ab+bc+ca=p^2+r^2+4Rr$ and $a+b+c=2p$, Q.E.D.

18. If $a_i>0$ $(i=1,2,\dots,n)$, then show that:

$$\left(\sum\frac{1}{\frac{1}{a_1}+\frac{1}{a_2}}\right)\left(\sum\frac{1}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}}\right)\cdots\left(\sum\frac{1}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}\right)<\left(\sum_{i=1}^{n}a_i\right)^{k-1}$$

where $k \in \{1, 2, \dots, n-1\}$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution: By HM-AM we get $\sum \frac{1}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_k}} \le \frac{a_1+a_2+\cdots+a_k}{k^2} < \frac{1}{k}\sum_{i=1}^{n} a_i$, so

$$LHS \le \frac{1}{2 \cdot 3 \cdot \ldots \cdot k}\left(\sum_{i=1}^{n} a_i\right)^{k-1} < \left(\sum_{i=1}^{n} a_i\right)^{k-1}$$

19. Prove that in all triangle ABC holds $(a^6+b^6+c^6)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \ge 48S^2$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution: By J. Radon's inequality we deduce that: $\sum a^6 = \sum (a^2)^3 \ge \frac{\left(\sum a^2\right)^3}{3^2}$ (1)

and by H. Bergström's inequality we obtain that: $\sum \frac{1}{a^2} \ge \frac{3^2}{\sum a^2}$ (2)

So, (1) and (2) yields that: $U = \left(\sum a^6\right)\left(\sum \frac{1}{a^2}\right) \ge \left(\sum a^2\right)^2$ (3)

By Weitzenböck's inequality we have that: $\sum a^4 \ge 4S\sqrt{3}$ (4)

Therefore, from (3) and (4) we obtain that: $U \ge \left(4S\sqrt{3}\right)^2 = 48S^2$

ABOUT AN INEQUALITY BY PHAN NGOC CHAU-I

By Marin Chirciu-Romania

In ΔABC the following relationship holds:

$$a^3+b^3+c^3 \ge 3r\left(\frac{a^3}{r_a}+\frac{b^3}{r_b}+\frac{c^3}{r_c}\right)$$

Proposed by Phan Ngoc Chau-Ho Chi Minh-Vietnam

Solution. **Lemma. In ΔABC the following relationship holds:**

$$\frac{a^3}{r_a}+\frac{b^3}{r_b}+\frac{c^3}{r_c} = \frac{2[s^2(2R+3r)-r(4R+r)^2]}{s}$$

Proof. Using $r_a = \frac{F}{s-a}$, we get:

$$\frac{a^3}{r_a}+\frac{b^3}{r_b}+\frac{c^3}{r_c}=\sum_{cyc}\frac{a^3}{F}=\frac{1}{F}\sum_{cyc}a^3(s-a)=\frac{1}{sr}\cdot 2r[s^2(2R+3r)-r(4R+r)^2]=$$

$$=\frac{2[s^2(2R+3r)-r(4R+r)^2]}{s}$$

Equality holds if and only if ΔABC is equilateral.

Let's get back to the main problem. Using Lemma and $a^3+b^3+c^3=2s(s^2-3r^2-6Rr)$, inequality becomes as:

$$2s(s^2-3r^2-6Rr)\geq 3r\cdot\frac{2[s^2(2R+3r)-r(4R+r)^2]}{s}\Leftrightarrow$$

$$s^2(s^2-3r^2-6Rr)\geq 3s^2r(2R+r)-3r^2(4R+r)^2\Leftrightarrow$$

$$s^2(s^2-12r^2-12Rr)+3r^2(4R+r)^2\geq 0$$

We distinguish the following cases:

Case I) If $(s^2-12r^2-12Rr)>0$, inequality is true.

Case II) If $(s^2-12r^2-12Rr)<0$, inequality becomes as:

$3r^2(4R+r)^2\geq s^2(12Rr+12r^2-s^2)$, which follows from Blundon-Gerretsen:

$$16Rr-5r^2\leq s^2\leq\frac{R(4R+r)^2}{2(2R-r)}\leq 4R^2+4Rr+3r^2$$

Remains to prove that: $3r^2(4R+r)^2\geq\frac{R(4R+r)^2}{2(2R-r)}\big(12Rr+12r^2-(16Rr-5r^2)\big)\Leftrightarrow$

$$6r(2R-r)\geq R(-4R+17r)\Leftrightarrow 4R^2-5Rr-6r^2\geq 0\Leftrightarrow$$

$(R-2r)(4R+3r)\geq 0$, which is true from $R\geq 2r$ $(Euler)$.

Equality holds if and only if triangle is equilateral.

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PROPOSED PROBLEMS

J.2001 In ΔABC the following relationship holds:

$$(\sin A+2\sin B)^4+(\sin B+2\sin C)^4+(\sin C+2\sin A)^4\le\frac{2187}{8}\left(1-\frac{r}{R}\right)$$

Proposed by Marian Ursărescu-Romania

J.2002

$$m_h=\frac{2ab}{a+b},m_g=\sqrt{ab},m_a=\frac{a+b}{2};a,b>0$$

Prove that:

$$2m_h+\sqrt{\frac{1}{2}\left(\left(m_a-m_g\right)^2+\left(m_g-m_h\right)^2+(m_h-m_a)^2\right)}\le m_g+m_a$$

Proposed by Daniel Sitaru,Oana Simona Dascălu-Romania

J.2003 In ΔABC the following relationship holds:

$$a^2+b^2+c^2\ge 4F\sqrt{\frac{abc}{(a+b)(b+c)(c+a)}}\cdot\sum_{cyc}\left(\csc C\cdot\sqrt{\frac{a+b}{c}}\right)$$

Proposed by Daniel Sitaru,Gilena Dobrică-Romania

J.2004 Solve for real numbers:

$$\begin{cases} x,y>0 \\ \dfrac{(x+y)^{10}}{(2x^2+y^2)(4x^3+y^3)(16x^5+y^5)}=\dfrac{2187}{128} \\ 2^x+\log_6 y=9 \end{cases}$$

Proposed by Daniel Sitaru,Daniela Dîrnu-Romania

J.2005 In ΔABC the following relationship holds:

$$\left(\frac{3\sin^2 A}{\cos A}+\frac{2\sin^2 B}{\cos B}+\frac{\sin^2 C}{\cos C}\right)\cos\left(\frac{\pi+2A+B}{6}\right)\ge 6\sin^2\left(\frac{\pi+2A+B}{6}\right)$$

Proposed by Daniel Sitaru,Sabina Subțirelu-Romania

J.2006 If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{xa\sqrt[3]{a}}{(y+z)\sqrt[3]{h_a^2}}+\frac{yb\sqrt[3]{b}}{(z+x)\sqrt[3]{h_b^2}}+\frac{zc\sqrt[3]{c}}{(x+y)\sqrt[3]{h_c^2}}\ge\sqrt[6]{18}\cdot\sqrt[3]{F}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2007 If $x,y,z>0$ and n_a,n_b,n_c –Nagel's cevians then in ΔABC holds:

$$\frac{y+z}{xh_a}\cdot a^2n_a+\frac{z+x}{yh_b}\cdot b^2n_b+\frac{x+y}{zh_c}\cdot c^2n_c\ge 8\sqrt{3}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2008 If $x,y,z>0,t\ge 0$ then in ΔABC the following relationship holds:

$$\frac{y+z+2t}{x+t}\cdot a^2+\frac{z+x+2t}{y+t}\cdot b^2+\frac{x+y}{z+t}\cdot c^2\ge 8\sqrt{3}\cdot F$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2009 Let $m\ge 0,x,y,z>0$ and $M\in Int(\Delta ABC),d_a=d(M,BC),d_b=d(M,CA),d_c=d(M,AB)$ then holds:

$$\frac{x^{m+1}a^{m+2}}{(y+z)^{m+1}d_a^m}+\frac{y^{m+1}b^{m+2}}{(z+x)^{m+1}d_b^m}+\frac{z^{m+1}c^{m+2}}{(x+y)^{m+1}d_c^m}\ge 2\left(\sqrt{3}\right)^{m+1}F$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2010 Let $M\in Int(\Delta ABC),d_a=d(M,BC),d_b=d(M,CA),d_c=d(M,AB),t\ge 0,x,y,z>0$ then:

$$\frac{x^{t+1}a^{3t+4}}{(y+z)^{t+1}d_a^t}+\frac{y^{t+1}b^{3t+4}}{(z+x)^{t+1}d_b^t}+\frac{z^{t+1}c^{3t+4}}{(x+y)^{t+1}d_c^t}\ge 8\left(\frac{4}{3}\right)^t F^{t+2}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2011 If $m\ge 0$ and $a,b,c>0$, then holds:

$$a^{m+1}\left(\frac{1}{b^m}+\frac{1}{c^m}\right)+b^{m+1}\left(\frac{1}{c^m}+\frac{1}{a^m}\right)+c^{m+1}\left(\frac{1}{a^m}+\frac{1}{b^m}\right)\ge 2(a+b+c)$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.2012 Let $a,b,c\in\mathbb{R}$ such that $|ax^2+bx+c|\le 1;\forall|x|\le 1$

a) Prove that: $\max\{|bx^2+cx+a|,|cx^2+bx+a|\}\le 1;\forall|x|\le 1$.

b) Find $a,b,c\in\mathbb{R},abc>0$ such that $\max\{|bx^2+cx+a|,|cx^2+bx+a|\}\le 1;\forall|x|\le 1$.

Proposed by Nguyen Van Canh-Vietnam

J.2013 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sin\frac{A}{2}}{2\cos\frac{B}{2}\cos\frac{C}{2}}=\sum_{cyc}\tan\frac{A}{2}+\sum_{cyc}\tan\frac{B}{2}\tan\frac{C}{2}-\prod_{cyc}\tan\frac{A}{4}$$

Proposed by Nguyen Van Canh-Vietnam

J.2014 Let $f(x)=x+\frac{9}{x+\alpha^2+2}-1$. Find all real numbers α such that:

$$2019\cdot\max\{f(x)\}-2018\cdot\min\{f(x)\}=2021;\forall x\in[-1,1]$$

Proposed by Nguyen Van Canh-Vietnam

J.2015 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{m_am_b}{r_c}}\le\frac{4R+r}{\sqrt{3r}}$$

Proposed by Nguyen Van Canh-Vietnam

J.2016 In ΔABC the following relationship holds:

$$m_a^2+m_b^2+m_c^2\ge w_a^2+w_b^2+w_c^2+\frac{r\cdot h_ah_bh_c}{4R\cdot m_am_bm_c}(R^2-4r^2)$$

Proposed by Nguyen Van Canh-Vietnam

J.2017 Let $a,b,c>0$ and $a^2+b^2+c^2=3$. Prove that:

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\ge 3$$

Proposed by Choy Fai Lam-Hong Kong

J.2018 Let $a,b,c>0$ and $a^2+b^2+c^2=3$. Prove that:

$$\frac{1}{\sqrt{6}}\left(\sqrt{\frac{a}{a+b}}+\sqrt{\frac{b}{b+c}}+\sqrt{\frac{c}{c+a}}\right)+\frac{9}{2\sqrt{a+b+c}}\ge 2\sqrt{3}$$

Proposed by Choy Fai Lam-Hong Kong

J.2019 Let $a,b,c>0$ and $abc=1$. Prove that:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{a+b}{c}\ge\frac{12}{\sqrt[4]{c^3}(b+c)(c+a)}$$

Proposed by Choy Fai Lam-Hong Kong

J.2020 Let $a,b,c>0$ and $abc=1$. Prove that:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{1}{8}\left(\frac{a+b}{c}\right)^2\ge\frac{6}{\sqrt[3]{c^4}(b+c)(c+a)}$$

Proposed by Choy Fai Lam-Hong Kong

J.2021 If in $\Delta ABC, \cos A+\cos B=1, \cos\left(\frac{B-C}{2}\right)=\sin\frac{B}{2}+\sin\frac{C}{2}$, then find:

$$\mu(A), \mu(B), \mu(C)$$

Proposed by Cristian Miu-Romania

J.2022 If $1 \leq a, b, c, d, e, f \leq 2$ then:

$$(a^2 + b^2 + c^2)(d^2 + e^2 + f^2) \leq (ad + be + cf)^2 + 27$$

Proposed by Daniel Sitaru-Romania

J.2023 In ΔABC the following relationship holds:

$$a^2b^2 + b^2c^2 + c^2a^2 \geq 2F(16F + \sqrt{3} \cdot \sqrt[3]{(a-b)^2(b-c)^2(c-a)^2}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.2024 If $M \in Int(\Delta ABC)$ and $R_a = MA, R_b = MB, R_c = MC, d_a = d(M, BC), d_b = d(M, CA), d_c = d(M, AB)$, then:

$$R_a^2 + R_b^2 + R_c^2 + d_a^2 + d_b^2 + d_c^2 \geq 4(d_a d_b + d_b d_c + d_c d_a)$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.2025 If $M \in Int(\Delta ABC), R_a = MA, R_b = MB, R_c = MC$ and $d_a = d(M, BC), d_b = d(M, CA), d_c = d(M, AB)$, then:

$$(R_a d_a + R_b d_b + R_c d_c)\left(\frac{1}{(d_a + d_b)^2} + \frac{1}{(d_b + d_c)^2} + \frac{1}{(d_c + d_a)^2}\right) \geq \frac{9}{2}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.2026 If $x, y, z > 0$ then in ΔABC holds:

$$\frac{x+y}{za} + \frac{y+z}{xb^2} + \frac{z+x}{y} \cdot a^3b^4c^2 \geq 8\sqrt{3} \cdot F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.2027 If $t \geq 0, n \in \mathbb{N}, n \geq 2$ and $g, x_k \in [1, \infty), k = \overline{1, n}, g = \frac{x_1 + x_2 + \cdots + x_k}{k}, k = \overline{1, n}$, then:

$$\sum_{k=1}^{n} \left(x_1^{x_k} + x_2^{x_k} + \cdots + x_n^{x_k}\right)^{t+1} \geq n^{t+2} \cdot g^{m(t+1)}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.2028 If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{x+y}{z}\sqrt{ab} + \frac{y+z}{x}\sqrt{bc} + \frac{z+x}{y}\sqrt{ca} \geq 4 \cdot \sqrt[4]{3} \cdot \sqrt{F}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.2029 In ΔABC the following relationship holds:

$$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}<2\sqrt{3}\cdot F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.2030 In ΔABC the following relationship holds:

$$\frac{yz}{h_a^2}+\frac{zx}{h_b^2}+\frac{xy}{h_c^2}\geq\frac{R^2}{27r^4}(x+y+z)^2;\forall x,y,z>0$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.2031 If $x,y,z>0$ then in ΔABC holds:

$$\frac{(y+z)a\sqrt{b}}{x\sqrt{h_b}}+\frac{(z+x)b\sqrt{c}}{y\sqrt{h_c}}+\frac{(x+y)c\sqrt{a}}{z\sqrt{h_a}}\geq 4\sqrt{6}\cdot\sqrt{F}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.2032 If $x,y,z>0$ then in ΔABC holds:

$$\frac{x+y}{z}a+\frac{y+z}{x}b+\frac{z+x}{y}c\geq 4\cdot\sqrt[4]{27}\cdot\sqrt{F}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.2033 In $\Delta ABC, n_a$ –Nagel's cevian and $M\in Int(\Delta ABC), x=MA, y=MB, z=MC$ holds:

$$(xn_a+yn_b+zn_c)^2\geq 12\cdot F^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.2034 If $m,n\in(0,\infty)$, then in ΔABC holds:

$$\frac{ma^2+nb^2}{(a+b-c)c}+\frac{mb^2+nc^2}{(b+c-a)a}+\frac{mc^2+na^2}{(c+a-b)b}\geq 3(m+n)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2035 If $m,n\geq 0$ then in ΔABC holds:

$$m\sin^6\frac{A}{2}+n\sin^3\frac{B}{2}\sin^3\frac{C}{2}\geq\frac{3r^2}{16R^2}\sqrt[3]{\frac{mn^2}{4}}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2036 If ABC is nonisosceles triangle, $m\geq 0$ and $x,y>0$, then:

$$\sum_{cyc} \frac{a^{3(m+1)}}{(bx+cy)^m (a-b)^{m+1} (a-c)^{m+1}} > \frac{2s}{(x+y)^m}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2037 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{(m_a + w_b)(w_a + h_c)}{h_b h_c} \geq 12$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.2038 In $\Delta ABC, M \in Int(\Delta ABC), x = MA, y = MB, z = MC$ holds:

$$(xm_a + ym_b + zm_c)^2 \geq 12 \cdot F^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

J.2039 In ΔABC the following relationship holds:

$$\sum_{cyc} \left[\left(\frac{\sin A \sin B}{\sin C} \right)^2 + \left(\frac{\cos A \cos B}{\cos C} \right)^2 \right] \geq 3$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2040 In ΔABC the following relationship holds:

$$\sum_{cyc} \cos \frac{A}{2} \left(\frac{1}{\sin B} + \frac{1}{\sin C} \right) \geq 6$$

Proposed by Neculai Stanciu-Romania

J.2041 Determine number $\overline{abcdefghıȷ}$ such that

$$\overline{abcdd} + \overline{defba} + \overline{gegd} + \overline{adg} + \overline{adg} = \overline{hıcaȷ}$$

where to different numbers correspond to different letters.

Proposed by Neculai Stanciu-Romania

J.2042 If $m \geq 0, u, v > 0$ with $2u - v > 0$ and α, β, γ are the measures of the angle of triangle ABC, then prove that:

$$\sum_{cyc} \frac{\sin \alpha}{\left(u \sin \beta + v\sqrt{\sin \alpha \sin \beta}\right)^m} \geq \left(\frac{3}{u+v}\right)^m (\sin \alpha + \sin \beta + \sin \gamma)^{1-m}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2043 If $x, y > 0$ then in ΔABC holds:

$$2s(s^2-6Rr-3r^2)x+(s^2+4Rr+r^2)y \geq 8\cdot\sqrt[4]{12x\cdot(sRry)^3}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2044 In ΔABC the following relationship holds:

$$s^2 \geq 3\sqrt{3}F+\frac{1}{6}(|a-b|^2+|b-c|^2+|c-a|^2)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2045 In ΔABC the following relationship holds:

$$\left(\frac{ab}{a+b}\right)^2+\left(\frac{bc}{b+c}\right)^2+\left(\frac{ca}{c+a}\right)^2 \geq 9r^2$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

J.2046 If $m \geq 0$ and $a,b,c,x>0$ then holds:

$$(a^{2m+2}+x^{2m+2})(b^{2m+2}+x^{2m+2})(c^{2m+2}+x^{2m+2}) \geq \frac{3^{m+1}x^{4m+4}}{2^{5m+2}}(a+b+c)^{2m+2}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

J.2047 If $m\geq 0, x,y,z>0, M\in Int(\Delta ABC)$ with area F, d_a, d_b, d_c −distances from M to the sides BC, CA, AB respectively, then holds:

$$\frac{x^{m+1}a^{m+2}}{d_a^m}+\frac{y^{m+1}b^{m+2}}{d_b^m}+\frac{z^{m+1}c^{m+2}}{d_c^m} \geq 4\sqrt{(xy+yz+zx)^{m+1}}\cdot F$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

J.2048 If $x,y,z>0$ then in ΔABC holds:

$$\frac{(x+y)\sqrt{a}}{z\sqrt{h_b}}+\frac{(y+z)\sqrt{b}}{x\sqrt{h_c}}+\frac{(z+x)\sqrt{c}}{y\sqrt{h_a}} \geq 2\sqrt{2}\cdot\sqrt[4]{27}$$

Proposed by D.M. Bătineţu-Giurgiu, Dan Nănuţi-Romania

J.2049 If $x,y,z>0$, then in ΔABC the following relationship holds:

$$\frac{x\sqrt{a}}{(y+z)\sqrt{h_a}}+\frac{y\sqrt{b}}{(z+x)\sqrt{h_b}}+\frac{z\sqrt{c}}{(x+y)\sqrt{h_c}} \geq \frac{\sqrt{54}}{2}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuţi-Romania

J.2050 Let $A_1B_1C_1, A_2B_2C_2$ triangles with sides a_k, b_k, c_k and the areas $F_k, k=\overline{1,2}$, then:

$$(a_1^2+a_2^2)(b_1^2+a_2^2)(c_1^2+a_2^2)+(a_1^2+b_2^2)(b_1^2+b_2^2)(c_1^2+b_2^2)+$$

$$+(a_1^2+c_2^2)(b_1^2+c_2^2)(c_1^2+c_2^2)\geq 108F_1F_2$$

Proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți-Romania

J.2051 If $t>0$, then in ΔABC with F area, holds:

$$(a^4+t^2)(b^4+t^2)+(b^4+t^2)(c^4+t^2)+(c^4+t^2)(a^4+t^2)\geq 64t^2F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.2052 If $x,y,z\geq 1, 3m=x+y+z$ then in ΔABC with area F, holds:

$$\sum_{cyc}(x^x+y^x+z^x)a^2\geq 12\sqrt{3}\cdot m^mF$$

Proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți-Romania

J.2053 If $x,y,z>0$ then in ΔABC holds:

$$\frac{ax+by}{zh_c}+\frac{by+cz}{xh_a}+\frac{cz+ax}{yh_b}\geq 4\sqrt{3}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

J.2054 If $a,b,c,x,y,z>0$, then holds:

$$(a^2+x^2)(b^2+x^2)(c^2+x^2)+(a^2+y^2)(b^2+y^2)(c^2+y^2)+$$

$$+(a^2+z^2)(b^2+z^2)(c^2+z^2)\geq\frac{1}{4}(x+y+z)^2(a+b+c)^2$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.2055 Let I –be the incentre of ΔABC and ΔMNP be the median triangle of ΔABC. Let A_1,B_1,C_1 –be the simmetrics of I to M,N,P and R_1,R_2,R_3 the circumradies of $\Delta BCA_1,\Delta CAB_1,\Delta ABC_1$. Prove that: $R_1^2+R_2^2+R_3^2\geq 3R^2$.

Proposed by Marian Ursărescu-Romania

J.2056 If $0<a\leq b\leq c<\frac{\pi}{2}$ then:

$$\frac{5}{\tan a}+\frac{3}{\tan b}+\frac{1}{\tan c}\geq\frac{27}{\tan a+\tan b+\tan c}$$

Proposed by Daniel Sitaru,Cristina Ene-Romania

J.2057 In ΔABC the following relationship holds:

$$\frac{2a^3+3b^3+5c^3}{2a^2+3b^2+5c^2}+\frac{3a^3+5b^3+2c^3}{3a^2+5b^2+2c^2}+\frac{5a^3+2b^3+3c^3}{5a^2+2b^2+3c^2}\geq 6\sqrt{3}r$$

Proposed by Daniel Sitaru,Iacob Meda-Romania

J.2058 If $x, y > 0$ then:

$$\frac{x}{x^2-x+1}+\frac{y}{y^2-y+1}+\frac{xy}{x^2y^2-xy+1} \le \frac{x^2}{x^2-x+1}+\frac{y^2}{y^2-y+1}+\frac{1}{x^2y^2-xy+1}$$

Proposed by Daniel Sitaru,Lucian Tuțescu-Romania

J.2059 Solve for real numbers: $x^3+\frac{x-1}{x^3+1}=\frac{x^3(x+1)}{x^4+1}+\frac{(x-1)(x^2+1)}{x^4}$

Proposed by Daniel Sitaru,Alina Tigae-Romania

J.2060 If $a, b, c > 0$ then: $abc+a^2+b^2+c^2+4 \ge 2(ab+bc+ca)$

Proposed by Daniel Sitaru,Rareș Tudorașcu-Romania

J.2061 Solve for complex numbers: $x^7+2x^6+5x^5+3x^4-16x^3-11x^2-20x-12=0$

Proposed by Daniel Sitaru,Elena Grigore -Romania

J.2062 If $x, y, z \ge 0$ then:

$$2\sum_{cyc} x^2(x^2+y^2) \ge \sum_{cyc} x(y^3+z^3)+xyz(x+y+z)$$

Proposed by Daniel Sitaru,Elena Alexie-Romania

J.2063 Solve for real numbers:

$$\sin^2 x\,(2\sin^2 x\cdot\sin^2 2x+4\cos^4 x+1)=\cos^2 x\,(2\cos^2 x\cdot\sin^2 2x+4\sin^4 x+1)$$

Proposed by Daniel Sitaru,Mihaela Dăianu-Romania

J.2064 Let $x, y, z > 0$ then:

$$\frac{x}{y+z}+\frac{y}{z+X}+\frac{z}{x+y} \ge \frac{9(x^2+y^2+z^2-xy-yz-zx)^2}{2(x+y+z)^4}+\frac{3}{2}$$

Proposed by Khang Nguyen-Vietnam

J.2065 If $u, v \in (0,1)$ and $M \in Int(\Delta ABC), x = MA, y = MB, z = MC$ then:

$$\sum_{cyc}\left(\frac{xy}{t(1-t^2)ab}+\frac{xz}{u(1-u^2)ac}\right)^4 \ge 27$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

J.2066 Let ΔABC is not acute triangle. Find then minimum value of the following expression:

$$\Omega=(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Phan Ngoc Chau-Vietnam

J.2067 Let I —be the incentre of ΔABC and ΔMNP be the median triangle of ΔABC. Let A_1, B_1, C_1 —be the simmetrics of I to M, N, P and R_1, R_2, R_3 the circumradies of $\Delta BCA_1, \Delta CAB_1, \Delta ABC_1$. Prove that: $\frac{1}{R_1^2}+\frac{1}{R_2^2}+\frac{1}{R_3^2} \geq \frac{3}{R^2}$.

Proposed by Marian Ursărescu-Romania

J.2068 If $0 < x, y < 1$ then:

$$\frac{x}{4\sqrt{x+y}-\sqrt{xy+y^2}}+\frac{y}{4\sqrt{x+y}-\sqrt{xy+x^2}} \leq \frac{x+y}{4\sqrt{x+y}-\sqrt{2xy}}$$

Proposed by Daniel Sitaru,Simonaa Radu-Romania

J.2069 In $\Delta ABC, \Delta A'B'C'$ the following relationship holds:

$$\frac{m_a^3 \cdot (a')^2}{a^2}+\frac{m_b^3 \cdot (b')^2}{b^2}+\frac{m_c^3 \cdot (c')^2}{c^2} \geq \frac{32s^6(r')^2}{243R^5}$$

Proposed by Daniel Sitaru,Claudiu Ciulcu-Romania

J.2070 If $\Delta A'B'C'$ is the pedal triangle of I —incenter in ΔABC and $x, y, z > 0$ then:

$$xIA + yIB + zIC \geq 4\left(\frac{yzIA'}{y+z}+\frac{zxIB'}{z+x}+\frac{xyIC'}{x+y}\right)$$

Proposed by Daniel SitaruMirea Mihaela Mioara-Romania

J.2071 Find $x \in \mathbb{Q}$ and $y \in \mathbb{Z}$ such that: $2020(x^2+y^2)+2019(x+y) = 2021xy$

Proposed by George Florin Șerban-Romania

J.2072 Solve for integers: $(6x+5y^2)(4z+x)(2y^2+3z) = 2021$

Proposed by George Florin Șerban-Romania

J.2073 Prove that: $\frac{1}{1+\frac{1}{x^3+\frac{1}{x^2}}} > \frac{1}{x+\frac{1}{x^2+\frac{1}{x^2}}} - \frac{1}{x^3+\frac{1}{x^2+\frac{1}{x}}}, x > 0$

Proposed by George Florin Șerban-Romania

J.2074 Prove that:

$$\prod_{k=1}^{n} k! \cdot k^k < \left(\frac{2n+1}{3}\right)^{n(n+1)}, n \in \mathbb{N}$$

Proposed by Florică Anastase-Romania

J.2075 Find A and prove that $2021 \in A$ if $\overline{abcd} \in A, \frac{a}{d+1} = \frac{c-b}{c} = \frac{a+b}{b+c}$

Proposed by George Florin Șerban-Romania

J.2076 If $x, y, z, t > 0$ such that $\frac{x^2}{1+x^2}+\frac{y^2}{1+y^2}+\frac{z^2}{1+z^2}+\frac{t^2}{1+t^2}=3$, then:

$$xy+xz+xt+yz+yt+zt \leq 2xyzt$$

Proposed by Marin Chirciu-Romania

J.2077 If $x, y, z, t > 0$ such that $\frac{x^3}{1+x^3}+\frac{y^3}{1+y^3}+\frac{z^3}{1+z^3}+\frac{t^3}{1+t^3}=3$, then:

$$xy\sqrt{xy}+xz\sqrt{xz}+xt\sqrt{xt}+yz\sqrt{yz}+zt\sqrt{zt} \leq 2xyzt\sqrt{xyzt}$$

Proposed by Marin Chirciu-Romania

J.2078 In $\Delta ABC, x, y, z > 0$ the following relationship holds:

$$x\cos\frac{A}{2}+y\cos\frac{B}{2}+z\cos\frac{C}{2} \leq \frac{1}{2}(xy+yz+zx)\sqrt{\frac{x+y+z}{xyz}}$$

Proposed by Bogdan Fuștei-Romania

J.2079 Let P be point in plane of ΔABC ant $t > 1$, then holds:

$$(PA\cdot PB)^t+(PB\cdot PC)^t+(PC\cdot PA)^t \geq \frac{(abc)^t}{\left(a^{\frac{t}{t-1}}+b^{\frac{t}{t-1}}+c^{\frac{t}{t-1}}\right)^{t-1}}$$

Proposed by Bogdan Fuștei-Romania

J.2080 In $\Delta ABC, M$ point in plane, the following relationship holds:

$$m_a\cdot AM+m_b\cdot BM+m_c\cdot CM \geq \frac{1}{2}(a^2+b^2+c^2)$$

Proposed by Bogdan Fuștei-Romania

J.2081 In $\Delta ABC, \omega$ –Brocard's angle, the following relationship holds:

$$\frac{1}{\sin\omega} \geq \frac{b}{c}+\frac{c}{b} \geq \frac{2m_a}{h_a}\cdot\sin A$$

Proposed by Bogdan Fuștei-Romania

J.2082 Let n_a, n_b, n_c –Nagel's cevian, then in ΔABC holds:

$$\frac{n_a^2\cdot a^3}{\sqrt{n_b n_c}}+\frac{n_b^2\cdot b^3}{\sqrt{n_c n_a}}+\frac{n_c^2\cdot c^3}{\sqrt{n_a n_c}} \geq 8\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

J.2083 If $m \geq 0, x, y, z > 0$ then in ΔABC with area F, holds:

$$x\left(\frac{1}{y}+\frac{1}{z}\right)(ab)^{m+1}+y\left(\frac{1}{z}+\frac{1}{x}\right)(bc)^{m+1}+z\left(\frac{1}{x}+\frac{1}{y}\right)(ca)^{m+1}\ge 2^{m+3}\left(\sqrt{3}\right)^{1-m}F^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

J.2084 If $x,y,z>0$, then in ΔABC with area F, holds:

$$\left(\frac{1}{y}+\frac{1}{z}\right)xa^2+\left(\frac{1}{z}+\frac{1}{x}\right)yb^2+\left(\frac{1}{x}+\frac{1}{y}\right)zc^2\ge 8\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

J.2085 Let $x\in(0,\infty)$, ΔABC with F area, $A_1\in(BC), B_1\in(CA), C_1\in(AB)$ such that $BA_1=xA_1C$, $CB_1=xB_1A, AC_1=xC_1B$ and $a_1=B_1C_1, b_1=C_1A_1, c_1=A_1B_1$, then:

$$aa_1+bb_1+cc_1\le\frac{4\sqrt{3x}}{x+1}F$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

J.2086 In any ΔABC holds:

$$a^2+b^2+c^2+\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\ge 4\sqrt{3}\left(a-\frac{1}{h_b}\right)^2+\left(b-\frac{1}{h_c}\right)^2+\left(1-\frac{1}{h_a}\right)^2$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

J.2087 Let $M\in Int(\Delta ABC)$ with F area and F_a − area of ΔMBC, F_b −area of ΔMCA, F_c −area of ΔMAB, then holds:

$$\frac{a^8}{F_a}+\frac{b^8}{F_b}+\frac{c^8}{F_c}\ge 256F^3$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.2088 In any ΔABC with F area, holds:

$$\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\ge\frac{\sqrt{3}}{F}+\frac{1}{2}\left(\left(\frac{1}{h_a}-\frac{1}{h_b}\right)^2+\left(\frac{1}{h_b}-\frac{1}{h_c}\right)^2+\left(\frac{1}{h_c}-\frac{1}{h_a}\right)^2\right)$$

Proposed by D.M. Bătinețu-Giurgiu Claudia Nănuți-Romania

J.2089 Let $M\in Int(\Delta ABC)$ with F area and F_a − area of ΔMBC, F_b −area of ΔMCA, F_c −area of ΔMAB, then holds:

$$\frac{a^4}{F_a}+\frac{b^4}{F_b}+\frac{c^4}{F_c}\ge 48F$$

Proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți-Romania

J.2090 Let $m \geq 0, M \in Int(\Delta ABC)$ with F area, $X \in [BC], Y \in [CA], Z \in [AB]$ and F_a −area of $\Delta MBC, F_b$ −area of $\Delta MCA, F_c$ −area of ΔMAB, then holds:

$$\frac{MX^{m+1} \cdot a^{5m+5}}{(F - F_a)^{m+1}} + \frac{MY^{m+1} \cdot b^{5m+5}}{(F - F_b)^{m+1}} + \frac{MZ^{m+1} \cdot c^{5m+5}}{(F - F_c)^{m+1}} \geq \frac{16^{m+1}}{3^m} F^{2m+2}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

J.2091 Let $m \geq 0, M \in Int(\Delta ABC)$ with F area, $X \in [BC], Y \in [CA], Z \in [AB]$ and F_a −area of $\Delta MBC, F_b$ −area of $\Delta MCA, F_c$ −area of ΔMAB, then holds:

$$\frac{MX^{m+1} \cdot a^{3m+3}}{(F - F_a)^{m+1}} + \frac{MY^{m+1} \cdot b^{3m+3}}{(F - F_b)^{m+1}} + \frac{MZ^{m+1} \cdot c^{3m+3}}{(F - F_c)^{m+1}} \geq 4^{m+1}\left(\sqrt{3}\right)^{1-m} F^{m+1}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

J.2092 If $m \geq 0, x, y, z > 0$ and $M \in Int(\Delta ABC)$ with F area and d_a, d_b, d_c −distances from M to the sides BC, CA, AB respectively, then holds:

$$\frac{x^{m+1} \cdot a^{m+2}}{d_a^m (y + z)^{m+1}} + \frac{y^{m+1} \cdot b^{m+2}}{d_b^m (z + x)^{m+2}} + \frac{z^{m+1} \cdot c^{m+2}}{d_c^m (x + y)^{m+1}} \geq 2\left(\sqrt{3}\right)^{m+1} F$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

J.2093 Let $M \in Int(\Delta ABC)$ with F area and F_a − area of $\Delta MBC, F_b$ −area of $\Delta MCA, F_c$ −area of ΔMAB, then holds:

$$\frac{x^{m+1} \cdot a^{2m+2}}{F_b^m (y + z)^{m+1}} + \frac{y^{m+1} \cdot b^{2m+2}}{F_c^m (z + x)^{m+1}} + \frac{z^{m+1} \cdot c^{2m+2}}{F_a^m (x + y)^{m}} \geq 2^{m+1}\left(\sqrt{3}\right)^{m+1} F$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

J.2094 If $x, y > 0$ hen in ΔABC with F area, holds:

$$\frac{a^5}{ax + y\sqrt{bc}} + \frac{b^5}{bx + y\sqrt{ca}} + \frac{c^5}{cx + y\sqrt{ab}} \geq \frac{16}{x + y} F^2$$

Proposed by D.M. Bătineţu-Giurgiu, Dan Nănuţi-Romania

J.2095 In ΔABC the following relationship holds: $(s - a)^2 + (s - b)^2 + (s - c)^2 \geq 9r^2$

Proposed by D.M. Bătineţu-Giurgiu, Dan Nănuţi-Romania

J.2096 If $x, y, z > 0$ then in ΔABC with F area, holds:

$$\left(\frac{x + y}{z}\sqrt{ab} + \frac{z}{x + y}c\right)^2 + \left(\frac{y + z}{x}\sqrt{bc} + \frac{x}{y + z}a\right)^2 + \left(\frac{z + x}{y}\sqrt{ca} + \frac{y}{z + x}b\right)^2 \geq 16\sqrt{3}F$$

Proposed by D.M. Bătineţu-Giurgiu, Dan Nănuţi-Romania

J.2097 If $m \geq 0$ then in ΔABC holds:

$$\frac{h_a^m}{(h_a-r)^m}+\frac{h_b^m}{(h_b-r)^m}+\frac{h_c^m}{(h_c-r)^m} \geq \frac{3^{m+1}}{2^m}$$

Proposed by D.M. Bătineţu-Giurgiu, Dan Nănuţi-Romania

J.2098 Let g_a, g_b, g_c −Gergonne's cevians, the following relationship holds:

$$\frac{g_a-2r}{h_a+r}+\frac{g_b-2r}{h_b+r}+\frac{g_c-2r}{h_c+r} \geq \frac{3}{4}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

J.2099 Let $m \geq 0$ and $M \in Int(\Delta ABC)$ with F area and d_a, d_b, d_c −distances from M to the sides BC, CA, AB respectively, then holds:

$$\frac{a^{m+2}}{d_a^m}+\frac{b^{m+2}}{d_b^m}+\frac{c^{m+2}}{d_c^m} \geq 2^{m+2}\left(\sqrt{3}\right)^{m+1}F$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

J.2100 Let $M \in Int(\Delta ABC)$ with F area and d_a, d_b, d_c −distances from M to the sides BC, CA, AB respectively, then holds:

$$\frac{1}{d_a}+\frac{1}{d_b}+\frac{1}{d_c} \geq \frac{3\sqrt[4]{27}}{\sqrt{F}}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

PROBLEMS FOR SENIORS

S.2001 Let $(a_n)_{n\geq 1}$ be sequence of real numbers such that $a_1 = a \in (0,1)$ and

$$a_{n+1}=\sum_{k=0}^{n}\binom{n}{k}a_k a_{n-k}\text{. Find: }\Omega=\lim_{n\to\infty}\binom{2n}{n}\frac{a_n}{n!}\frac{(1-a)^n}{a}$$

Proposed by Florică Anastase-Romania

S.2002 Solve for real numbers:

$$\begin{cases}2x^2+3y^2+z^2=7\\ x^2+y^2+z^2=\sqrt{2}z(x+y)\end{cases}$$

Proposed by Daniel Sitaru,Roxana Vasile-Romania

S.2003$z_1z_2,z_3\in\mathbb{C}^*$ different in pairs, $|z_1|=|z_2|=|z_3|=1, A(z_1),B(z_2),C(z_3)$. Prove that:

$$\sum_{cyc}\sqrt{|(2z_1-z_2-z_3)(2z_2-z_1-z_3)|}=9\Rightarrow AB=BC=CA$$

Proposed by Marian Ursărescu-Romania

S.2004 If $a,b,c>0,p,q,r>1,pq+qr+rp=pqr$ then: $abcpqr\le qra^p+rpb^q+pqc^r$

Proposed by Daniel Sitaru,Luiza Cremeneanu-Romania

S.2005 For $x,y,z\in\mathbb{R}$,

$$A=\begin{pmatrix}\sin^2x-\cos^2y & \cos^2z & \cos^2z\\ \cos^2x & \sin^2y-\cos^2z & \cos^2x\\ \cos^2y & \cos^2y & \sin^2z-\cos^2x\end{pmatrix}$$

$$B=\begin{pmatrix}\cos^2x-\sin^2y & \sin^2z & \sin^2z\\ \sin^2x & \cos^2y-\sin^2z & \sin^2x\\ \sin^2y & \sin^2y & \cos^2z-\sin^2x\end{pmatrix}$$

Prove that: $\det(AB)\ge 0$.

Proposed by Daniel Sitaru,Cătălin Nicola-Romania

S.2006 If $x,y\in\left(0,\frac{\pi}{2}\right)$ then:

$$\tan^3x+\frac{1}{\tan^3x}+\tan^3y+\frac{1}{\tan^3y}\ge\tan x+\frac{1}{\tan x}+\tan y+\frac{1}{\tan y}$$

Proposed by Daniel Sitaru,Lavinia Trincu-Romania

S.2007 If $n\ge 2,n\in\mathbb{N},a\in(0,\pi)$, then:

$$\left(\frac{2}{n}\right)^n\left(\sum_{k=0}^{n}\cos\frac{a}{2^k}\right)^{n-1}\ge\frac{\sin a}{n^2\sin\frac{a}{2^n}}\left(\sum_{k=1}^{n}\sec\frac{a}{2^k}\right)$$

Proposed by Florică Anastase-Romania

S.2008 If $m, n \in \mathbb{N} - \{0\}$ then:

$$\frac{2^{mn+1}-1}{m(mn+1)} + \log 2 > \frac{2^n-1}{n} + \frac{1}{m}$$

Proposed by Daniel Sitaru,Ileana Stanciu-Romania

S.2009 If $0 \le x, y, z < 1$ then:

$$\frac{x^2}{\sqrt{1-x^2}} + \frac{y^2}{\sqrt{1-y^2}} + \frac{z^2}{\sqrt{1-z^2}} \ge \frac{xy}{\sqrt{1-xy}} + \frac{yz}{\sqrt{1-yz}} + \frac{zx}{\sqrt{1-zx}}$$

Proposed by Daniel Sitaru,Dan Mitricoiu-Romania

S.2010 If $a, b, c, x, y, z > 0$, then holds:

$$\frac{a^2}{xb^2+yca} + \frac{b^2}{xc^2+yab} + \frac{c^2}{xa^2+ybc} \ge \frac{3}{2(x+y)} \cdot \frac{ab+bc+ca}{a^2+b^2+c^2}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.2011 If $u, v, w, x, y > 0$ and $a, b, c > 0, abc = t^3$, then holds:

$$\frac{(xa+yb)^2}{ua+vb+wc} + \frac{(xb+yc)^2}{ub+vc+wa} + \frac{(xc+ya)^2}{uc+va+wb} \ge \frac{3t(x+y)^2}{u+v+w}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.2012 If $x, y \ge 0, x+y > 0$, then holds:

$$\frac{a^3}{ax+by} + \frac{b^3}{bx+cy} + \frac{c^3}{cx+ay} \ge \frac{a^2+b^2+c^2}{x+y}; \forall a, b, c > 0$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.2013 If $m, x, y \ge 0, x+y > 0$ then holds:

$$\frac{a^{m+2}}{(ax+by)^m} + \frac{b^{m+2}}{(bx+cy)^m} + \frac{c^{m+2}}{(bx+ay)^m} \ge \frac{a^2+b^2+c^2}{(x+y)^m}; \forall a, b, c > 0$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.2014 If $u, v, x, y, z > 0$, then holds:

$$\frac{1+x^2}{u+vy+uz^2} + \frac{1+y^2}{u+vz+ux^2} + \frac{1+z^2}{u+vx+uy^2} \ge \frac{6}{2u+v}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.2015 If $m, x, y, z > 0$, then holds:

$$\frac{1+x^2}{1+my+z^2}+\frac{1+y^2}{1+mz+x^2}+\frac{1+z^2}{1+mx+y^2}\geq\frac{6}{m+2}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.2016 If $a,b,c,x,y>0$, then holds:

$$\frac{a}{xb+yc}+\frac{b}{xc+ya}+\frac{c}{xa+yb}+\frac{z^4(ab+bc+ca)}{(x+y)(a^2+b^2+c^2)}\geq\frac{4z}{x+y}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.2017 If $x,y,z>0,t\geq0$ then in ΔABC the following relationship holds:

$$\frac{y+z+2t}{x+t}\cdot\frac{1}{a}+\frac{z+x+2t}{y+t}\cdot\frac{1}{b}+\frac{x+y+2t}{z+t}\cdot\frac{1}{c}\geq\frac{2\sqrt{3}}{R}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

S.2018 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f\left(\frac{x^2-1}{x^2+2}\right)=f(y^2+1)+x^3y,\forall x,y\in\mathbb{R}$.

Proposed by Nguyen Van Canh-Vietnam

S.2019 Find all continuous functions $f:\{-202;2020]\to\mathbb{R}$ such that

$$2020f(x)+2021f'(x)=2022,f(-2020)=f(2020)=\frac{2022}{2020}$$

Proposed by Nguyen Van Canh-Vietnam

S.2020 In ΔABC the following relationship holds:

$$\frac{729}{(4R+r)^4}\leq\sum_{cyc}\frac{1}{m_am_b}\cdot\sum_{cyc}\frac{1}{w_aw_b}\leq\frac{1}{9r^4}$$

Proposed by Nguyen Van Canh-Vietnam

S.2021 In $\Delta ABC,p_a$ –Spieker's cevian, the following relationship holds:

$$\sum_{cyc}p_a^2+2r(R-2r)\leq\sum_{cyc}n_a^2\leq\sum_{cyc}r_a^2$$

Proposed by Nguyen Van Canh-Vietnam

S.2022 Let $f_m(x)=x^3-x+m,g_m(x)=x^4-x^3+mx-1$. Find all positive real numbers m such that $\min f_m(x)\cdot\max g_m(x)+\max f_m(x)\cdot\min g_m(x)=6,\forall x\in[0,1]$

Proposed by Nguyen Van Canh-Vietnam

S.2023 In $\Delta ABC,n_a$ –Nagel's cevian, the following relationship holds:

$$n_a^2+n_b^2+n_c^2\geq 2(a^2+b^2+c^2)-\frac{1}{2}(ab+bc+ca)$$

Proposed by Nguyen Van Canh-Vietnam

S.2024 If $x,y>0$ then:

$$4(x+1)^{x+1}\cdot(y+1)^{y+1}\cdot(x+y)^{x+y}\leq x^x\cdot y^y\cdot(x+y+2)^{x+y+2}$$

Proposed by Daniel Sitaru,Maria Lavinia Popa-Romania

S.2025 If $1\leq x,y\leq 2, 3\leq z,t\leq 4$ then:

$$\frac{x^2+y^2+1}{xz+yt+3}+\frac{\sqrt{6}}{3}\leq\frac{3+xz+yt}{9+z^2+t^2}+\frac{11}{12}$$

Proposed by Daniel Sitaru,Sorin Pîrlea-Romania

S.2026 Let $a,b,c>0$ and $a+b+c=3$. Prove that:

$$\sqrt{a+b}+\sqrt{b+c}+\backslash sqt\ (c+a)+\frac{a^2}{\sqrt{b+c}}+\frac{b^2}{\sqrt{c+a}}+\frac{c^2}{\sqrt{a+b}}\geq\frac{9\sqrt{2}}{2}$$

Proposed by Choy Fai Lam-Hong Kong

S.2027 In acute $\Delta ABC, AD, BE, CF$ −altitudes, H −orthocenter. Prove that:

$$\left(\frac{AH}{HD}\right)^4+\left(\frac{BH}{HE}\right)^4+\left(\frac{CH}{HF}\right)^4\geq 48$$

Proposed by George Apostolopoulos- Greece

S.2028 If $t\geq 0, n\in\mathbb{N}, n\geq 2$ and $m, x_k\in[1,\infty), k=\overline{1,n}$, then:

$$\sum_{k=1}^{n}\left(x_1^{x_k}+x_2^{x_k}+\cdots+x_n^{x_k}\right)^{t+1}\geq n^{t+2}\cdot m^{m(t+1)}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

S.2029 If $x,y,z>0$, then in ΔABC holds:

$$\frac{bc}{x^2}+\frac{ca}{y^2}+\frac{ab}{z^2}+4(xy\cdot c^2+yz\cdot a^2+zx\cdot b^2)\geq 16\sqrt{3}\cdot F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

S.2030 If $m\geq 0$, then:

$$\frac{x^{3m+3}}{(x^3+x^2y)^{m+1}}+\frac{y^{3m+3}}{(x^3+y^2z)^{m+1}}+\frac{z^{3m+3}}{(y^3+z^2x)^{m+1}}\geq\frac{3}{2^{m+1}}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

S.2031 In ΔABC the following relationship holds: $\frac{a}{h_b+h_c}+\frac{b}{h_c+h_a}+\frac{c}{h_a+h_b}\geq\sqrt{3}$

Proposed by D.M. Bătinețu-Giurgiu-Romania

S.2032 If $m\geq 0$ then in ΔABC holds:

$$\left(\frac{a^2b^2}{bc+ca-ab}\right)^{m+1}+\left(\frac{b^2c^2}{ca+ab-bc}\right)^{m+1}+\left(\frac{c^2a^2}{ab+bc+ca}\right)^{m+1}\geq 2^{2m+2}\left(\sqrt{3}\right)^{1-m}\cdot F^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

S.2033 If $x,y,z>0, x\geq y$ then in ΔABC holds:

$$\frac{xh_a-yr}{zh_a+yr}+\frac{xh_b-yr}{zh_b+yr}+\frac{xh_c-yr}{zh_c+yr}\geq\frac{3(3x-y)}{y+3z}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

S.2034 Solve for real numbers: $\sin x+\cos x+\sec x\cdot\csc x=2+\sqrt{2}$

Proposed by Daniel Sitaru,Dorina Goiceanu-Romania

S.2035 If $a,b>0$ then:

$$\frac{\sqrt{2^{a+b}}+\sqrt{3^{a+b}}+\sqrt{5^{a+b}}}{2^{\sqrt{ab}}+3^{\sqrt{ab}}+5^{\sqrt{ab}}}\geq\frac{\frac{1}{2^{\sqrt{ab}}}+\frac{1}{3^{\sqrt{ab}}}+\frac{1}{5^{\sqrt{ab}}}}{\frac{1}{\sqrt{2^{a+b}}}+\frac{1}{\sqrt{3^{a+b}}}+\frac{1}{\sqrt{5^{a+b}}}}$$

Proposed by Daniel Sitaru,Iulia Sanda-Romania

S.2036 Solve for real numbers:

$$\log_x e\cdot(\log x)^{-1}+\log_{\frac{e}{x}} e\cdot\left(\log\frac{e}{x}\right)^{-1}=8$$

Proposed by Daniel Sitaru,Nicolae Radu-Romania

S.2037 If $a,b,x,y>0$ then:

$$\frac{\left(\sqrt{ab}+\sqrt{xy}\right)\left(\frac{a+b}{2}+\frac{x+y}{2}\right)\left(\sqrt{\frac{a^2+b^2}{2}}+\sqrt{\frac{x^2+y^2}{2}}\right)}{\left(\sqrt{ab}+\frac{x+y}{2}\right)\left(\frac{a+b}{2}+\sqrt{\frac{x^2+y^2}{2}}\right)\left(\sqrt{\frac{a^2+b^2}{2}}+\sqrt{xy}\right)}\leq 1$$

Proposed by Daniel Sitaru,Mihaela Stăncele-Romania

S.2038 If $x,y,z>0$ then: $x^8+y^8+z^8+15\geq x^3+y^3+z^3+5\sqrt{3(xy+yz+zx)}$

Proposed by Daniel Sitaru,Mihai Ionescu-Romania

S.2039 If $x, y, z, m \in [1, \infty), 3m = x + y + z$ then in ΔABC holds:

$$a^{2x} + b^{2x} + c^{2x} + a^{2y} + b^{2y} + c^{2y} + a^{2z} + b^{2z} + c^{2z} \geq 4^m \left(\sqrt{3}\right)^{2-m} \cdot F^m$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

S.2040 In $\Delta ABC, g_a$ −Gergonne's cevian, $M \in Int(\Delta ABC), x = MA, y = MB, z = MC$ holds:

$$x^2 g_a^2 + y^2 g_b^2 + z^2 g_c^2 \geq 4F^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

S.2041 If $M \in Int(\Delta ABC), x = MA, y = MB, z = MC$ then:

$$x^2 h_a^2 + y^2 h_b^2 + z^2 h_c^2 \geq 4 \cdot F^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

S.2042 In ΔABC the following relationship holds:

$$\frac{a^2}{s-a} + \frac{b^2}{s-b} + \frac{c^2}{s-c} \geq 4s$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

S.2043 If $m, t \geq 0; x, y, z > 0$ then in ΔABC holds:

$$\frac{(x+y)^m a^{t+1}}{z^m h_b^{t+1}} + \frac{(y+z)^m b^{t+1}}{z^m h_c^{t+1}} + \frac{(z+x)^m c^{t+1}}{y^m h_a^{t+1}} \geq 2^{m+t+1} \left(\sqrt{3}\right)^{1-t}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

S.2044 If $g, m, x, y, z \in [1, \infty), g = \sqrt[3]{xyz}; 3m = x + y + z$ then in ΔABC holds:

$$\frac{x^x + y^x + z^x}{a} + \frac{x^y + y^y + z^y}{b^2} + (x^z + y^z + z^z) a^3 b^4 c^2 \geq 12\sqrt{3} \cdot g^m \cdot F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

S.2045 If $m, x, y, z \in [1, \infty), 3m = x + y + z$ then in ΔABC holds:

$$\frac{x^x + y^x + z^x}{h_a^2} + \frac{x^y + y^y + z^y}{h_b} + (x^z + y^z + z^z) bc \geq 6\sqrt{3} \cdot m^m$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

S.2046 In $\Delta ABC, M \in Int(\Delta ABC)$ the following relationship holds:

$$b \cdot AM + c \cdot BM + a \cdot CM \geq \sqrt{a^2 b^2 + b^2 c^2 + c^2 a^2}$$

Proposed by Bogdan Fuștei-Romania

S.2047 In non-obtuse ΔABC the following relationship holds:

$$m_a + m_b + m_c \geq 2R + 5r$$

Proposed by Bogdan Fuștei-Romania

S.2048 In $\Delta ABC, x, y, z > 0$ the following relationship holds:

$$\sum_{cyc} \sqrt{\frac{x}{y+z}} \cdot \cos\frac{A}{2} \leq \sqrt{\frac{(x+y+z)^2}{(x+y)(y+z)(z+x)}}$$

Proposed by Bogdan Fuștei-Romania

S.2049 Let $M \in Int(\Delta ABC)$ the following relationship holds:

$$\sum_{cyc} \frac{n_a}{h_a^2} \cdot AM \geq \frac{R}{r}$$

Proposed by Bogdan Fuștei-Romania

S.2050 In ΔABC the following relationship holds:

$$\frac{m_a w_a + m_b w_b + m_c w_c}{h_a h_b + h_b h_c + h_c h_a} \geq \sqrt{\frac{m_a m_b m_c}{h_a h_b h_c}}$$

Proposed by Bogdan Fuștei-Romania

S.2051 If $a, b \in \mathbb{N}^*, a < b$, then determine all pairs (a, b) which satisfy $a^2 + b^2 = ab + 2(a + b)$

Proposed by Neculai Stanciu-Romania

S.2052 If $A = \begin{pmatrix} -1 & 14 & 4 \\ -1 & -2 & 1 \\ 2 & -1 & 3 \end{pmatrix}$, then find $A^n, n \in \mathbb{N}^*$.

Proposed by Neculai Stanciu-Romania

S.2053 Let $z \in \mathbb{C}$ such that $|z| = 1$ $Re(z) \geq 0$ and $Im(z) > 0$. Prove that there exist a natural number $n \geq 3$ such that any three of the numbers $|z+1|, |z+i|, |z^{n-1} - 1|$,

$|z^n - 1|, |z^n - i|$ are the sides of an acute triangle.

Proposed by Marius Drăgan, Neculai Stanciu-Romania

S.2054 If $a, y \in \mathbb{R} - \{0,1\}$ and $\{a, b, c, d\} = \left\{x, \frac{1}{x}, y, \frac{1}{y}\right\}$, then prove that

$a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$ and compute $\Omega = \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} + \frac{1}{1-d}$.

Proposed by Neculai Stanciu-Romania

S.2055 If

$$\sum_{cyc}\sqrt{\frac{m_a}{(r_b+m_a)(r_c+m_a)}}=\sum_{cyc}\frac{1}{\sqrt{r_a}+\sqrt{r_b}}$$

then show that triangle ABC is equilateral.

Proposed by Alex Szoros-Romania

S.2056

$$4, 9, 36, 81, 100, 576, 625, \ldots$$

I have some perfect squares and I observe that each perfect square has an odd number of distinct factors. For example: $d(4)=3, d(36)=9, d(81)=5$

Is it true that a perfect square always has an odd number of distinct factors?

Proposed by Naren Bhandari-Nepal

S.2057 If $0\le a,b<\frac{\pi}{2}$ then:

$$3\left(\tan\left(\frac{a+b}{2}\right)+\tan\left(\sqrt{ab}\right)\right)\ge 2\tan\left(\sqrt{\frac{a^2+b^2}{2}}\right)+4\tan\left(\frac{2ab}{a+b}\right)$$

Proposed by Seyran Ibrahimov-Azerbaijan

S.2058 Let $z_1,z_2,z_3\in\mathbb{C}^*$ −different in pairs, $A(z_1),B(z_2),C(z_3),|z_1|=|z_2|=|z_3|=1$.

Prove that:

$$\sum_{cyc}\frac{|z_2+z_3-2z_1|}{|z_2-z_3|^2}=3\Rightarrow AB=BC=CA$$

Proposed by Marian Ursărescu-Romania

S.2059 If $a,b,c>0$ such that $ab+bc+ca=abc$ then prove: $\frac{a+b+c}{3}\sqrt{\frac{a+b+c+9}{6abc}}\ge 1$

Proposed by Neculai Stanciu-Romania

S.2060 Let $z_1,z_2,z_3\in\mathbb{C}^*$ −different in pairs, $A(z_1),B(z_2),C(z_3),|z_1|=|z_2|=|z_3|=1$.

Prove that:

$$\sum_{cyc}\frac{|(z_1-z_2)|z_1-z_3|+(z_1-z_3)|z_1-z_2||}{|z_2-z_3|^2}=\sum_{cyc}|z_1-z_2|\Rightarrow AB=BC=CA$$

Proposed by Marian Ursărescu-Romania

S.2061 In ΔABC the following relationship holds:

$$\frac{a^2}{m_a^2}+\frac{b^2}{m_b^2}+\frac{c^2}{m_c^2}\geq 4$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

S.2062 Let $m\geq 0, M\in Int(\Delta ABC), F-$ area of $\Delta ABC, F_a$ $-$area of MBC, F_b $-$area of ΔMAB and d_a, d_b, d_c $-$distances from M to the sides BC, CA and AB respectively, then:

$$\frac{a^{m+1}\cdot b^{2m+1}}{d_a^m\cdot F_a^m}+\frac{b^{m+1}\cdot c^{2m+1}}{d_c^m\cdot F_b^m}+\frac{c^{m+1}\cdot a^{2m+1}}{d_c^m\cdot F_c^m}\geq 2^{3m+2}\cdot\left(\sqrt{3}\right)^{2m+1}\cdot F$$

Proposed by D.M. Bătineţu-Giurgiu, Florică Anastase-Romania

S.2063 If $M\in Int(\Delta ABC)$ with area F, F_a $-$area of $\Delta MBC, F_b$ $-$area of $\Delta MCA, F_c$ $-$area of MAB, then:

$$\frac{a^4}{F_a}+\frac{b^4}{F_b}+\frac{c^4}{F_c}\geq 48F$$

Proposed by D.M. Bătineţu-Giurgiu, Florică Anastase-Romania

S.2064 If $x,y,z>0$, then in ΔABC with area F, holds:

$$\frac{x}{\sqrt{yz}}\cdot ab+\frac{y}{\sqrt{zx}}\cdot bc+\frac{z}{\sqrt{xy}}\cdot ca\geq 4\sqrt{3}\cdot F$$

Proposed by D.M. Bătineţu-Giurgiu, Florică Anastase-Romania

S.2065 Let $x,y>0, A_1A_2\dots A_n, n\geq 3, A_{n+1}=A_1$ a convex polygon with F area and

$M\in Int(A_1A_2\dots A_n)$. Let $a_k=A_kA_{k+1}, k\in\overline{1,n}$ lengths of sides and F_k area of $MA_kA_{k+1}, k\in\overline{1,n}$, then holds:

$$\sum_{k=1}^{n}\frac{a_k^4}{xF_k+yF_{k+1}}\geq\frac{16}{x+y}\cdot F\cdot\tan^2\frac{\pi}{n}$$

Proposed by D.M. Bătineţu-Giurgiu, Florică Anastase-Romania

S.2066 Let $x,y>0, m\geq 0, A_1A_2\dots A_n, n\geq 3, A_{n+1}=A_1$ a convex polygon with F area and

$M\in Int(A_1A_2\dots A_n)$. Let $a_k=A_kA_{k+1}, k\in\overline{1,n}$ lengths of sides and F_k area of $MA_kA_{k+1}, k\in\overline{1,n}$, then holds:

$$\sum_{k=1}^{n}\frac{a_k^{2m\neq 2}}{(xF_k+yF_{k+1})^m}\geq\frac{2^{2m+2}}{(x+y)^m}\cdot F\cdot\left(\tan\frac{\pi}{n}\right)^{m+1}$$

Proposed by D.M. Bătineţu-Giurgiu, Florică Anastase-Romania

S.2067 Let $t, x, y, z > 0$ and ΔABC with F area such that $ax + by + cz = tF$, then:

$$\frac{a^{m+2}}{x^m} + \frac{b^{m+2}}{y^m} + \frac{c^{m+2}}{z^m} \geq \frac{4^{m+1}\left(\sqrt{3}\right)^{m+1}}{t^m} \cdot F, \forall m \geq 0$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

S.2068 Let $m \geq 0$ and ΔABC with F area, then holds:

$$\frac{a^{m+2}}{r_a^m} + \frac{b^{m+2}}{r_b^m} + \frac{c^{m+2}}{r_c^m} \geq 2^{m+2}\left(\sqrt{3}\right)^{m+1}\left(\frac{R}{2R-r}\right)^m \cdot F$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

S.2069 If $x, y, z > 0$ and ΔABC with F area, then holds:

$$(x+y)ab + (y+z)bc + (z+x)ca \geq 8\sqrt{xy+yz+zx} \cdot F$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

S.2070 If $x, y, z > 0$ then in ΔABC with F area holds: $\frac{xa^2+yb^2}{z} + \frac{yb^2+zc^2}{x} + \frac{zc^2+xa^2}{y} \geq 8\sqrt{3}F$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

S.2071 If $x, y, z > 0$, then in $D\ ABC$ with area F, holds:

$$\frac{(ax+by)c}{z} + \frac{(by+cz)a}{x} + \frac{(cz+ax)b}{y} \geq 8\sqrt{3}F$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

S.2072 If $m \geq 0, x, y, z > 0$ then in ΔABC with F area, holds:

$$(x^{m+1} + y^{m+1})(ab)^{m+1} + (y^{m+1} + z^{,=m+1})(bc)^{m+1} + (z^{m+1} + x^{m+1})(ca)^{m+1} \geq$$

$$\geq \frac{2^{2m+3}}{3^m}(xy+yz+zx)^{\frac{m+1}{2}} \cdot F^{m+1}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

S.2073 If the sequence $(a_n)_{n\geq 1}$ is defined by $a_0 = a_2 = 0, a_1 = 1$ and $a_{n-1} + a_{n-2} + a_{n-3} = 3a_n$ then prove that $\lim_{n\to\infty} |a_n| \leq \frac{1}{3}$.

Proposed by Neculai Stanciu-Romania

S.2074 In ΔABC, the following relationship holds:

$$\left(4R + r - a\sqrt{3}\right)bc + \left(4R + r - b\sqrt{3}\right)ca + \left(4R + r - c\sqrt{3}\right)ab \geq 12\sqrt{3}rF$$

Proposed by D.M. Bătineţu-Giurgiu, Dan Nănuţi-Romania

S.2075 If $a, b, c > 0$ then

$$\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{b+c}{\sqrt{(a+b)(a+c)}}$$

Proposed by Marin Chirciu-Romania

S.2076 In ΔABC the following relationship holds:

$$2\sqrt{6}\leq\sum_{cyc}\sqrt{\sec^2\frac{B}{2}+\sec^2\frac{C}{2}}\leq\sqrt{6\left(2+\frac{R}{r}\right)}$$

Proposed by Marin Chirciu-Romania

S.2077 Let $a, b\in\mathbb{Z}, a=2m, b=2n, m, n\in\mathbb{Z}$. Prove that exists $p, q\in 8\mathbb{Z}$ such that:

$$(a^2+b^2)^3=p^2+q^2$$

Proposed by Laura and Gheorghe Molea-Romania

S.2078 Let $a, b, c > 0$. Prove that:

$$3\sum_{cyc}a^5+\sum_{cyc}a^2b^2(a+b)\geq 5(ab^4+bc^4+ca^4)\geq 5abc(a^2+b^2+c^2)$$

Proposed by Nguyen Van Canh-Vietnam

S.2079 Solve for integers $q^3=p^2+5$. Solve for natural numbers $19^c=2^a+5^b$.

Proposed by Hikmat Mammadov-Azerbaijan

S.2080 In ΔABC the following relationship holds:

$$9\leq\sum_{cyc}\sqrt{\frac{n_a}{n_b+n_c}}\cdot\sum_{cyc}\sqrt{\frac{n_b+n_a}{n_c}}\leq\frac{27R^2+8s^2}{36r^2}$$

Proposed by Nguyen Van Canh-Vietnam

S.2081 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{n_a^2+r_a^2}{2m_a+n_a-h_a}\geq 4R+r$$

Proposed by Bogdan Fuștei-Romania

S.2082 Solve for natural numbers: $x^3+y^4=2022$

Proposed by George Florin Șerban-Romania

S.2083 $M \in Int(\Delta ABC)$, the following relationship holds:

$$\sum_{cyc} \frac{n_a}{h_a^2} \cdot AM \geq \frac{s\sqrt{6}}{\sqrt{(\sum h_a)^2 - \frac{3}{4}(\max\{h_a, h_b, h_c\} - \min\{h_a, h_b, h_c\})^2}}$$

Proposed by Bogdan Fuștei-Romania

S.2084 In acute ΔABC the following relationship holds:

$$m_a + m_b + m_c \geq \sum_{cyc} \frac{g_a^2}{h_a}$$

Proposed by Bogdan Fuștei-Romania

S.2085 If $a, b, c \in \mathbb{R}, 0 \leq \alpha \leq 1$ then:

$$|a + b + c|^\alpha \leq \frac{1}{3} \sum_{cyc} (|a + b|^\alpha + |c|^\alpha)$$

Proposed by Seyran Ibrahimov-Azerbaijan

S.2086 Which type of triangles ABC verify $b \cos A + c \cos B + a \cos C = s$?

Proposed by Neculai Stanciu-Romania

S.2087 Solve for integers:

$$\frac{1201x^2 + 799x + 15578}{x^2 + x + 20} + \frac{109x^2 + 71x + 388}{x^2 + x + 6} + \frac{1729x^2 + 1151x + 27046}{x^2 + x + 24} = 2021$$

Proposed by Neculai Stanciu, George Florin Șerban-Romania

S.2088 Solve for real numbers:

$$x\sqrt{3a^2 - 1 + x} + \sqrt{12a^2 + 2 - 2x} = 3a\sqrt{x^2 + 2}, a > 1, a \text{ –fixed.}$$

Proposed by Marin Chirciu-Romania

S.2089 If $x, y, z > 0$ such that $x + y \leq 2, y + z \leq 2, z + x \leq 2$, then:

$$\sum_{cyc} \frac{1}{x + 1} \leq \sum_{cyc} \frac{1}{1 + \sqrt{yz}}$$

Proposed by Marin Chirciu-Romania

S.2090 For $x_1, x_2, \dots, x_{2021} > 0$, prove that:

$$\sqrt{\frac{x_1^2 + 1}{2}} + \sqrt{\frac{x_2^2 + 1}{2}} + \dots \sqrt{\frac{x_{2021}^2 + 1}{2}} \geq \sqrt{2021(x_1 + x_2 + \dots + x_{2021})}$$

Proposed by Neculai Stanciu-Romania

S.2091 If $x, y > 0$, then prove that:

$$\frac{x^2}{y}+\frac{y^2}{x}\geq\sqrt{2(x^2+y^2)}+\frac{3(x^2-y^2)^2}{4xy\left(\sqrt{2(x^2+y^2)}+x+y\right)}$$

Proposed by Neculai Stanciu-Romania

S.2092 In ΔABC the following relationship holds:

$$8\prod_{cyc}\cos\frac{A-B}{2}\leq\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^4+3\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2+4$$

Proposed by Adil Abdullayev-Azerbaijan

S.2093 In ΔABC the following relationship holds: $\max\{s_a, s_b, s_c\}\geq\min\{w_a, w_b, w_c\}$

Proposed by Adil Abdullayev-Azerbaijan

S.2094 In ΔABC the following relationship holds:

$$\left(\frac{m_a+m_b}{a+b}\right)^2+\left(\frac{m_b+m_c}{b+c}\right)^2+\left(\frac{m_c+m_a}{c+a}\right)^2\geq\frac{9}{4}$$

Proposed by Adil Abdullayev-Azerbaijan

S.2095 In acute ΔABC the following relationship holds:

$$\cos(A-B)\cos(B-C)\cos(C-A)\leq\cos^2\left(\frac{A-B}{2}\right)$$

Proposed by Adil Abdullayev-Azerbaijan

S.2096 In ΔABC the following relationship holds:

$$\frac{R}{r}+\frac{16abc}{(a+b)(b+c)(c+a)}\geq 3+\frac{\left(r_a^2+r_b^2+r_c^2\right)}{r_ar_b+r_br_c+r_cr_a}$$

Proposed by Adil Abdullayev-Azerbaijan

S.2097 Let K be the symmedian point in $\Delta ABC, D, E, F$ —circumcenters of $\Delta BCK, \Delta CAK, \Delta ABK$.

Prove that:

$$\frac{[ABC]}{[DEF]}\leq\frac{8}{9}+2\left(\frac{2r}{R}\right)^2$$

Proposed by Adil Abdullayev-Azerbaijan

S.2098 In ΔABC the following relationship holds:

$$\frac{R}{2r} \ge \sqrt{\frac{m_a m_b + m_b m_c + m_c m_a}{w_a w_b + w_b w_c + w_c w_a}}$$

Proposed by Adil Abdullayev-Azerbaijan

S.2099 In ΔABC the following relationship holds:

$$\frac{m_a m_b m_c (m_a + m_b + m_c)}{9F^2} \ge \sqrt{\frac{m_a m_b m_c}{h_a h_b h_c}}$$

Proposed by Adil Abdullayev-Azerbaijan

S.2100 In ΔABC the following relationship holds:

$$\frac{3(ab + bc + ca)}{a^2 + b^2 + c^2} \le \frac{m_a m_b + m_b m_c + m_c m_a}{m_a^2 + m_b^2 + m_c^2} + 2$$

Proposed by Adil Abdullayev-Azerbaijan

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

UNDERGRADUATE PROBLEMS

U.1292 If $0 < a \le b$ then:

$$\int_a^b \frac{x^{19}}{\sqrt{1 + x^{30}}} dx \ge \log \sqrt[10]{\frac{2 + b^{20}}{2 + a^{20}}}$$

Proposed by Daniel Sitaru-Romania

U.1293 If $x_m = \lim\limits_{n\to\infty} \sum_{k=0}^{m} \left\{\sqrt{n^2 + (2k+1)n + k^2 + k}\right\}, m, n \in \mathbb{N}^*, m > 4$ then:

$$m\sqrt[m]{m+1} < \sum_{k=1}^{m} \frac{1}{x_k} < 2\sqrt{m}$$

Proposed by Florică Anastase-Romania

U.1294 $z_1z_2, z_3 \in \mathbb{C}^*$ different in pairs, $|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$. Prove that:

$$\sum_{cyc}\left|\frac{z_1+z_2}{z_1-z_2}\right|^2 = 1 \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu-Romania

U.1295 Find:

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\sum_{i=1}^{n}\frac{1}{(i+1)(n-i+1)}\binom{2i}{i}\binom{2n-2i}{n-i}}$$

Proposed by Daniel Sitaru,Alecu Orlando-Romania

U.1296 Find:

$$\Omega = \int_0^{\frac{\pi}{6}}\frac{\sin x\cdot\sin\left(x+\frac{\pi}{3}\right)\cdot\sin\left(x+\frac{2\pi}{3}\right)}{\sin 3x+\cos 3x}dx$$

Proposed by Daniel SitaruCătălin Pană-Romania

U.1297 Find $f, g: (1,\infty) \to \mathbb{R}$ such that $f(x) = ax - xg'(x)\cdot\log x$ and

$$g(x) = ax - xg'(x)\cdot\log x\,, x > 1, a \in \mathbb{R}$$

Proposed by Florică Anastase-Romania

U.1298 Let $0 < a < b, m = \frac{a+b}{2}$ and $f: [a,b] \to \mathbb{R}$ derivable with derivative continuous on $[a,b]$ such that $f(m) = 0$, then prove:

$$\int_a^b\left(f'(x)\right)^2dx \ge \frac{12}{(b-a)^3}\left(\int_a^b f(x)dx\right)^2$$

Proposed by Florică Anastase-Romania

U.1299 Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$f\left(f(x+y)\right) = f(x)f(y) + x^\alpha y^\beta; \forall x, y \in \mathbb{R}, \alpha, \beta > 0$$

Proposed by Nguyen Van Canh-Vietnam

U.1300 In $\Delta ABC, n_a$ —Nagel's cevian, g_a —Gergonne's cevian, the following relationship holds:

$$\sum_{cyc}(n_a+m_a) \le \sqrt{3}\left(\sqrt{\sum_{cyc}g_a^2+\frac{s^4(R^2-4r^2)}{3r^4}}+\sqrt{\sum_{cyc}w_a^2+\frac{s^2(R^3-8r^2)}{2r^2}}\right)$$

Proposed by Nguyen Van Canh-Vietnam

U.1301 In $\Delta ABC, g_a$ —Gergonne's cevian, the following relationship holds:

$$\sqrt[3]{\prod_{cyc} m_a^2} + \frac{1}{3}\sum_{cyc}\left|m_a^2 - m_b^2\right| \geq \frac{1}{3}\sum_{cyc} m_a^2 \geq \frac{1}{3}\left(\sum_{cyc} g_a^2 + \frac{r(R-2r)}{4}\right)$$

Proposed by Nguyen Van Canh-Vietnam

U.1302 Let $x, y, z, t > 0, \alpha \geq \beta \geq 1$ such that $\sum_{cyc} x = \sum_{cyc} x^4$ then:

$$\left(\sum_{cyc} x\right)^{2\alpha}\left(\prod_{cyc} x\right)^{2\beta} + \alpha^{\sum_{cyc} x}\beta^{\left(\sum_{cyc} x\right)^2} \leq 16^\alpha + \alpha^4\beta^{16}$$

Proposed by Nguyen Van Canh-Vietnam

U.1303 In ΔABC the following relationship holds:

$$\sum_{cyc}(h_a^2 + w_a^2) + 4(R^2 - 4r^2) \geq 2\sum_{cyc} m_a^2$$

Proposed by Nguyen Van Canh-Vietnam

U.1304 In $\Delta ABC, p_a$ —Spieker's cevian, $\alpha \geq 6$. Prove that:

$$\sum_{cyc} n_a^2 \leq \sum_{cyc} p_a^2 + \alpha(R^2 - 4r^2)$$

Proposed by Nguyen Van Canh-Vietnam

U.1305 Find all numbers $\alpha > 0$ such that:

$$\frac{x+y+z}{\sqrt[3]{xyz}} + \frac{\alpha xyz}{x^2y + y^2z + z^2x + xyz} \geq 4; x, y, z > 0$$

Proposed by Nguyen Van Canh-Vietnam

U.1306 In $\Delta ABC, n_a$ —Nagel's cevian, the following relationship holds:

$$\sum_{cyc} m_a^2 + 3r(R - 2r) \leq \sum_{cyc} n_a^2$$

Proposed by Nguyen Van Canh-Vietnam

U.1307 In $\Delta ABC, n_a$ —Nagel's cevian, the following relationship holds:

$$\sum_{cyc} n_a^2 \leq \sum_{cyc} m_a^2 + 7(R^2 - 4r^2)$$

Proposed by Nguyen Van Canh-Vietnam

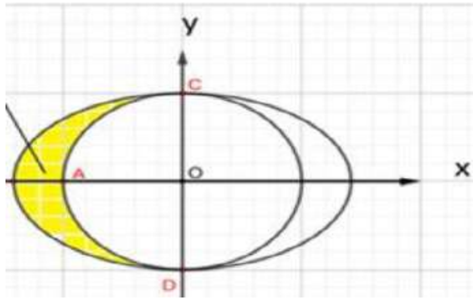
U.1308 In $\Delta ABC, n_a$ —Nagel's cevian, the following relationship holds:

$$\sum_{cyc} \min\left\{n_a, m_a + \frac{|b-c|}{2}\right\} \le \sqrt{3\sum_{cyc} m_a^2 + 21(R^2 - 4r^2)}$$

Proposed by Nguyen Van Canh-Vietnam

U.1309 In xOy, let $A(-a,0), E(-b,), C(0,a), D(0,-a)$

with $b > a > 1$. Find the maximum and minimum of the expression $P = x^3 + y^3 + x^2 + y^2; \forall (x,y) \in S$



Proposed by Nguyen Van Canh-Vietnam

U.1310 Let $k \in \mathbb{Z}_+, 0 < \alpha \le \beta \le \gamma \le \delta, \frac{1}{\alpha} + \frac{2}{\beta} + \frac{\delta}{\gamma} \ge 3, \frac{2}{\beta} + \frac{\delta}{\gamma} \ge 2$. Prove that:

$$\alpha^k + \beta^k + \gamma^k - \delta^k \le 2^k + 1$$

Proposed by Nguyen Van Canh-Vietnam

U.1311 For $n \in \mathbb{N}, n \ge k \ge 1$ and $x \in \left(0, \frac{\pi}{2}\right)$ prove that:

$$\sum_{j=0}^{k-1} \left(1 - \frac{j}{k}\right)\binom{n}{j} x^{2j} \cos^{2n} x \le \left(1 - \frac{\sin^2 x}{k}\right)^n$$

Proposed by Florică Anastase-Romania

U.1312 Prove that:

$$\psi^{(1)}\left(\frac{1}{6}\right) - \psi^{(1)}\left(\frac{5}{6}\right) = 10\psi^{(1)}\left(\frac{1}{3}\right) - \frac{20}{3}\pi^2$$

Proposed by Fao Ler-Iraq

U.1313 Let $f(y,n) = F_x[\frac{\sin(nx)}{x}](y)$ the prove the summation:

$$\sum_{n=1}^{m} \cos(2\pi n) f\left(\frac{\pi}{n}, n\right)^k = \left(\frac{\pi}{2}\right)^{\frac{k}{2}} (m-1)$$

where $m \ge 2, k \ge 1$ and $F_x[f](y)$ —is Fourier transform.

Proposed by Srinivasa Raghava-AIRMC-India

U.1314 Find a closed form:

$$\Omega=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{\pi^n}\cdot\left(\frac{\pi}{e}\right)^k$$

Proposed by Daniel Sitaru,Ramona Nălbaru-Romania

U.1315 Let $f:[0,1]\to\mathbb{R}, f$ −continuous. Prove that:

$$\int_0^1 f(x)dx=9\Rightarrow\int_0^1 f^2(x)dx\geq 1+4\int_0^1 xf(x)dx$$

Proposed by Daniel Sitaru,Luiza Dumitrescu-Romania

U.1316

$$\Omega_1=1-\frac{\pi}{2}+\sum_{n=2}^{\infty}\left(-\frac{1}{\pi}\right)^n\cdot\frac{1}{n+1},\Omega_2=1-\frac{\pi}{2}+\sum_{n=2}^{\infty}\left(-\frac{1}{e}\right)^n\cdot\frac{1}{n+1}$$

$$A.\ \Omega_1<\Omega_2\quad B.\Omega_1=\Omega_2\quad C.\Omega_1>\Omega_2$$

Proposed by Daniel Sitaru,Camelia Dană-Romania

U.1317 Find:

$$\Omega=\lim_{n\to\infty}\int_0^1\left([nx]\cdot\left|x-\left[x+\frac{1}{2}\right]\right|\right)dx\,,[*]-GIF.$$

Proposed by Daniel Sitaru,Ileana Duma-Romania

U.1318$0<a\leq b<1, f:[0,1]\to[0,1], f$ −continuous. Prove that:

$$2\int_{\sqrt{ab}}^{\frac{a+b}{2}}xf(x)dx\geq\left(\int_{\sqrt{ab}}^{\frac{a+b}{2}}f(x)dx\right)\left(\int_0^{\frac{a+b}{2}}f(x)dx+\int_0^{\sqrt{ab}}f(x)dx\right)$$

Proposed by Daniel Sitaru,Gigi Zaharia-Romania

U.1319 Find:

$$\Omega=\lim_{n\to\infty}\left(n-\sum_{k=1}^{n}\frac{(e-1)n}{n+(e-1)k}\right)$$

Proposed by Daniel Sitaru,Nedelcu Elena-Romania

U.1320 Let $h:\mathbb{N}\to\mathbb{N}$ be a strictly increasing function such that $h(3m)=h\Big(h\big(h(m)\big)\Big)=3h(m)$, for all natural numbers. Find: $h(2021)$.

Proposed by Hikmat Mammadov-Azerbaijan

U.1321 Find a closed form:

$$\Omega\int_0^1 x^2\tan^{-1}(2x)\log^2(3x)\,dx$$

Proposed by Ose Favour-Nigeria

U.1322 Find:

$$\Omega(n)=\int_1^n([x]^2\cdot\{x\}+[x]\cdot\{x\}^2)dx\,,n\in\mathbb{N},[*]-\text{GIF},\{x\}=x-[x]$$

Proposed by Togrul Ehmedov-Azerbaijan

U.1323 Find:

$$\Omega=\lim_{n\to\infty}\int_0^5\frac{(1-x)x^{n+4}}{1+x^{3n}}dx$$

Proposed by Daniel Sitaru,Marian Ciuperceanu-Romania

U.1324 Prove that:

$$\int_0^1\frac{(\tan^{-1}x)^3}{x+1}dx=\frac{\pi(24\pi C-63\zeta(3)+2\pi^2\log 2)}{256}$$

Proposed by Fao Ler-Iraq

U.1325 Let $x\in\mathbb{Z}_+$. Prove that:

$$\int_0^\infty\frac{\sin(x^{-n})\log x}{x}dx=\frac{\pi\gamma}{2n^2}$$

where γ is the Euler-Mascheroni constant.

Proposed by Max Wong-Hong Kong

U.1326 Find a closed form:

$$\Omega=\int_0^1\frac{\sin^{-1}(\sqrt{x})\cdot\log^2 x\cdot\log^2(1-x)}{x(1-x)}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

U.1327 Find a closed form:

$$\Omega=\int_0^1\frac{\sin^{-1}\left(\frac{2x}{1+x^2}\right)\cdot\tan^{-1}x}{1+x}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

U.1328 If $m \in \mathbb{R}_+ = [0, \infty)$ then in ΔABC holds:

$$\frac{a^{2m+1} + b^{2m+1}}{(ab)^m} + \frac{b^{2m+1} + c^{2m+1}}{(bc)^m} + \frac{c^{2m+1} + a^{2m+1}}{(ca)^m} \geq 4 \cdot \sqrt[4]{27} \cdot \sqrt{F}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1329 If $x, y, z > 0$ then in ΔABC holds:

$$\frac{x+y}{z} \cdot \frac{(r_a + r_b)(r_a + r_c)}{r_b r_c} + \frac{y+z}{x} \cdot \frac{(r_b + r_c)(r_b + r_a)}{r_c r_a} + \frac{z+x}{y} \cdot \frac{(r_c + r_a)(r_c + r_b)}{r_a r_b} \geq 24$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1330 If $x, y > 0$ then in ΔABC holds:

$$h_a^2 \left(\frac{x}{b} + \frac{y}{c}\right)^2 + h_b^2 \left(\frac{x}{c} + \frac{y}{a}\right)^2 + h_c^2 \left(\frac{x}{a} + \frac{y}{b}\right)^2 \geq \frac{18xy \cdot r}{R}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1331 In ΔABC the following relationship holds:

$$a + b + c \geq 2 \cdot \sqrt[4]{27} \cdot \sqrt{F} + \frac{1}{2}\left(\left(\sqrt{a} - \sqrt{b}\right)^2 + \left(\sqrt{b} - \sqrt{c}\right)^2 + \left(\sqrt{c} - \sqrt{a}\right)^2\right)$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1332 If $x, y > 0, M \in Int(\Delta ABC), R_a = MA, R_b = MB, R_c = MC$ and $d_a = d(M, BC), d_b = d(M, CA), d_c = d(M, AB)$, then:

$$x^2\left(R_a^2 + R_b^2 + R_c^2\right) + y^2\left(d_a^2 + d_b^2 + d_c^2\right) \geq 4xy(d_a d_b + d_b d_c + d_c d_a) + \sum_{cyc} (xR_a - yd_a)^2 \geq$$

$$\geq 4xy(d_a d_b + d_b d_c + d_c d_a)$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1333 If $x, y, z > 0$ then in ΔABC holds:

$$\frac{x}{(y+z)h_a} + \frac{y}{(z+x)h_b} + \frac{z}{(x+y)h_c} \geq \frac{\sqrt[4]{27}}{2\sqrt{F}}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1334 If $a, b, c, t, x, y, z > 0$ and $m \geq 0$ then:

$$\frac{(ta^2 + xb^2)^{m+1}}{\left(c(ya + zb)\right)^m} + \frac{(tb^2 + xc^2)^{m+1}}{\left(a(yb + zc)\right)^m} + \frac{(tc^2 + xa^2)^{m+1}}{\left(b(yc + za)\right)^m} \geq \frac{(t+x)^{m+1}}{(y+z)^m}(a^2 + b^2 + c^2)$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1335 If we have the function

$$S(n)=\frac{1}{4}\sin\left(\frac{n\pi}{2}\right)+\frac{4}{9}\sin\left(\frac{2n\pi}{3}\right)+\frac{9}{16}\sin\left(\frac{3n\pi}{4}\right)$$

then prove the sum relation:

$$\sum_{n=1}^{\infty}\frac{(-1)^n S(n)S'(n)}{n^3}=\frac{20301013\pi^4}{1289945088},\qquad S'(n)=\frac{\partial S(n)}{\partial n}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.1336 Find:

$$\Omega=\int_0^{\infty}\frac{\sqrt{x}\log x}{x^2-x+1}dx$$

Proposed by Vasile Mircea Popa-Romania

U.1337

$$\Omega(n)=\sum_{k=1} e^{4\left(2-\frac{k}{n}\right)}\cdot\sum_{k=1}^{n} e^{6\left(2-\frac{k}{n}\right)}-\left(\sum_{k=1}^{n} e^{5\left(2-\frac{k}{n}\right)}\right)^2$$

Find:

$$\Omega=\lim_{n\to\infty} n^7\cdot\Omega(n)\cdot\sin\left(\frac{1}{n^4}\right)\cdot\tan\left(\frac{1}{n^5}\right)$$

Proposed by Daniel Sitaru-Romania

U.1338 If $m,n,p,s\in\mathbb{N}-\{0\}$ –fixed, then find:

$$\Omega(m,n,p,s)=\lim_{n\to\infty}\sum_{1\le i<j\le n}\left(\frac{1}{i^{m+r}}-\frac{1}{j^{m+r}}\right)\left(\frac{1}{i^{p+s}}-\frac{1}{j^{p+s}}\right)$$

Proposed by Daniel Sitaru-Romania

U.1339 If $x,y,z>0$ then:

$$\frac{\sqrt[3]{2y(x+y)^2}}{\sqrt{y(2x+y)}}+\frac{\sqrt[3]{2z(y+z)^2}}{\sqrt{z(2y+z)}}+\frac{\sqrt[3]{2x(z+x)}}{\sqrt{x(2z+x)}}\ge 2\sqrt{3}$$

Proposed by Daniel Sitaru-Romania

U.1340 Solve for complex numbers: $x^4+(1+i)x^3+2ix^2+(i-1)x-1=0$

Proposed by Daniel Sitaru-Romania

U.1341 Without any software:

$$\Omega=\log_2(\log_2 e)+\log(\log\pi)+\log_{\pi}(\log_{\pi}2)$$

$$A.\Omega < 0 \quad B.\Omega = 0 \quad C.\Omega > 0$$

Proposed by Daniel Sitaru-Romania

U.1342 If $0 \le x \le \frac{\pi}{2}$ then:

$$\frac{1}{\sin x} + \frac{2}{\pi x} \le \left(1 - \frac{2}{\pi}\right)^2 + \frac{1}{x} + \frac{2}{\pi}$$

Proposed by Daniel Sitaru-Romania

U.1343 If $\alpha, \beta, \gamma, \delta \in \left(0, \frac{\pi}{2}\right), 16 \sin\alpha \cdot \sin\beta \cdot \sin\gamma \cdot \sin\delta = 9$ then:

$$8\sqrt{3} + \sum_{cyc} \frac{1}{\left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}\right)^2} \le 16$$

Proposed by Daniel Sitaru-Romania

U.1344 Solve for real numbers: $x^{32} + x^{16} + y^2 = 2\sqrt{2}x^{12}y$

Proposed by Daniel Sitaru,Carina Viespescu-Romania

U.1345 Solve for real numbers:

$$\begin{cases} x, y, z, t > 0 \\ 8x^4 + 64y^4 + 216z^4 + 1728t^4 = 1 \\ x + y + z + t = 1 \end{cases}$$

Proposed by Daniel Sitaru-Romania

U.1346 Solve for real numbers:

$$\begin{cases} x, y, z > 0 \\ x^3 + y^3 + z^3 + 3\left(\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z}\right) = 12 \\ xyz = 1 \end{cases}$$

Proposed by Daniel Sitaru-Romania

U.1347 Solve for real numbers:

$$\frac{1}{1 + \tan^4 x} + \frac{1}{10} = \frac{2}{1 + 3\tan^2 x}$$

Proposed by Daniel Sitaru-Romania

U.1348 If $a, b, c > 0, a + b + c = 3$ then:

$$\frac{3(1+b)(1+c) + 3(1+c)(1+a) + 3(1+a)(1+b)}{a^7(1+b)(1+c) + b^7(1+c)(1+a) + c^7(1+a)(1+b)} \le \frac{1}{a^6} + \frac{1}{b^6} + \frac{1}{c^6}$$

Proposed by Daniel Sitaru-Romania

U.1349 Solve for real numbers:

$$\begin{cases} x, y, z > 1 \\ \log_x z + \log_y x + \log_z y = \log_{xy}(zy) + \log_{yz}(xz) + \log_{zx}(yx) \\ x + y + z = 6 \end{cases}$$

Proposed by Daniel Sitaru-Romania

U.1350 Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n}\left(1 + \frac{1}{5}\prod_{k=1}^{n} \frac{2k+1}{2k}\right)\left(\prod_{k=0}^{n} \frac{2k+6}{2k+5}\right)^{-1}$$

Proposed by Daniel Sitaru-Romania

U.1351 Find without softwares:

$$\Omega = \int_0^{\pi} \frac{\sin^{2020} x + \sin^2 x}{1 + \sin^{2020} x + \cos^{2020} x} dx$$

Proposed by Radu Diaconu-Romania

U.1352 Find a closed Form:

$$\Omega = \int_0^{\infty} \frac{\sqrt[14]{x}}{(1 + x^4)^2} dx$$

Proposed by Radu Diaconu-Romania

U.1353 In $\Delta ABC, I$ —incenter, $ID \perp BC, IE \perp CA, IF \perp AB, D \in (BC), E \in (CA), F \in (AB)$,

I_a, I_b, I_c —excenters. Prove that: $\sum_{cyc} \frac{EF}{\sin\frac{A}{2}} + \prod_{cyc} \frac{EF}{\sin\frac{A}{2}} = \frac{1+4r^2}{R} \cdot [I_a I_b I_c]$

Proposed by Radu Diaconu-Romania

U.1354 In acute ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{\sin^3 A}{\mu(B)} + \frac{\cos^3 A}{\mu(C)}\right) + \prod_{cyc}\left(\frac{\sin^3 A}{\mu(B)} + \frac{\cos^3 A}{\mu(C)}\right) > \frac{9}{2\pi} + \frac{1}{\prod_{cyc}(\pi - \mu(A))}$$

Proposed by Radu Diaconu-Romania

U.1355 In acute $\Delta ABC, \cos A \cos B \cos C = \frac{1}{32}$. Prove that:

$$\frac{1}{s} < \frac{1}{\pi^2}\sum_{cyc} \frac{\mu^2(A)}{h_a \cdot \cos A} < \frac{R}{r^2}$$

Proposed by Radu Diaconu-Romania

U.1356 In acute ΔABC the following relationship holds:

$$\frac{2}{27R}<\frac{1}{\pi}\left(\sum_{cyc}\frac{\mu(A)}{m_a\cdot\tan A}\right)<\frac{R}{4\sqrt{3}r^2}$$

Proposed by Radu Diaconu-Romania

U.1357 Find:

$$\Omega(a)=\int_1^2\frac{\log(ax)}{x^2+3x+2}dx\,;a\geq 1$$

Proposed by Radu Diaconu-Romania

U.1358 Let ΔDEF be the orthic triangle of acute ΔABC. Prove that:

$$3\min\{a,b,c\}\leq 2(DE+EF+FD)\leq 3\max\{a,b,c\}$$

Proposed by Radu Diaconu-Romania

U.1359 In ΔABC the following relationship holds:

$$\left(6\sqrt{3}+1\right)^2r^2\leq\begin{vmatrix}\sqrt{a} & \sqrt{b} & \sqrt{c} & \sqrt{r}\\ \sqrt{c} & -\sqrt{r} & -\sqrt{a} & \sqrt{b}\\ \sqrt{b} & -\sqrt{a} & \sqrt{a} & -\sqrt{c}\\ \sqrt{r} & \sqrt{c} & -\sqrt{b} & -\sqrt{a}\end{vmatrix}\leq\frac{\left(6\sqrt{3}+1\right)^2R^2}{4}$$

Proposed by Radu Diaconu-Romania

U.1360 In $\Delta ABC, P\in(ABC)$, the following relationship holds:

$$\left(\sum_{cyc}\frac{\left(\frac{PA}{a}\right)^3}{\left(\frac{PA}{a}\right)^2+\frac{PA\cdot PB}{ab}+\left(\frac{PB}{b}\right)^2}\right)\left(\sum_{cyc}a\sin^4\frac{A}{2}\right)\geq 2r\left(\frac{5}{32}+\frac{r^2}{8R^2}\right)$$

Proposed by Radu Diaconu-Romania

U.1361 Find:

$$\Omega=\int_0^1\frac{ax+b\cos^m(2n\pi x)}{a+2b\cos^m(2n\pi x)}dx, a\geq 1, b\geq 1, m\geq 1, n\in\mathbb{N}^*$$

Proposed by Radu Diaconu-Romania

U.1362 In $\Delta ABC, p_a$ −Spieker's cevian, the following relationship holds:

$$\sum_{cyc} p_a r_a \geq \sqrt{\frac{2r}{R}} s^2$$

Proposed by Soumava Chakraborty-India

U.1363 In $\Delta ABC, p_a$ −Spieker's cevian, the following relationship holds:

$$\sum_{cyc} p_a^2 \leq \sum_{cyc} m_a^2 + \frac{1}{2}\sum_{cyc} m_a |b-c|$$

Proposed by Soumava Chakraborty-India

U.1364 In $\Delta ABC, p_a$ −Spieker's cevian, the following relationship holds:

$$s^2 - 12Rr - 3r^2 \geq \frac{8Rr}{\sum ab}\left(\sum_{cyc} p_a^2 - s^2\right)$$

Proposed by Soumava Chakraborty-India

U.1365 In $\Delta ABC, S$ −Spieker center, the following relationship holds:

$$\sum_{cyc} \frac{AS^2}{bc + 2as} \geq \frac{1}{4}$$

Proposed by Soumava Chakraborty-India

U.1366 In $\Delta ABC, S$ −Spieker center, the following relationship holds:

$$\sum_{cyc} AS^2 \geq 4r(2R - r)$$

Proposed by Soumava Chakraborty-India

U.1367 In $\Delta ABC, p_a$ −Spieker's cevian, the following relationship holds:

$$\sum_{cyc} \frac{p_a w_a}{h_a m_a} \leq \frac{3R}{2r}$$

Proposed by Soumava Chakraborty-India

U.1368 In $\Delta ABC, p_a$ −Spieker's cevian, the following relationship holds:

$$3(R-r)s^2 - r\sum_{cyc} r_a^2 \geq R\sum_{cyc}(2p_a - m_a)^2$$

Proposed by Soumava Chakraborty-India

U.1369 In $\Delta ABC, v_a, v_b, v_c$ −cevians through Bevan's point, the following relationship holds:

$$4Rs\sum_{cyc}\frac{a}{b+c}\geq 12Rs-\sum_{cyc}v_a(b+c)$$

Proposed by Soumava Chakraborty-India

U.1370 In $\Delta ABC, p_a$ —Spieker's cevian, the following relationship holds:

$$\frac{1}{3}\sum_{cyc}\frac{2p_a-m_a+h_a}{h_a}\leq\frac{R}{r}$$

Proposed by Soumava Chakraborty-India

U.1371 In $\Delta ABC, v_a$ —Bevan's cevian, the following relationship holds:

$$r^2\sum_{cyc}\frac{AV}{v_a}+R(R-2r)\geq 2Rr$$

Proposed by Soumava Chakraborty-India

U.1372 In $\Delta ABC, v_a$ —Bevan's cevian, the following relationship holds:

$$\sum_{cyc}\frac{v_a}{s-a}\geq\frac{27R}{2s}$$

Proposed by Soumava Chakraborty-India

U.1373 In $\Delta ABC, AD, BE, CF$ —medians, S —Spieker center, the following relationship holds:

$$\sum_{cyc}\frac{AS^2-DS^2}{DS^2}\geq\frac{6(2R-r)}{R}$$

Proposed by Soumava Chakraborty-India

U.1374 In $\Delta ABC, p_a$ —Spieker's cevian, the following relationship holds:

$$\sum_{cyc}\frac{p_a}{m_a}\leq\sum_{cyc}\frac{m_a}{w_a}$$

Proposed by Soumava Chakraborty-India

U.1375 In $\Delta ABC, v_a$ —Bevan's cevian, the following relationship holds:

$$\sum_{cyc}v_a(b+c)\cos\frac{A}{2}\geq 2s^2$$

Proposed by Soumava Chakraborty-India

U.1376 In $\Delta ABC, AD, BE, CF$ —medians, S —Spieker center, the following relationship holds:

$$\sum_{cyc} bc\left(\frac{AD^2 - DS^2}{DS^2}\right) \geq 4s^2$$

Proposed by Soumava Chakraborty-India

U.1377 In $\Delta ABC, V$ –Bevan's point, the following relationship holds:

$$Rs\sum_{cyc}\frac{b^2c^2 - 4r^2s^2}{(s-a)AV^2} \leq (R+r)s^2 + r(4R+r)^2$$

Proposed by Soumava Chakraborty-India

U.1378 In ΔABC the following relationship holds:

$$\left(a - \sqrt{ab} + b\right)^2 + \left(b - \sqrt{bc} + c\right)^2 + \left(c - \sqrt{ca} + a\right)^2 \geq 4\sqrt{3} \cdot F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1379 If $x, u \in [0, \infty)$ and $x + u = 2$ then in ΔABC holds: $a^x b^u + b^x c^u + c^x a^u \geq 4\sqrt{3} \cdot F$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1380 If $g, m, x, y, z \in [1, \infty), g = \sqrt[3]{xyz}, 3m = x + y + z$ then in ΔABC holds:

$$(x^x + y^x + z^x)a^2 + (x^y + y^y + z^y)b^2 + (x^z + y^z + z^z)c^2 \geq 4\sqrt{3} \cdot g^m \cdot F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1381 If $m \geq 0$ then in ΔABC holds:

$$\frac{1}{\left(\sqrt{h_a h_b}\right)^m} + \frac{1}{\left(\sqrt{h_b h_c}\right)^m} + \frac{1}{\left(\sqrt{h_c h_a}\right)^m} \geq \frac{\left(\sqrt{2}\right)^m \cdot 3^{m+1}}{\left(\sqrt{F}\right)^m\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^m}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1382 If $m \geq 0, x, y > 0$ then in ΔABC holds:

$$(a^{m+1} + b^{m+1} + c^{m+1})\left(\frac{a^{m+1}}{(bx+cy)^m} + \frac{b^{m+1}}{(cx+ay)^m} + \frac{c^{m+1}}{(ax+by)^m}\right) \geq \frac{2^{m+2}\left(\sqrt{3}\right)^{6-m}\left(\sqrt{F}\right)^{m+1}}{(x+y)^m}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1383 If $m, n > 0$ and $t \geq 0$ then in ΔABC holds:

$$ms^{2t+2} + nr^{2t+2} + n(4Rr)^{t+1} \geq \frac{3 \cdot 6^t + n}{2^t}\left(\sqrt{3}\right)^{t+1} \cdot F^{t+1}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1384 If $m, n > 0$ and $t \geq 0$ then in ΔABC holds:

$$ms^{2t+2} + nr^{2t+2} + n(4Rr)^{t+1} \geq \frac{3 \cdot 6^t + n}{2^t}\left(\sqrt{3}\right)^{t+1} \cdot F^{t+1}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

U.1385 In ΔABC the following relationship holds:

$$\frac{b+c}{a+\sqrt{(a+2b)(a+2c)}} + \frac{c+a}{b+\sqrt{(b+2c)(b+2a)}} + \frac{a+b}{c+\sqrt{(c+2a)(c+2b)}} \geq \frac{3}{2}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

U.1386 If $m \in [0, \infty)$ and $M \in Int(\Delta ABC), x = MA, y = MB, z = MC$ then:

$$(x \cdot m_a)^{2m+1} + (y \cdot w_b)^{2m+1} + (z \cdot h_c)^{2m+1} \geq 2^{2m+1} \cdot F^{2m+1} \cdot \left(\sqrt{3}\right)^{1-2m}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

U.1387 If $x, y, z \in (0,1)$, then:

$$\left(\frac{x}{1-y^2} + \frac{y}{1-z^2} + \frac{z}{1-x^2}\right)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{27\sqrt{3}}{8}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

U.1388 If $m \geq 0; u, v > 0, M \in Int(\Delta ABC), x = MA, y = MB, z = MC$, then:

$$\sum_{cyc}\left(\frac{x}{a}\left(\frac{uy}{b} + \frac{vz}{c}\right)\right)^{m+1} \geq \frac{(u+v)^{m+1}}{3^m}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

U.1389 If $n \geq 0, m, p, t > 0, M \in Int(\Delta ABC), x = MA, y = MB, z = MC$, then:

$$\sum_{cyc}\left(n \cdot \frac{x^3yx^2}{a^3bz^2} + p \cdot \frac{y^3za^2}{b^3cx^2} + t \cdot \frac{z^3xb^2}{c^3ay^2}\right)^{n+1} \geq \frac{(m+p+t)^{m+1}}{3^n}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

U.1390 If $n \in \mathbb{N}, n \geq 2$ and $a_k \in (0,1); \forall k = \overline{1,n}, \sum_{k=1}^{n} a_k = 1$, then:

$$\sum_{k=1}^{n} \frac{1}{a_k^2(1-a_k^2)} \geq \frac{(n+1) \cdot 3\sqrt{3}}{2} \cdot \sum_{k=1}^{n} \frac{1}{1+a_k}$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

U.1391 In ΔABC the following relationship holds:

$$3(a^2+b^2+c^2) \geq 8\sqrt{3}F + \sum_{cyc}(a-b+c)^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1392 If $u,v \geq 0, u+v>0$ and $m,x,y,z \in [1,\infty), 3m = x+y+z$ then in ΔABC holds:

$$((ux+vy)^x+(uy+vz)^x+(uz+vx)^x)a^2+((ux+vy)^y+(uy+vz)^y+(uz+vx)^y)b^2$$
$$+((ux+vy)^z+(uy+vz)^z+(uz+vx)^z)c^2 \geq 12\sqrt{3}(u+v)^m \cdot m^m \cdot F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1393 If $x,y>0$ then in ΔABC holds:

$$(x^2+y^2)(a^2+b^2+c^2) \geq 8xy\sqrt{3}\cdot F + (xa-yb)^2+(xb-yc)^2+(xc-ya)^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1394 If $m,t \geq 0; x,y,z>0$ then in ΔABC holds:

$$\frac{x+y}{2}\cdot\frac{a^m b^{t+2}}{c^{m+t}}+\frac{y+z}{x}\cdot\frac{b^m c^{t+2}}{a^{m+t}}+\frac{z+x}{y}\cdot\frac{c^m a^{t+2}}{b^{m+t}} \geq 8\sqrt{3}\cdot F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1395 Find:

$$\Omega(m) = \lim_{n\to\infty}\left(\left(\sqrt[n+1]{(n+1)!}\right)^{m+1}-\left(\sqrt[n]{n!}\right)^{m+1}\right)\cdot \tan^m\frac{\pi}{n}.$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1396 If $m>0$ then in ΔABC holds:

$$\left(\frac{a^{\frac{m}{2}}+b^{\frac{m}{2}}+c^{\frac{m}{2}}}{3}\right)^{\frac{2}{m}} \geq \frac{2}{\sqrt[4]{3}}\cdot\sqrt{F}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1397 If $x,y,z \in \left(0,\frac{\pi}{2}\right)$ then in ΔABC holds:

$$\frac{\tan y+\tan z}{\sin x}\cdot a+\frac{\tan z+\tan x}{\sin y}\cdot b+\frac{\tan x+\tan y}{\sin z}\cdot c > 4\cdot\sqrt[4]{27}\cdot\sqrt{F}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1398 If $u,v \geq 0, x,y>0$ then in ΔABC holds:

$$\sum_{cyc}(x^{u+1}+y^{u+1})h_b^u h_c^u\cdot\frac{a^{2u+w+2}}{(bc)^v}\geq 2^{u+1}\cdot(x+y)^{u+1}\cdot\sqrt{3}\cdot F^{2n+1}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1399 Let $(a_n)_{n\geq1}$ sequence of real numbers strictly positive such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a>0$ and $(x_n)_{n\geq1}$, $x_n=\sum_{k=1}^{n}\tan^{-1}\left(\frac{1}{k^2-k+1}\right)$. Find: $\lim\limits_{n\to\infty}\left(\frac{\pi^2}{4}-x_n^2\right)\sqrt[n]{a_n}$.

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1400 Let $(a_n)_{n\geq1},(b_n)_{n\geq1}$ sequences of real numbers such that $\lim\limits_{n\to\infty}a_n=a\in\mathbb{R},b>0,\forall n\in\mathbb{N}^*$, $\lim\limits_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n}=b>0$ and $\exists t>0$ such that $\lim\limits_{n\to\infty}(a_n-a)\left(\sqrt[n]{b_n}\right)^t=c\in\mathbb{R}$. Find:

$$\lim_{n\to\infty}(1+a_n-a)^{n^t}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1401 If $(x_n)_{n\geq1},x_n=\sum_{k=1}^{n}\tan^{-1}\left(\frac{1}{k^2-k+1}\right)$ then find:

$$\Omega=\lim_{n\to\infty}\left(\frac{\pi^2}{4}-x_n^2\right)\sqrt[n]{(2n-1)!!}.$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1402 Find:

$$\lim_{n\to\infty}\frac{(n+1)\cdot\sqrt[n+1]{(2n+1)!!}-n\cdot\sqrt[n]{(2n-1)!!}}{\sqrt[n]{n!}}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1403 If $m,t\geq0$ and $x,y,z>0$ then in ΔABC holds:

$$\sum_{cyc}\frac{x+y}{z}\cdot\frac{a^{m+1}b^{t+1}}{c^{m+t}}\geq 8\sqrt{3}\cdot F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1404 If $m,t\geq0$ and $x,y,z>0$ then in ΔABC holds:

$$\sum_{cyc}\frac{x+y}{z}\cdot\frac{a^m b^{t+4}}{c^{m+t}}\geq 32\cdot F^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1405 If $(a_n)_{n\geq 1}$ is a sequence of real numbers strictly positive such that $\lim\limits_{n\to\infty}\frac{n\cdot a_{n+1}}{a_n}=a>0$, then find $\lim\limits_{n\to\infty}\sqrt[n]{a_n\cdot n!}$.

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1406 If $(a_n)_{n\geq 1}$ is a sequence of real numbers strictly positive such that $\lim\limits_{n\to\infty}\frac{n\cdot a_{n+1}}{a_n}=a>0$, then find

$$\lim_{n\to\infty} n^2\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right).$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1407 Prove that:

$$\sum_{n=1}^{\infty}\frac{F_{2n}H_n}{4^n n}=\frac{1}{\sqrt{5}}\left(Li_2\left(\frac{2}{\sqrt{5}}-1\right)-Li_2\left(-1-\frac{2}{\sqrt{5}}\right)\right)$$

where F_n, H_n and $Li_2(x)$ are nth Fibonacci and Harmonic number and dilogarithm respectively.

Proposed by Naren Bhandari-Nepal

U.1408 If Knuth's up arrow notation, $x\uparrow\uparrow n=\underbrace{x^{x^{\cdot^{\cdot^{x}}}}}_{n}$ for $x\neq 0$, then prove that:

$$\lim_{n\to\infty^+}\lim_{x\to 1}\left(\zeta(x)-\frac{1}{x\uparrow\uparrow n-1}\right)=1+\gamma$$

$$\lim_{n\to\infty^+}\lim_{x\to 1}\left(\zeta(x\uparrow\uparrow n)-\frac{1}{x\uparrow\uparrow n-1}\right)=\gamma$$

where $\zeta(x)$ is Riemann zeta function and γ is Euler-Mascheroni constant.

Proposed by Naren Bhandari-Nepal

U.1409 Find:

$$\Omega(a)=\int_{\frac{1}{a}}^{a}\frac{\sin^2 x\cdot\tan\left(\frac{1}{x^2}\right)}{(1+x^2)\left(\sin^2\left(\frac{1}{x}\right)\tan\left(\frac{1}{x^2}\right)+\sin^2 x\tan(x^2)\right)}dx$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

U.1410 If $(x_n)_{n\geq 1}$ is a positive real sequence such that $x_n+x_{n+1}=x_{n+3}$, then find:

$$\Omega=\lim_{n\to\infty}\frac{x_n^2+x_{n+2}^2+x_{n+6}^2}{x_{n+1}^2+x_{n+3}^2+x_{n+4}^2+x_{n+5}^2}$$

Proposed by Neculai Stanciu-Romania

U.1411 If $f:\mathbb{R}\to\mathbb{R}, f(x)=\sum_{k=0}^{n} a_k x^k$, $a_k\geq 0, \forall k=\overline{1,n}$ with $f(4)=8$ and $f(9)=18$, then determine $\max\{f(6)\}$ and the function which realize this maximum.

Proposed by Neculai Stanciu-Romania

U.1412 Find:

$$\Omega=\lim_{n\to\infty}\frac{\sum_{k=1}^{n}(2n^2+2n+k)^2}{\sum_{k=1}^{n+1}(2n^2+2n-k+1)^2}$$

Proposed by Neculai Stanciu-Romania

U.1413 Find the closed form:

$$\Omega=\int_0^{\frac{\pi}{4}} x\tan x\log(cos\, x)\, dx$$

Proposed by Naren Bhandari-Bajura-Nepal

U.1414 Prove that:

$$\frac{8}{\pi}\int_0^{\frac{\pi}{2}} x\cot x\log(1-\sin^4 x)\,dx=\frac{5\pi^2}{12}-\frac{24}{5}\log^2 2-4Li_2\left(\frac{1}{\sqrt{2}}\right)+2Li_2\left(\frac{2-\sqrt{2}}{4}\right)-$$

$$-\log^2\left(1+\sqrt{2}\right)+3\log 2\log\left(1+\sqrt{2}\right)$$

where $Li_2(x)=\sum_{k=1}^{\infty}\frac{x^k}{k^2}$ is dilogarithm function.

Proposed by Naren Bhandari-Bajura-Nepal

U.1415 Prove that:

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(4n+5)(2n+2)}=1-\frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{8\Gamma\left(\frac{7}{4}\right)}=1-\frac{\Gamma^2\left(\frac{1}{4}\right)}{6\sqrt{2\pi}}$$

where $\Gamma(z)=\int_0^{\infty} x^{z-1}e^{-x}\,dx$ is gamma function for $\Re(z)>0$.

Proposed by Naren Bhandari-Bajura-Nepal

U.1416 If we have the equation $f(a,b,x)=\frac{ax^3-bx+1}{ax^3+bx+1}$ then find the non-trivial solutions for the following equation in x, $f\left(\frac{a+b}{2},\sqrt{ab},x\right)=f(a,b,x)$.

Proposed by Srinivasa Raghava-AIRMC-India

U.1417 If $1<a\leq b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{1+xy} \leq (b-a)\tan^{-1}\left(\frac{b-a}{1+ab}\right)$$

Proposed by Daniel Sitaru-Romania

U.1418 Find:

$$\Omega(n) = \int_0^n \log\left(\sqrt{n+x} - \sqrt{n-x}\right) dx, n \in \mathbb{N} - \{0\}$$

Proposed by Daniel Sitaru-Romania

U.1419 Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n^{n\alpha}} \cdot \left\{\sum_{k=1}^{n} \left(\binom{n}{k} \cdot \left[\sum_{i=1}^{n} i^{n-k}\right] + \frac{2k}{n}\right)\right\}^{\alpha}, \alpha \in \mathbb{R}$$

Proposed by Florică Anastase-Romania

U.1420 Prove that:

$$\prod_{k=1}^{n} k! \cdot k^{n-k+1} \leq \left(\frac{n+2}{3}\right)^{n(n+1)}, n \in \mathbb{N}$$

Proposed by Florică Anastase-Romania

U.1421 Prove or disprove the following:

$$\sum_{n=1}^{\infty} (-1)^{\frac{n^2+n+2}{2}} e^{-\pi n^2 x} = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$$

Proposed by Artan Ajredini-Serbie

U.1422 For $k \in \mathbb{N}$ fixed and $\alpha > 0$ find:

$$\Omega = \lim_{n\to\infty} \frac{1}{\sqrt{n^\alpha}} \left(\frac{\prod_{i=1}^{k}(n+k+i)}{\prod_{i=1}^{k}(n+i)}\right)^{n^\alpha}$$

Proposed by Florică Anastase-Romania

U.1423 Prove:

$$\int_0^\infty \sum_{n=1}^{\infty} \frac{dx}{2n^3x^2(n-2) + (nx+x)^2 + 2 - nx^2} = \frac{\pi^3}{8}$$

Proposed by Ankush Kumar Parcha-India

U.1424 Find:

$$\Omega=\int_0^\infty \frac{\sqrt{x}\tan^{-1}x}{x^3+1}dx$$

Proposed by Vasile Mircea Popa-Romania

U.1425 Find:

$$\Omega=\int_0^1 x\frac{\log^2 x}{x^8+x^4+1}dx$$

Proposed by Hussain Reza Zadah-Afghanistan

U.1426 Find:

$$\Omega=\int_0^{\frac{\pi}{4}}(\cos 2x)\cdot 2^{[\sin x+\cos x]}\,dx,[*]-GIF$$

Proposed by Mohammad Rostami-Afghanistan

U.1427 $MA=\frac{a+b}{2}, MG=\sqrt{ab}, MQ=\sqrt{\frac{a^2+b^2}{2}}, a,b>0, f:[0,\infty)\to\mathbb{R}$ increasing and concave.

Prove that:

$$\int_{MG}^{MQ} f(x)\,dx\ge\frac{1}{4}(MA-MG)\big(f(MQ)+3f(MG)\big)+\frac{1}{4}(MQ-MA)\big(3f(MQ)+f(MA)\big)$$

Proposed by Seyran Ibrahimov-Azerbaijan

U.1428 Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n^4}\cdot\sum_{1\le i<j\le n}\sin^2\frac{(j-i)\pi}{n}\cdot\sum_{1\le i<j\le n}\cos^2\frac{(j-i)\pi}{n}$$

Proposed by Neculai Stanciu-Romania

U.1429 In acute $\Delta ABC, r_i i=1,2,3$ Malfatti's radies, holds:

$$\frac{\sqrt{r_1r_2}+\sqrt{r_2r_3}+\sqrt{r_3r_1}}{r}+\frac{s}{2r}\ge\frac{3}{2}+\sum_{cyc}\frac{m_a}{\sqrt{m_bm_c}}$$

Proposed by Bogdan Fuștei-Romania

U.1430 Find a closed form:

$$\Omega=\int_0^1\frac{\tan^{-1}x\cdot\log(1+x^2)}{x}dx$$

Proposed by Ajentunmobi Abdulqoyyum-Nigeria

U.1431 Find a closed form:

$$\Omega=\int_0^{\frac{\pi}{2}}\cos 2x\cdot\tan\frac{x}{2}\,dx$$

Proposed by Ajentunmobi Abdulqoyyum-Nigeria

U.1432 Prove that:

$$\sum_{d|n}\sigma_k(d)\cdot\phi\left(\frac{n}{d}\right)=n\cdot\sigma_{k-1}(n)$$

where $\sigma_k(n),\phi(n)$ is divisors function and Euler's totient function respectively.

Proposed by Amrit Awasthi-India

U.1433 If $(a_n)_{n\ge1}, a_{n+1}=3a_n-a_{n-1}, a_1=4, a_2=11$, then prove that a_n divides a_{n+1}^2-5 and a_{n+1} divides a_n^2-5.

Proposed by Neculai Stanciu-Romania

U.1434 If $f(\cot x)=\sin 2x+\cos 2x\,, x\in(0,\pi)$ then find:

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\cos x\cdot f(\sin x)\,dx$$

Proposed by Neculai Stanciu-Romania

U.1435 If $x\ge0$ and $(a_n)_{n\ge1}$ a sequence of real numbers strictly positive such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a>0$ then find $\lim\limits_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}^x}-\sqrt[n]{a_n^x}\right)\left(\sqrt[n]{(2n-1)!!}\right)^{1-x}$.

Proposed by D.M. Bătinețu-Giurgiu-Romania

U.1436 If $x\ge0$ and $(a_n)_{n\ge1},(b_n)_{n\ge1}$ are sequences of real numbers strictly positive such that

$\lim\limits_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a>0$ and $\lim\limits_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n}=b>0$ then find $\lim\limits_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}^x}-\sqrt[n]{a_n^x}\right)\cdot\sqrt[n]{b_n^{1-x}}$.

Proposed by D.M. Bătinețu-Giurgiu-Romania

U.1438 If $m,t\ge0$ and $x,y,z>0$ then in ΔABC holds:

$$\frac{(x+y)^m a^{t+1}b}{z^m h_b^t}+\frac{(y+z)^m b^{t+1}c}{x^m h_c^t}+\frac{(z+x)^m c^{t+1}a}{y^m h_a^t}\ge 2^{m+t+2}\left(\sqrt{3}\right)^{1-t}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu-Romania

U.1439 If $m,t\ge0$ then in ΔABC holds:

$$\frac{s^2+r_ar_b}{s^2-r_ar_b}\cdot\frac{a^{m+4}b^t}{c^{m+t}}+\frac{s^2+r_br_c}{s^2-r_br_c}\cdot\frac{b^{m+4}c^t}{a^{m+t}}+\frac{s^2+r_cr_a}{s^2-r_cr_a}\cdot\frac{c^{m+4}a^t}{b^{m+t}}\geq 32\cdot F^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1440 If $A_1B_1C_1$ and $A_2B_2C_2$ are triangles with R_1 and R_2 circumradii, then:

$$\left(\frac{1}{a_1^2}+\frac{1}{b_1^2}+\frac{1}{c_1^2}\right)\left(\frac{1}{a_1a_2}+\frac{1}{b_1b_2}+\frac{1}{c_1c_2}\right)\geq\frac{4}{R_1^2(R_1+R_2)^2}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1441 In $\Delta A_kB_kC_k, k=\overline{1,n}, h_{a_k}, h_{b_k}, h_{c_k}$ are altitudes, s_k semiperimeters, F_k areas and R_k circumradii, then:

$$h_{a_1}h_{a_2}+h_{b_1}h_{b_2}+h_{c_1}h_{c_2}\geq\frac{4F_1F_2}{(R_1+R_2)^2}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1442 If $g,x,y,z\in[1,\infty), 3g=x+y+z$ then in ΔABC holds:

$$\frac{x^x+y^x+z^x}{h_a^2}+\frac{x^y+y^y+z^y}{h_b}\cdot c^2+(x^z+y^z+z^z)\cdot b\geq 6\sqrt{3}\cdot g^g$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1443 If $x,y,z>0; x+z\geq y$ then in ΔABC holds:

$$\left(xa-y\sqrt{ab}+zc\right)^2+\left(xb-y\sqrt{bc}+zc\right)^2+\left(xc-y\sqrt{ca}+za\right)^2\geq 4\sqrt{3}(x-y+z)^2F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1444 If $x,y,z>0$ and $u,v,w\geq 0$ such that $u+v+w=2m>0$ then in ΔABC holds:

$$\frac{x+y}{z}\cdot a^ub^vc^w+\frac{y+z}{z}\cdot b^uc^va^w+\frac{z+x}{y}\cdot c^ua^vb^w\geq 2^{2m+1}\left(\sqrt{3}\right)^{2-m}\cdot F^m$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1445 If $m\in\mathbb{N}^*$ and $x,y,z>0$ then in ΔABC holds:

$$3m+\left(\frac{(x+y)a}{z}\right)^{m+1}+\left(\frac{(y+z)b}{x}\right)^{m+1}+\left(\frac{(z+x)c}{y}\right)^{m+1}\geq 4\cdot\sqrt[4]{27}\cdot\sqrt{F}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1446 Let $g,m,,x,y,z\in[1,\infty), g=\sqrt[3]{xyz}$ and $3m=x+y+z$, then in ΔABC holds:

$$(x^x + y^x + z^x)ab + (xY + y^y + z^y)bc + (x^z + y^z + z^z)ca \geq 12\sqrt{3} \cdot g^m \cdot F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1447 Let $g, m, , x, y, z \in [1, \infty), g = \sqrt[3]{xyz}$ and $3m = x + y + z$, then in ΔABC holds:

$$(x^x + y^x + z^x)a^4 + (xY + y^y + z^y)b^4 + (x^z + y^z + z^z)c^4 \geq 48 \cdot g^m \cdot F^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1448 If $x, y, z > 0$ then in ΔABC holds:

$$\frac{x+y}{z} \cdot \frac{(h_a + h_b)(h_a + h_c)}{h_b h_b} + \frac{y+z}{x} \cdot \frac{(h_b + h_c)(h_b + h_a)}{h_c h_a} + \frac{z+x}{y} \cdot \frac{(h_c + h_a)(h_c + h_b)}{h_a h_b} \geq 24$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1449 Let $n \in \mathbb{N}, n \geq 3; g, m, x_k \in [1, \infty), g = \sqrt[3]{xyz}, 3m = x + y + z$ and $A_1A_2 \dots A_n$ a convex polygon by F area and with sides $a_k = A_kA_{k+1}, k = \overline{1, n}, A_{n+1} = A_1$ then:

$$\left(\sum_{k=1}^{n} \left(x_1^{x_k} + x_2^{x_k} + \dots + x_n^{x_k}\right)\right) \sum_{k=1}^{n} \left(a_k - \sqrt{a_k a_{k+1}} + a_{k+1}\right)^2 \geq 4n^2 \cdot g^m \cdot F \cdot \tan\frac{\pi}{n}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1450 If $m \geq 0, n \in \mathbb{N}, n \geq 3, xy, z > 0, x + z \geq y$ and $A_1A_2 \dots A_n$ a convex polygon by F area and with sides $a_k = A_kA_{k+1}, k = \overline{1, n}, A_{n+1} = A_1$, then:

$$\sum_{k=1}^{n} (xa_k^2 - ya_k a_{k+1} + za_{k+1}^2)^{m+1} \geq \frac{4^{m+!}(x - y + z)^{m+1} \cdot F^{m+1}}{n^m} \cdot \tan^{m+1}\frac{\pi}{n}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1451 If $x, y > 0$ then in ΔABC holds:

$$(x^2 + y^2)(a + b + c) \geq 4 \cdot \sqrt[4]{27} \cdot xy\sqrt{F} + \left(x\sqrt{a} - y\sqrt{b}\right)^2 + \left(x\sqrt{b} - y\sqrt{c}\right)^2 + \left(x\sqrt{c} - y\sqrt{a}\right)^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1452 If $u, v \geq 0; m, x, y, z \in [1, \infty), 3m = x + y + z$ then in ΔABC holds:

$$(x^x + y^x + z^x)\frac{a^u b^{v+4}}{c^{u+v}} + (x^y + y^y + z^y)\frac{b^u c^{v+4}}{a^{u+v}} + (x^z + y^z + z^z)\frac{c^u a^{v+4}}{b^{u+v}} \geq 48m^m \cdot F^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1453 If $u, v \geq 0, u + v > 0$ and $m, x, y, z \in [1, \infty), 3m = x + y + z$ then in ΔABC holds:

$$((ux + vy)^x + (uy + vz)^x + (uz + vx)^x)a^4$$
$$+ ((ux + vy)^y + (uy + vz)^y + (uz + vx)^y)b^4$$

$$+((ux+vy)^z+(uy+vz)^z+(uz+vx)^z)c^4 \geq 48(u+v)^m \cdot m^m \cdot F^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1454 If $x, y > 0$ then in ΔABC holds:

$$\frac{a^7}{ax+by}+\frac{b^7}{ax+by}+\frac{b^7}{cx+ay} \geq \frac{64}{\sqrt{3}(x+y)} \cdot F^3$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1455 If $a, b, c, d, x, y > 0$ then:

$$a^2\left(\frac{x^2}{b}+\frac{y^2}{c}\right)+b^2\left(\frac{x^2}{c}+\frac{y^2}{d}\right)+c^2\left(\frac{x^2}{d}+\frac{y^2}{a}\right)+d^2\left(\frac{x^2}{a}+\frac{y^2}{b}\right) \geq \frac{(x+y)^2}{2}(a+b+c+d)$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1456 If $a, b, c, m, n > 0$ then:

$$\frac{a^2+b^2}{ma+nb}+\frac{b^2+c^2}{mb+nc}+\frac{c^2+a^2}{mc+na} \geq \frac{2(a+b+c)}{m+n}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1457 If $x, y, z > 0$ then in ΔABC holds:

$$\frac{x+y}{z}\left(a-\sqrt{ab}+b\right)^2+\frac{y+z}{x}\left(b-\sqrt{bc}+c\right)^2+\frac{z+x}{y}\left(c-\sqrt{ca}+a\right)^2 \geq 8\sqrt{3} \cdot F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1458 If $m, x, y, z \in [1, \infty), 3m = x+y+z$ then in ΔABC holds:

$$\frac{x^x+y^x+z^x}{a}+\frac{x^y+y^y+z^y}{b^2}+(x^z+y^z+z^z)a^3b^4c^2 \geq 12\sqrt{3} \cdot m^m F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1459 If $n, t \geq 0$ and $m, x, y, z \in [1, \infty), 3m = x+y+z$ then in ΔABC holds:

$$(x^x+y^x+z^x)\frac{a^{n+1}b^{t+1}}{c^{n+t}}+(x^y+y^y+z^y)\frac{b^{n+1}c^{t+1}}{a^{n+t}}+(x^z+y^z+z^z)\frac{c^{n+1}a^{t+1}}{b^{n+t}}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1460 If $a, b, m \geq 0$ and $x, y, u, v > 0$ then:

$$\left(a \cdot \frac{x}{y}+b \cdot \frac{u}{v}\right)^{m+1}+\left(a \cdot \frac{y}{x}+b \cdot \frac{v}{u}\right)^{m+1} \geq 2(a+b)^{m+1}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1461 Let $(a_n)_{n\ge1},(b_n)_{n\ge1}$ sequences of real numbers strictly positive such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a>0$ and $\lim\limits_{n\to\infty}\frac{n^2\cdot b_{n+1}}{b_n}=b>0$ then find $\lim\limits_{n\to\infty}\sqrt[n]{b_n}\left((n+1)^2\cdot\sqrt[n+1]{a_{n+1}}-n^2\cdot\sqrt[n]{a_n}\right)$.

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1462 Let $(a_n)_{n\ge1}$ be sequence of real numbers strictly positive such that $\lim\limits_{n\to\infty}\frac{n\cdot a_{n+1}}{a_n}=a>0$ then find $\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt[n+1]{a_{n+1}}}-\frac{1}{\sqrt[n]{a_n}}\right)$.

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1463 Let $a_n=\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!},n\in\mathbb{N}^*-\{1\}$ and $(b_n)_{n\ge1}$ a sequence of real numbers strictly positive such that $\lim\limits_{n\to\infty}\frac{b_{n+1}}{b_n\cdot\sqrt[n]{a_n}}=b>0$. Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}}-\sqrt[n]{b_n}\right)$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1464 Let $t\ge0$ and $(a_n)_{n\ge1},(b_n)_{n\ge1}$ sequence of real numbers strictly positive such that:

$$\lim_{n\to\infty}\frac{a_{n+1}}{n^t\cdot a_n}=a>0,\ \lim_{n\to\infty}\frac{b_{n+1}}{n^{t+1}\cdot b_n}=b>0.\text{ Find }\lim_{n\to\infty}\frac{\sqrt[n+1]{b_{n+1}}-\sqrt[n]{b_n}}{\sqrt[n]{a_n}}.$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1465 If $f:\mathbb{R}\to\mathbb{R}$ an odd function and continuous on $\mathbb{R},a\in(0,\infty)$ then find:

$$\int_{-a}^{a}\frac{x^3\big((f\circ f)(x)+x\big)}{x^2+1}dx$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1466 In ΔABC the following relationship holds:

$$s^2\ge3\sqrt{3}\cdot F+\frac{1}{4}((a-b)^2+(b-c)^2+(c-a)^2)$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1467 If $x,y,z>0$ then in ΔABC holds:

$$\frac{x}{h_a^2}+\frac{y}{h_b^2}+\frac{z}{h_c^2}\ge\frac{1}{2F}\cdot\sqrt{\frac{xy}{\sin^2\frac{C}{2}}+\frac{yz}{\sin^2\frac{A}{2}}+\frac{zx}{\sin^2\frac{B}{2}}}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1468 If $x,y,z>0$ then in ΔABC holds:

$$\frac{(x+y)a}{h_b}+\frac{(y+z)b}{h_c}+\frac{(z+x)c}{h_a}-\left(\frac{ax}{h_a}+\frac{by}{h_b}+\frac{cz}{h_c}\right)\ge$$

$$\geq 4 \cdot \sqrt{xy \cdot \sin^2\frac{C}{2} + yz \cdot \sin^2\frac{A}{2} + zx \cdot \sin^2\frac{B}{2}}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1469 If $m, n \geq 0$ then in ΔABC holds:

$$\left(\frac{1}{b^{2m}} + \frac{1}{c^{2m}}\right) \cdot \frac{a^{2m+2n+2}}{(b^2+c^2)^n} + \left(\frac{1}{c^{2m}} + \frac{1}{a^{2m}}\right) \cdot \frac{b^{2m+2n+2}}{(c^2+a^2)^n} +$$

$$+\left(\frac{1}{a^{2m}} + \frac{1}{b^{2m}}\right) \cdot \frac{c^{2m+2n+2}}{(a^2+b^2)^n} \geq 3^{2-n} \cdot \sqrt{3} \cdot F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1470 If $x, y > 0$ and in $\Delta ABC, g_a$ −Gergonne's cevian, then holds:

$$a^4\left(x^2 g_b^2 + y^2 g_c^2\right) + b^4(x^2 g_c^2 + y^2 g_a^2) + c^4\left(x^2 g_a^2 + y^2 g_b^2\right) \geq 8\sqrt{3}(x+y)^2 \cdot F^3$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1471 If $x, y > 0$ and $a_k \in (0, \infty), k = \overline{1,n}, n \in \mathbb{N}^* - \{1\}$, then:

$$\sum_{k=1}^{n} a_k^2 \left(\frac{x^2}{a_{k+1}} + \frac{y^2}{a_{k+2}}\right) \geq \frac{(x+y)^2}{2} \cdot \sum_{k=1}^{n} a_k\, , a_{n+1} = a_1, a_{n+2} = a_2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1472 If $m, n, p, t > 0$ then in ΔABC holds:

$$\frac{m+n}{p+t} \cdot a + \frac{n+p}{t+m} \cdot b + \frac{p+t}{m+n} \cdot c \geq \sqrt[4]{27} \cdot \sqrt{F}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1473 If $t, x, y, z > 0$ then:

$$\frac{t^2+x^2}{t^2+ty+z^2} + \frac{t^2+y^2}{t^2+tz+x^2} + \frac{t^2+z^2}{t^2+tx+y^2} \geq 2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1474 If $g, m, x, y, z \in [1, \infty), g = \sqrt[3]{xyz}, 3m = x + y + z$ then in ΔABC holds:

$$(x^x + y^x + z^x)(a^2 - ab + b^2)^2 + (x^y + y^y + z^y)(b^2 - bc + c^2)^2 +$$

$$+(x^z + y^z + z^z)(c^2 - ca + a^2)^2 \geq 48 \cdot g^m \cdot F^2$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1475 If $m,n \geq 0$ and $a,b,c,m+n>0$ then:

$$\frac{(a^2+bc)^2}{mab+nac}+\frac{(b^2+2ca)^2}{mbc+nab}+\frac{(c^2+2ab)^2}{mac+nbc} \geq \frac{3(a+b+c)^2}{m+n}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1476 In ΔABC the following relationship holds:

$$a+b+c \geq \left(\sqrt{a}-\sqrt{b}\right)^2+\left(\sqrt{b}-\sqrt{c}\right)^2+\left(\sqrt{c}-\sqrt{a}\right)^2+2\cdot\sqrt[4]{27}\cdot\sqrt{F}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1477 If $x,y,z>0$ then in ΔABC holds: $(ax+by+cz)^2 \geq 4\sqrt{3}(xy+yz+zx)\cdot F$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1478 If $m,n \geq 0$ and $a,b,c,m+n>0$ then:

$$\frac{a^2}{mb^2+nab}+\frac{b^2}{mc^2+nbc}+\frac{c^2}{ma^2+nca} \geq \frac{(a+b+c)^2}{(m+n)(a^2+b^2+c^2)}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1479 If $A_kB_kC_k, k=\overline{1,3}$ are three triangles with circumradii $R_k, k=\overline{1,3}$, then:

$$\frac{1}{a_1a_2a_3}+\frac{1}{b_1b_2b_3}+\frac{1}{c_1c_2c_3} \geq \frac{9\sqrt{3}}{R_1+R_2+R_3}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1480 If $m,x,y,z>0$ and $u,v,w \geq 0, 3m=x+y+z, u+v+w=2t>0$ then in ΔABC holds:

$$(x^x+y^x+z^x)a^ub^vc^w+(x^y+y^y+z^y)a^vb^wc^u+(x^z+y^z+z^z)a^wb^vc^u \geq$$
$$\geq 4^t\left(\sqrt{3}\right)^{4-t}\cdot m^m\cdot F^t$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1481 If $n \in \mathbb{N}, n \geq 3$ and $m, x_k \in [1,\infty), k=\overline{1,n}, m=\frac{x_1+x_2+\cdots+x_n}{n}$, then:

$$\sum_{k=1}^{n}\left(x_1^{x_k}+x_2^{x_k}+\cdots+x_n^{x_k}\right) \geq n^2\cdot m^m$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1482 Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ sequences of real numbers strictly positive such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n\cdot\sqrt[n]{(2n-1)!!}}=a>0$ and $\lim\limits_{n\to\infty}\frac{b_{n+1}}{b_n\cdot\sqrt[n]{a_n}}=b>0$. Find: $\lim\limits_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}}-\sqrt[n]{b_n}\right)$.

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1483 Let $(a_n)_{n\geq 1}$ sequence of real numbers strictly positive such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a$ and let be $b_n(x)=n^{\sin^2 x}\left(\sqrt[n+1]{a_{n+1}^{\cos^2 x}}+\sqrt[n]{a_n^{\cos^2 x}}\right), \forall x\in\mathbb{R}, n\in\mathbb{N}^*$. Find:

$$\lim_{n\to\infty} b_n(x).$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1484 If $m,n>0$ then in ΔABC holds:

$$\frac{(a^2+bc)^2}{mab+nac}+\frac{(b^2+2ca)^2}{mbc+nab}+\frac{(c^2+2ab)^2}{mac+nbc}\geq\frac{36\sqrt{3}}{m+n}\cdot F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1485 If $x,y>0$ then in ΔABC holds:

$$\left(\frac{x}{a}+\frac{y}{b}\right)^2 w_c^2+\left(\frac{x}{b}+\frac{y}{c}\right)^2 w_a^2+\left(\frac{x}{c}+\frac{y}{a}\right)^2 w_b^2\geq\frac{18xy\cdot r}{R}$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1486 In ΔABC the following relationship holds:

$$2\left(\frac{g_a^2}{h_a^2}+\frac{g_b^2}{h_b^2}+\frac{g_c^2}{h_c^2}\right)+\frac{2r}{R}\geq 7$$

Proposed by Adil Abdullayev-Azerbaijan

U.1487 In ΔABC the following relationship holds:

$$\left(\frac{R}{2r}\right)^2\geq\frac{m_am_bm_c(m_a+m_b+m_c)}{9F^2}\geq\frac{1}{4}+\frac{m_a^3+m_b^3+m_c^3}{4m_am_bm_b}\geq\left(\frac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a}\right)^2$$

Proposed by Adil Abdullayev-Azerbaijan

U.1488 In ΔABC the following relationship holds:

$$\left(\sqrt[3]{\frac{r_a}{r_b}}+\sqrt[3]{\frac{r_b}{r_a}}\right)\left(\sqrt[3]{\frac{r_b}{r_c}}+\sqrt[3]{\frac{r_c}{r_a}}\right)\left(\sqrt[3]{\frac{r_c}{r_a}}+\sqrt[3]{\frac{r_a}{r_c}}\right)\leq\frac{4R}{r}$$

Proposed by Adil Abdullayev-Azerbaijan

U.1489 In ΔABC the following relationship holds:

$$\frac{2(a^2+b^2)}{(a+b)^2}+\frac{2(b^2+c^2)}{(b+c)^2}+\frac{2(c^2+a^2)}{2(c+a)^2}\leq\frac{3n_ag_ar_a}{h_ah_bh_c}$$

Proposed by Adil Abdullayev-Azerbaijan

U.1490 In $\Delta ABC, DE = m_a, EF = m_b, FD = m_c, R_m, r_m$ —circumradii and inradii in ΔDEF. Prove that:

$$\frac{R_m}{2r_m} \geq \frac{m_a}{h_a}$$

Proposed by Adil Abdullayev-Azerbaijan

U.1491 In ΔABC the following relationship holds:

$$\frac{r_a^2 - r_b^2}{\cos^2\frac{A}{2}} + \frac{r_b^2 - r_c^2}{\cos^2\frac{B}{2}} + \frac{r_c^2 - r_a^2}{\cos^2\frac{C}{2}} \geq 0$$

Proposed by Adil Abdullayev-Azerbaijan

U.1492 In acute ΔABC the following relationship holds:

$$\cos(A - B) + \cos(B - C) + \cos(C - A) \leq \sum_{cyc} \frac{2m_a m_b}{m_a^2 + m_b^2}$$

Proposed by Adil Abdullayev-Azerbaijan

U.1493 In ΔABC the following relationship holds:

$$4\left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\right) + \frac{15 w_a w_b w_c}{r_a r_b r_c} \geq 27$$

Proposed by Adil Abdullayev-Azerbaijan

U.1494 In ΔABC the following relationship holds:

$$2 + \sum_{cyc}\left(\frac{r_a^2}{bc} + \frac{bc}{r_a^2}\right) \geq \frac{4R}{r}$$

Proposed by Adil Abdullayev-Azerbaijan

U.1495 In ΔABC the following relationship holds:

$$48\prod_{cyc} \sin\frac{A}{2} \leq 3 + \cos(A - B) + \cos(B - C) + \cos(C - A)$$

Proposed by Adil Abdullayev-Azerbaijan

U.1496 In $\Delta ABC, N_a$ —Nagel's point, D, E, F —circumcenters of $\Delta BCN_a, \Delta CAN_a, \Delta ABN_a$. If AM, BK, CN —Gergonne's cevians then:

$$\frac{[ABC]}{4 \cdot [MNK]} = 1 + \sqrt{\frac{[DEF]}{[ABC]}}$$

Proposed by Adil Abdullayev-Azerbaijan

U.1497 Let be $M \in Int(\Delta ABC)$ with area F and $x = AM, y = BM, z = CM$, then:

$$\frac{xy \cdot h_a}{b} + \frac{yz \cdot h_b}{c} + \frac{zx \cdot h_c}{a} \geq 2F$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

U.1498 If $x, y, z > 0$ then in ΔABC holds:

$$xy \cdot h_a h_b + yz \cdot h_b h_c + zx \cdot h_c h_a \leq \frac{1}{4}(ax + by + cz)^2$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

U.1499 If $x, y, z > 0$, then in ΔABC with F area holds:

$$\frac{xy \cdot h_a}{b} + \frac{yz \cdot h_b}{c} + \frac{zx \cdot h_c}{a} \leq \frac{(ax + by + cz)^2}{8F}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

U.1500 If $x, y, z > 0$ then in ΔABC with F area holds:

$$\frac{(x+y)b}{h_a} + \frac{(y+z)c}{h_b} + \frac{(z+x)a}{h_c} - \left(\frac{ax}{h_a} + \frac{bx}{h_b} + \frac{cz}{h_c}\right) \geq 4\sqrt{xy \sin^2\frac{C}{2} + yz \sin^2\frac{B}{2} + zx \sin^2\frac{C}{2}}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

U.1501 If $x, y, z > 0$ then in ΔABC with F area holds:

$$\left(\frac{x}{h_a^2} + \frac{y}{h_b^2} + \frac{z}{h_c^2}\right)^2 \geq \frac{xy + yz + zx}{F^2}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

U.1502 If $x, y, z > 0$, then in ΔABC with F area holds:

$$\frac{ax + by}{z} + \frac{by + cz}{x} + \frac{cz + ax}{y} \geq 4 \cdot \sqrt[4]{27} \cdot \sqrt{F}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

U.1503 If $x, y, z > 0$ then in ΔABC with F area and a, b, c the lengths of sides holds:

$$a + b + c \geq 2 \cdot \sqrt[4]{27} \cdot \sqrt{F} + \frac{2}{x + y + z}\left(a \cdot \frac{x^2 - yz}{x} + b \cdot \frac{y^2 - zx}{y} + c \cdot \frac{z^2 - xy}{z}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

U.1504 Let $X \in Int(\Delta ABC)$ and d_a, d_b, d_c −distances from M to the sides BC, CA, AB. If

$M \in BC, N \in CA, P \in AB$, then holds:

$$\frac{AM}{d_a}+\frac{BN}{d_b}+\frac{CP}{d_c}\ge 9$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

U.1505 In $\Delta ABC, n_a$ −Nagel's cevian, g_a −Gergonne's cevian, holds:

$$n_a^2+n_b^2+n_c^2+g_a^2+g_b^2+g_c^2+2r(r_a+r_b+r_c)\ge 8\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

U.1506 For $m\ge 0$, find:

$$\Omega=\lim_{n\to\infty}\left(\left(\sqrt[n+1]{(2n+1)!!}\right)^{m+1}-\left(\sqrt[n]{(2n-1)!!}\right)^{m+1}\right)\cdot\tan^m\frac{\pi}{n}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

U.1507 If $x>0$, then in ΔABC with F area holds: $xa^2+b^2+x^2c^2\ge 4x\sqrt{3}F$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

U.1508 In ΔABC with F area holds: $1936a^2+44b^2+c^2\ge 176\sqrt{3}F$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

U.1509 If $x>0$, then in ΔABC with F area holds:

$$\frac{x^2}{h_a}+\frac{x}{h_b}+\frac{1}{h_c}\ge\frac{x\sqrt[4]{27}}{\sqrt{F}}$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

U.1510 If $m\ge 0$, then in ΔABC holds:

$$\frac{(h_a+r)^{m+1}}{h_a(h_a-r)^m}+\frac{(h_b+r)^{m+1}}{h_b(h_b-r)^m}+\frac{(h_c+r)^{m+1}}{h_c(h_c-r)^m}\ge 2^{m+2}$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

U.1511 If $x,y,t>0$ and $x+y=2$, then in ΔABC with F area holds:

$$ta^2+t^2b^xc^x+b^xc^x\ge 4t\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

U.1512 In ΔABC with F area holds:

$$a^2+b^2+c^2+ab+bc+ca\ge 8\sqrt{3}F+\left(a-\sqrt{bc}\right)^2+\left(b-\sqrt{ca}\right)^2+\left(c-\sqrt{ab}\right)^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

U.1513 Let $t \geq 0, (H_n)_{n\geq 1}, H_n = \sum_{k=1}^{n} \frac{1}{k}$ and $(a_n)_{n\geq 1}$ a sequence of real numbers strictly positive such that $\lim_{n\to\infty} \frac{a_{n+1}}{n\cdot a_n} = a > 0$. Find:

$$\Omega = \lim_{n\to\infty} e^{-tH_n}\left(\left(\sqrt[n+1]{a_{n+1}}\right)^{t+1} - \left(\sqrt[n]{a_n}\right)^{t+1}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

U.1514 For $m \geq 0$, find:

$$\Omega = \lim_{n\to\infty}\left(\left(\sqrt[n+1]{(n+1)!}\right)^{m+1} - \left(\sqrt[n]{n!}\right)^{m+1}\right)\sin^m\frac{\pi}{n}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

U.1515 Let $t \in (0,2], M \in AC$ of ΔABC, then holds:

$$\frac{w_a - tr}{h_a + r} + \frac{BM - tr}{h_a + r} + \frac{m_c - tr}{h_c + r} \geq \frac{3(3-t)}{4}$$

Proposed by D.M. Bătineţu-Giurgiu, Claudia Nănuţi-Romania

U.1516 If $x, y \in \left[0, \frac{\pi}{2}\right)$ then in ΔABC with F area holds:

$$\frac{e^x + e^y}{1 + \sin z}\cdot a^4 + \frac{e^y + e^z}{1 + \sin x}\cdot b^4 + \frac{e^z + e^x}{1 + \sin y}\cdot c^4 \geq 32F^2$$

Proposed by D.M. Bătineţu-Giurgiu, Florică Anastase-Romania

U.1517 Let $t > 0, u \geq 0$ then in ΔABC with F area holds:

$$a^{t(u+1)} + b^{t(u+1)} + c^{t(u+1)} \geq 2^{t(u+1)}\left(\sqrt{3}\right)^{2-t(u+1)}\left(\sqrt[4]{3}\right)^{t(u+1)}F^{\frac{t(u+1)}{2}}$$

Proposed by D.M. Bătineţu-Giurgiu, Dan Nănuţi-Romania

U.1518 If $x, y, z > 0$ then in ΔABC with F area holds:

$$\frac{x}{\sqrt{yz}}\cdot a + \frac{y}{\sqrt{zx}}\cdot b + \frac{z}{\sqrt{xy}}\cdot c \geq 2\cdot\sqrt[4]{27}\cdot\sqrt{F}$$

Proposed by D.M. Bătineţu-Giurgiu, Dan Nănuţi-Romania

U.1519 If $x, y, z > 0$ then in ΔABC with F area holds:

$$\frac{e^x + x^2}{y + z}\cdot a^4 + \frac{e^{y+y^2}}{z + x}\cdot b^4 + \frac{e^{z+z^2}}{x + y}\cdot c^4 > 24F^2$$

Proposed by D.M. Bătineţu-Giurgiu, Dan Nănuţi-Romania

U.1520 Let $M \in Int(\Delta ABC)$ and $x = MA, y = MB, z = MC$, then holds:

$$\frac{xa}{h_a\sqrt{yz}}+\frac{yb}{h_b\sqrt{zx}}+\frac{zc}{h_c\sqrt{xy}}\geq 2\sqrt{3}$$

Proposed by D.M. Bătineţu-Giurgiu,Gheorghe Boroica-Romania

U.1521 If $x,y,z>0$ then in ΔABC with F area holds:

$$\frac{e^{x^2}}{y+z}\cdot a+\frac{e^{y^2}}{z+x}\cdot b+\frac{e^{z^2}}{x+y}\cdot c>2\sqrt[4]{27}\cdot\sqrt{F}$$

Proposed by D.M. Bătineţu-Giurgiu -Romania

U.1522 In ΔABC with F area holds:

$$e^{a^4}+e^{b^4}+e^{c^4}\geq 8\sqrt{3}\cdot F$$

Proposed by D.M. Bătineţu-Giurgiu-Romania

U.1523 In ΔABC with F area holds:

$$e^{a^2}+e^{b^2}+e^{c^2}\geq 4\sqrt[4]{27}\cdot\sqrt{F}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

U.1524 Let $M\in Int(\Delta ABC)$ and $x=MA,y=MB,z=MC$, then:

$$\frac{x}{\sqrt{yz}}\cdot a^2+\frac{y}{\sqrt{zx}}\cdot b^2+\frac{z}{\sqrt{xy}}\cdot c^2\geq 4\sqrt{3}\cdot F$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

U.1525 In ΔABC holds:

$$\frac{a\sin B}{(\sin^2 C+\sin^2 A)h_a}+\frac{b\sin^2 C}{(\sin^2 A+\sin^2 B)h_c}+\frac{c\sin A}{(\sin^2 B+\sin^2 C)h_a}>\sqrt{3}$$

Proposed by D.M. Bătineţu-Giurgiu, Daniel Sitaru-Romania

U.1526 Let $m\geq 0, M\in Int(\Delta ABC)$ and $F=[ABC],F_a=[MBC],F_b=[MCA],F_c=[MAB]$, then:

$$\frac{1+F_a^{2m+2}}{(F-F_a)^{m+1}}+\frac{1+F_b^{2m+2}}{(F-F_b)^{m+1}}+\frac{1+F_c^{2m+2}}{(F-F_c)^{m+1}}\geq\frac{3}{2^m}$$

Proposed by D.M. Bătineţu-Giurgiu, Mihaly Bencze-Romania

U.1527 Let $m\geq 0, A_1A_2\dots A_8$ a convex octagon with F area with lengths sides $a_k=A_kA_{k+1}$,

$k=\overline{1,8}, A_9=A_1, M\in Int(A_1A_2\dots A_8)$. If $d_k=d(M,A_kA_{k+1}), k=\overline{1,8}$ is distance from M to the side $A_kA_{k+1}, k=\overline{1,8}$, then:

$$\sum_{k=1}^{8}\frac{a_k^{m+2}}{d_k^m}\geq 2^{m+2}\cdot F\cdot \tan^{m+1}\frac{\pi}{8}$$

Proposed by D.M. Bătineţu-Giurgiu, Mihaly Bencze-Romania

U.1528 If $x,y,z>0$ then in ΔABC with F area holds:

$$\left(\frac{1}{y}+\frac{1}{z}\right)\cdot\frac{x}{h_bh_c}+\left(\frac{1}{z}+\frac{1}{x}\right)\cdot\frac{y}{h_ch_a}+\left(\frac{1}{x}+\frac{1}{y}\right)\cdot\frac{z}{h_ah_b}\geq\frac{2\sqrt{3}}{F}$$

Proposed by D.M. Bătineţu-Giurgiu, Mihaly Bencze-Romania

U.1529 Let $x\in(0,\infty)$ and ΔABC with F area, the points $A_1\in(BC),B_1\in(CA),B_1\in(AB)$ such that $BA_1=x\cdot A_1C, CA_1=x\cdot B_1A, AC_1=x\cdot C_1B, a_1=B_1C_1, b_1=C_1A_1, c_1=A_1B_1$, then:

$$aa_1+bb_1+cc_1\geq 4\sqrt{3}\cdot\frac{\sqrt{x^2-x+1}}{x+1}\cdot F$$

Proposed by D.M. Bătineţu-Giurgiu, Mihaly Bencze-Romania

U.1530 Let $x,y,z>0$ such that $xyz\geq 1$, then in ΔABC with F area holds:

$$xa^2+yb^2+zc^2\geq 4\sqrt{3}\cdot F$$

Proposed by D.M. Bătineţu-Giurgiu, Mihaly Bencze-Romania

U.1531 If $x,y,z>0$ then in ΔABC with F area holds:

$$\frac{x+y}{h_ah_b}+\frac{y+z}{h_bh_c}+\frac{z+x}{h_ch_a}-\left(\frac{x}{h_a^2}+\frac{\frac{y}{h_b^2}z}{h_c^2}\right)\geq\frac{1}{2F}\sqrt{xy\sin^2\frac{C}{2}+yz\sin^2\frac{A}{2}+zx\sin^2\frac{B}{2}}$$

Proposed by D.M. Bătineţu-Giurgiu, Mihaly Bencze-Romania

U.1532 If $m\geq 0$ then in ΔABC with F area holds:

$$\frac{a^{m+2}b}{h_a^mh_b^{2m+1}}+\frac{b^{m+2}c}{h_b^mh_c^{2m+1}}+\frac{c^{m+2}a}{h_c^mh_a^{2m+1}}\geq\frac{2^{m+3}}{3^mF^{m-1}}$$

Proposed by D.M. Bătineţu-Giurgiu, Florică Anastase-Romania

U.1533 Let $m\geq 0$ and $x,y,z>0$ then in ΔABC with F area holds:

$$\frac{xa^{m+2}}{\sqrt{yz}h_a^m}+\frac{yb^{m+2}}{\sqrt{zx}h_b^m}+\frac{zc^{m+2}}{\sqrt{xy}h_c^m}\geq 2^{m+2}\left(\sqrt{3}\right)^{1-m}F$$

Proposed by D.M. Bătineţu-Giurgiu, Florică Anastase-Romania

U.1534 Let $m,n\geq 0$ and $x,y,z>0$ then in ΔABC with F area holds:

$$\frac{xa^{m+2}}{\sqrt{yz}h_a^{m+2n}}+\frac{yb^{m+2}}{\sqrt{zx}h_b^{m+2n}}+\frac{zc^{m+2}}{\sqrt{xy}h_c^{m+2n}}\geq 2^{m+2}\left(\sqrt{3}\right)^{1-m-n}\cdot F^{1-n}$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

U.1535 In $\Delta ABC, \Delta A_1B_1C_1$ with areas F, F_1 holds:

$$\frac{aa_1^2}{h_a}+\frac{bb_1^2}{h_b}+\frac{cc_1^2}{h_c}\geq 8F_1$$

Proposed by D.M. Bătinețu-Giurgiu, Florică Anastase-Romania

U.1536 Let $M\in Int(\Delta ABC)$ and d_a, d_b, d_c −distances from M to the sides BC, CA, BA, then holds:

$$\frac{ad_b}{\sqrt{d_cd_a}}+\frac{bd_c}{\sqrt{d_ad_b}}+\frac{cd_a}{\sqrt{d_bd_c}}\geq 2\sqrt{3}\cdot\sqrt{(4R+r)r}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

U.1537 In ΔABC with F area holds:

$$\frac{a^3}{r_a}+\frac{b^3}{r_b}+\frac{c^3}{r_c}\geq\frac{24s}{2R-r}\cdot F$$

Proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți-Romania

U.1538 In ΔABC with s semiperimeter holds:

$$\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}\geq\frac{6\sqrt{3}r}{2R-r}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți-Romania

U.1539 In ΔABC the following relationship holds:

$$\left(\frac{R}{2r}\right)^n+\left(\frac{w_aw_bw_c}{r_ar_br_c}\right)^n\geq 2, n>0$$

Proposed by Adil Abdullayev-Azerbaijan

U.1540 If $x, y, z>0$ then:

$$\sum_{cyc}\frac{x}{y+z}+4\prod_{cyc}\frac{x}{y+z}\geq 2+\prod_{cyc}\frac{x}{y+z}\cdot\frac{(x+y)(y+z)(z+x)-8xyz}{(x+y)(y+z)(z+x)-4xyz}$$

Proposed by Adil Abdullayev-Azerbaijan

U.1541 In ΔABC the following relationship holds:

$$\frac{3}{2}\leq\sum_{cyc}\frac{a^2}{b^2+c^2}<1+\frac{R}{4r}$$

Proposed by Marin Chirciu-Romania

U.1542 If $a, b, c>0, k\geq 2$ then:

$$\frac{a}{\sqrt{ka+b}}+\frac{b}{\sqrt{kb+c}}+\frac{c}{\sqrt{kc+a}}\le\sqrt{\frac{3(a+b+c)}{k+1}}$$

Proposed by Marin Chirciu-Romania

U.1543 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt[3]{\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}}\le\frac{3}{2}\sqrt[6]{3}$$

Proposed by Marin Chirciu-Romania

U.1544 In ΔABC the following relationship holds:

$$\frac{1}{F}\left(2-\frac{r}{R}\right)^2\le\sum_{cyc}\frac{\cot\frac{A}{2}}{a^2}\le\frac{1}{F}\left(\frac{R^2}{r^2}-\frac{R}{r}+\frac{r^2}{R^2}\right)$$

Proposed by Marin Chirciu-Romania

U.1545 Solve: $2^{i3^{ix}}=3^{ix}, x\in\mathbb{R}$.

Proposed by Jalil Hajimir- Canada

U.1546 Solve for real numbers:

$$([x]+1)^x+([x]+2)^x+([x]+3)^x=[x]5^x, [*]-GIF.$$

Proposed by Jalil Hajimir- Canada

U.1546 Find:

$$f^{(2021)}(z)\text{ if }f(z)=\frac{1}{2z+3}+e^{3z^4}\cos(5z)$$

where $f^{(n)}(z)$ –denotes the n^{th} derivative of $f(z)$.

Proposed by Jalil Hajimir- Canada

U.1547 Solve for real numbers: $\sqrt{2(x^2+1)}+\log x=2x$

Proposed by Jalil Hajimir- Canada

U.1548 Solve for real numbers:

$$\sqrt{x[x]}+\sqrt{\frac{[x]}{x}}=5; [*]-GIF.$$

Proposed by Jalil Hajimir- Canada

U.1549 Find:

$$\Omega=\lim_{x\to1}\frac{\Gamma(2x)\Gamma(3x)-6\Gamma(x)}{\sqrt[5]{x}-1}$$

Proposed by Jalil Hajimir- Canada

U.1550 Find without softwares:

$$\Omega=\int_0^\infty\frac{2^{\frac{x}{2}}}{1+e^x}dx$$

Proposed by Jalil Hajimir- Canada

U.1551 Find without any software:

$$\Omega=\int_{1399}^{2020}(-1)^{[x]}\cdot(2x-[x])dx, [*]-GIF.$$

Proposed by Jalil Hajimir- Canada

U.1552 If $a,b,c>0$ and $a+b+c=1$ then: $13(a^3+b^3+c^3)-12(a^4+b^4+c^4)\geq 1$

Proposed by Marin Chirciu-Romania

U.1553 If $a,b,c>0$ such that $abc=1$ and $\lambda\geq 0$ then:

$$\sum_{cyc}\frac{(b^2 2+\lambda c^2)^2}{a+bc}\geq\frac{3}{2}(\lambda+1)^2$$

Proposed by Marin Chirciu-Romania

U.1554 If $a,b,c>0$ such that $abc=1$ and $n\in\mathbb{N}^*, n\geq 3$ then:

$$\sum_{cyc}\frac{(b+c)^n}{a^3+\lambda}\geq\frac{3\cdot 2^n}{\lambda+1};\lambda\geq 2$$

Proposed by Marin Chirciu-Romania

U.1555 Find:

$$\Omega=\int_1^2\frac{x^2+e^x}{x^2+xe^x}dx$$

Proposed by Marin Chirciu-Romania

U.1556 In ΔABC the following relationship holds:

$$\frac{6}{3+\cos(A-B)+\cos(B-C)+\cos(C-A)}\leq\frac{R}{2r}$$

Proposed by Adil Abdullayev-Azerbaijan

U.1557 In acute $\Delta ABC, o_a$ −circumcevian, the following relationship:

$$\frac{w_a^2}{h_a^2}+\frac{w_b^2}{h_b^2}+\frac{w_c^2}{h_c^2}=2\left(\frac{o_a}{o_a+h_a}+\frac{o_b}{o_b+h_b}+\frac{o_c}{o_c+h_c}\right)$$

Proposed by Adil Abdullayev-Azerbaijan

U.1558 In ΔABC the following relationship holds:

$$\left(\frac{\cos A}{s_a}\right)^2+\left(\frac{\cos B}{o_b}\right)^2+\left(\frac{\cos C}{s_c}\right)^2\geq\frac{1}{3R^2}$$

Proposed by Adil Abdullayev-Azerbaijan

U.1559 In ΔABC the following relationship holds:

$$\frac{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}{\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}}\leq\left(\frac{R}{2r}\right)^2$$

Proposed by Adil Abdullayev-Azerbaijan

U.1560 In ΔABC the following relationship holds:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}+\frac{3r(R-2r)}{8R^2}$$

Proposed by Adil Abdullayev-Azerbaijan

U.1561 In ΔABC the following relationship holds:

$$\left(\frac{2m_a}{h_a}\right)^2\geq\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)^2+\frac{m_a^2(m_b-m_c)^2\,(m_a^2-m_b^2-m_c^2)^2}{9m_b^2m_c^2F^2}$$

Proposed by Adil Abdullayev-Azerbaijan

U.1562 Find:

$$\Omega=\lim_{x\to 2}\frac{1+\cos\big(\pi\Gamma(x)\big)}{(x-2)^2}$$

Proposed by Jalil Hajimir--Canada

U.1563 In ΔABC the following relationship holds:

$$1+\frac{r_a}{h_a}+\frac{r_b}{h_b}+\frac{r_c}{h_c}\geq 2\left(\frac{m_a m_b m_c}{r_a r_b r_c}+\frac{w_a w_b w_c}{h_a h_b h_c}\right)$$

Proposed by Adil Abdullayev-Azerbaijan

U.1564 In ΔABC the following relationship holds:

$$\left(\frac{a+c}{a+b}\right)^2+\left(\frac{b+a}{b+c}\right)^2+\left(\frac{c+b}{c+a}\right)^2+5\geq\frac{8r_a r_b r_c}{w_a w_b w_c}$$

Proposed by Adil Abdullayev-Azerbaijan

U.1565 Solve for real numbers:

$$[x^2]+x^2=2x[x],[x]-GIF.$$

Proposed by Jalil Hajimir--Canada

U.1566 Solve for real numbers:

$$x[x]+x^2[x^2]=20,[*]-GIF.$$

Proposed by Jalil Hajimir--Canada

U.1567 Prove:

$$\frac{\tan^{-1}x}{x}+\frac{\tan^{-1}y}{y}\geq\frac{\pi}{2},0<x,y\leq 1$$

Proposed by Jalil Hajimir--Canada

U.1568 Prove:

$$\sqrt{[x](y-[y])}+\sqrt{[y](x-[x])}\leq\sqrt{xy},x,y\geq 0,[*]-GIF.$$

Proposed by Jalil Hajimir--Canada

U.1569 Evaluate:

$$\Omega=\int\frac{\sqrt[3]{x^2}}{x^2+1}dx$$

Proposed by Jalil Hajimir--Canada

U.1570 Solve:

$$2\leq\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+\left[\frac{3}{x}\right]\leq 5,[*]-GIF.$$

Proposed by Jalil Hajimir--Canada

U.1571 Solve for real numbers:

$$\sqrt[3]{2x+1}+\sqrt[5]{8x+4}=4$$

Proposed by Jalil Hajimir--Canada

U.1572 Find the values of a and n if:

$$\lim_{x\to 0}\frac{\sin\left(\frac{3\pi}{2}\left(\frac{\sin x}{x}\right)\right)+\sin\left(\frac{\pi}{2}\left(\frac{\sin x}{x}\right)\right)}{ax^n}=1$$

Proposed by Jalil Hajimir--Canada

U.1573 Let a,b and c be positive real numbers such that $abc=1$. Prove that:

$$\frac{a}{a^5+1}+\frac{b}{b^5+1}+\frac{c}{c^5+1}\leq\frac{3}{2}$$

Proposed by Jalil Hajimir--Canada

U.1574 Calculate the surface area of:

$$S=\{(x,y,z)|x^2+y^2=z^2,-1\leq z\leq 1\}$$

Proposed by Jalil Hajimir-Canada

U.1575 Let $a, b, c > 0$. Find for what values of k the following inequality holds:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{k\sqrt[3]{abc}}{a+b+c} \ge 3+k$$

Proposed by Jalil Hajimir--Canada

U.1576 Prove that:

$$|\sin x_1| + |\sin x_2| + \cdots + |\sin x_n| \ge 1 - |\cos(x_1 + x_2 + \cdots + x_n)|$$

where $x_1, x_2, \dots, x_n \in \mathbb{R}, n \ge 2, n \in \mathbb{N}$.

Proposed by Jalil Hajimir--Canada

U.1577 Find without softwares:

$$\Omega = \int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)^2}\,dx$$

Proposed by Jalil Hajimir-Canada

U.1578 In ΔABC the following relationship holds:

$$\frac{(m_a+m_b)(m_b+m_c)(m_c+m_a)}{8m_am_bm_c} \le \left(\frac{R}{2r}\right)^2$$

Proposed by Adil Abdullayev-Azerbaijan

U.1579 Find without softwares:

$$\Omega = \int_{-\infty}^\infty \frac{dx}{1+x^{2020}}$$

Proposed by Adil Abdullayev-Azerbaijan

U.1580 In acute ΔABC the following relationship holds:

$$\cos(A-B)\cos(B-C)\cos(C-A) \le \frac{4abc}{a^3+b^3+c^3+abc}$$

Proposed by Adil Abdullayev-Azerbaijan

U.1581 In ΔABC the following relationship holds:

$$16(m_a^2m_b^2 + m_b^2m_c^2 + m_c^2m_a^2) = 9(a^2b^2 + b^2c^2 + c^2a^2)$$

Proposed by Adil Abdullayev-Azerbaijan

U.1582 In ΔABC the following relationship holds:

$$\frac{(m_a+m_b)(m_b+m_c)(m_c+m_a)}{m_am_bm_c} + \frac{9F^2}{m_am_bm_c(m_a+m_b+m_c)} \ge 9$$

Proposed by Adil Abdullayev-Azerbaijan

U.1583 In ΔABC the following relationship holds:

$$8\cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2} \le \left(\frac{h_a}{w_a}\right)^4 + 3\left(\frac{h_a}{w_a}\right)^2 + 4$$

Proposed by Adil Abdullayev-Azerbaijan

U.1584 In ΔABC the following relationship holds:

$$3(r_a^2r_b+r_b^2r_c+r_c^2r_a)(r_ar_b^2+r_br_c^2+r_cr_a^2)\geq s^6$$

Proposed by Adil Abdullayev-Azerbaijan

U.1585 In ΔABC the following relationship holds:

$$\frac{m_a}{m_b+m_c}+\frac{m_b}{m_c+m_a}+\frac{m_c}{m_a+m_b}+\frac{9F^2}{2m_am_bm_c(m_a+m_b+m_c)}\leq 2$$

Proposed by Adil Abdullayev-Azerbaijan

U.1586 In ΔABC the following relationship holds:

$$\frac{m_a}{m_b+m_c}+\frac{m_b}{m_c+m_a}+\frac{m_c}{m_a+m_b}\leq\frac{1}{2}+\sqrt{\frac{m_am_bm_c(m_a+m_b+m_c)}{9F^2}}$$

Proposed by Adil Abdullayev-Azerbaijan

U.1587 In ΔABC the following relationship holds:

$$\frac{r_a^2}{\cos^2\frac{A}{2}}+\frac{r_b^2}{\cos^2\frac{B}{2}}+\frac{r_c^2}{\cos^2\frac{C}{2}}\geq 2R(4R+r)$$

Proposed by Adil Abdullayev-Azerbaijan

U.1588 If $a,b,c\geq 0$ such that $ab+bc+ca=1$ then find the minimum value of the expression:

$$P=\frac{a^4}{b+c}+\frac{b^4}{c+a}+\frac{c^4}{a+b}+\frac{80}{a+b+c}$$

Proposed by Marin Chirciu-Romania

U.1589 In ΔABC the following relationship holds:

$$18F\leq\sum_{cyc}r_a(r_b+r_c)\cot\frac{A}{2}\leq 2F\left(\frac{4R}{r}+1\right)$$

Proposed by Marin Chirciu-Romania

U.1590 In $\Delta ABC, N$ –ninepoint center, S_p –Spieker's point, I –incenter, O –circumcenter, G –centroid. Prove that: $[IGN]=[OGS_p]$.

Proposed by Adil Abdullayev-Azerbaijan

U.1591 If $a,b,c>0$ and $0\leq\lambda\leq 5$ then:

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{8\lambda abc}{(a+b)(b+c)(c+a)}\geq\lambda+3$$

Proposed by Marin Chirciu-Romania

U.1592 Find:

$$\Omega=\int_0^1\frac{x^3}{(x-1)^3+3x-5}dx$$

Proposed by Marin Chirciu-Romania

U.1593 In ΔABC the following relationship holds:

$$12r\leq\sum_{cyc}(b+c)\tan\frac{A}{2}\leq 6R$$

Proposed by Marin Chirciu-Romania

U.1594 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{s-a}{h_a-2r}\leq 2\sum_{cyc}\frac{s-a}{r_a-r}$$

Proposed by Marin Chirciu-Romania

U.1595 If $a_1, a_2, \dots, a_n > 0, n \geq 3$ such that $a_1 a_2 \cdot \dots \cdot a_n = 1$ then:

$$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_n}{a_1}\geq a_1^{\frac{2}{n-1}}+a_2^{\frac{2}{n-1}}+\cdots+a_n^{\frac{2}{n-1}}$$

Proposed by Marin Chirciu-Romania

U.1596 If $a, b, c \geq 0$ such that $a+b+c=2$ and $ab+bc+ca=1$ then:

$$\sum_{cyc}\frac{1}{(1+a^2)(1+b^2)}\geq\frac{5}{4}$$

Proposed by Marin Chirciu-Romania

U.1597 If $a, b, c, x, y, z > 0$ such that $x+y+z=3$ and $\lambda \geq 2$ then:

$$\frac{a}{a+\lambda bx}+\frac{b}{b+\lambda cy}+\frac{c}{c+\lambda az}\geq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

U.1598 Solve for real numbers:

$$\begin{cases}\sqrt{x^2-2x+2a}\cdot\log_a(2a-y)=x\\ \sqrt{y^2-2y+2a}\cdot\log_a(2a-z)=y\,; a>1\\ \sqrt{z^2-2z+2a}\cdot\log_a(2a-x)=z\end{cases}$$

Proposed by Marin Chirciu-Romania

U.1599 Find:

$$\Omega=\int_0^{\frac{\pi}{3}}\frac{x\tan x+1}{x+\cos x}dx$$

Proposed by Marin Chirciu-Romania

U.1600 Let m and n be a positive integers. Prove that if n is a multiple of 3, then $(X^m-1)^n-1$ is divisible by $X^{2m}-X^m+1$.

Proposed by Marin Chirciu-Romania

U.1601 In ΔABC the following relationship holds:

$$\lambda\sum_{cyc}\sqrt{bc}+\mu\sum_{cyc}\sqrt{\frac{b^2+c^2}{2}}\leq 3(\lambda+\mu)\sqrt{3}R; \lambda,\mu\geq 0$$

Proposed by Marin Chirciu-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

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