TRIANGLE SOLUTION

AND

INEQUALITIES OF ALEXANDRU SZOROS

By ***Le Van***

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ABSTRACT

Trigonometric inequalities in triangles may not be a new topic in the world of mathematics, but its problems have never been too old. Otherwise, they are created and challenged it day by day all around the world, and so are the inequalities proposed by teacher Alexandru Szoros. And it is truly my pleasure to represent in this article those problems, which I solved by using the triangle solution method.

1. Introduction

In triangle ABC

$$\sin\frac{A}{2} \le \frac{abc}{(p-a)(a+b)(a+c)}$$

$$\left(\frac{R}{2}\right)^2 \ge \frac{r_a r_b + r_b r_c + r_c r_a}{27} \ge r^2$$

$$\frac{4R}{a} \ge \frac{b}{l_c} + \frac{c}{l_b} \ge \frac{8r}{a}$$

Alexandru Szoros

In this article, I focus on solving these problems by using acknowledgements of triangle solution. In lieu of representing theories and solutions separately, I would discuss on them in parallel sections. The three problems proposed by Alexandru Szoros are represented in Section 2, Section 3, and Section 4, respectively. In Section 5, I represent more discussions of solving techniques and tricks. And Section 6 contains the conclusion of this article.

2. A. Szoros' Inequality I

Problem: Given triangle ABC, prove that

$$\sin\frac{A}{2} \le \frac{abc}{(p-a)(a+b)(a+c)} \qquad (I)$$

Solution: Note that, if we put

$$f(a;b;c) = LHS(I)$$
$$g(a;b;c) = RHS(I)$$

Then $degf(a;b;c) = degg(a;b;c) = 0$.

This notation reminds me to rewrite $RHS(I)$ under trigonometric form. And fortunately, the law of sine indicates that, in any triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

And this law results in a consequence, if

$$h(a;b;c) = \frac{P(a;b;c)}{Q(a;b;c)}$$

where $degP = degQ = n$, then

$$h(a;b;c) = h(\sin A\,;\ \sin B\,;\ \sin C)$$

Indeed

$$h(a;b;c) = \frac{(2R)^n P(\sin A\,;\ \sin B\,;\ \sin C)}{(2R)^n Q(\sin A\,;\ \sin B\,;\ \sin C)} = h(\sin A\,;\ \sin B\,;\ \sin C)$$

Hence, we are able to rewrite $RHS(I)$:

$$\frac{abc}{(p-a)(a+b)(a+c)} = \frac{2abc}{(b+c-a)(a+b)(a+c)}$$

$$= \frac{2\sin A\sin B\sin C}{(\sin B+\sin C-\sin A)(\sin A+\sin B)(\sin A+\sin C)}$$

$$= \left(\frac{4\sin\frac{A}{2}\cos\frac{A}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2} - 2\sin\frac{A}{2}\cos\frac{A}{2}}\right)\left(\frac{2\sin\frac{C}{2}\cos\frac{C}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}\right)\left(\frac{2\sin\frac{B}{2}\cos\frac{B}{2}}{2\sin\frac{A+C}{2}\cos\frac{A-C}{2}}\right)$$

$$= \left(\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\cos\frac{A}{2}}\right)\left(\frac{\sin\frac{C}{2}\cos\frac{C}{2}}{\cos\frac{C}{2}\cos\frac{A-B}{2}}\right)\left(\frac{\sin\frac{B}{2}\cos\frac{B}{2}}{\cos\frac{B}{2}\cos\frac{A-C}{2}}\right)$$

$$= \frac{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A-B}{2}\cos\frac{A-C}{2}\left(\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right)}$$

$$= \frac{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A-B}{2}\cos\frac{A-C}{2}}$$

$$= \frac{\sin\frac{A}{2}}{\cos\frac{A-B}{2}\cos\frac{A-C}{2}}$$

And obviously, since

$$\cos\frac{A-B}{2}\cos\frac{A-C}{2} \le 1$$

We get

$$RHS(I) \ge \sin\frac{A}{2} \ge LHS(I)$$

QED. Equality holds when $\cos\frac{A-B}{2}\cos\frac{A-C}{2} = 1$, in other words, $A = B = C = \frac{\pi}{3}$. ■

3. A. Szoros' Inequality II

<u>Problem:</u> Given triangle ABC, prove that

$$\left(\frac{R}{2}\right)^2 \ge \frac{r_a r_b + r_b r_c + r_c r_a}{27} \ge r^2 \quad (II)$$

3.1.About inradius and exradii of a triangle

In this subsection, I revise the calculation of inradius and exradii of a triangle.

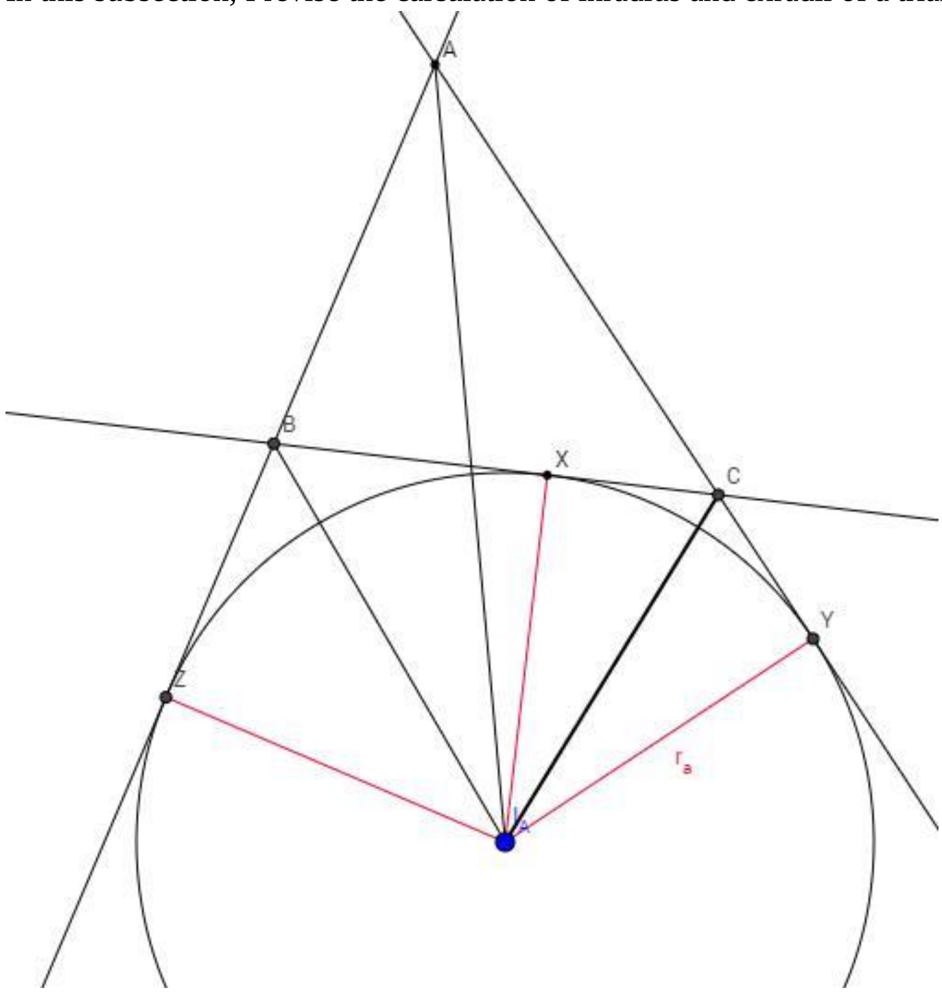

Figure 1. Excircle (I_A; r_a) of triangle ABC

From Figure 1, we get

$$\begin{aligned} S &= S_{ABI_A} + S_{ACI_A} - S_{BCI_A} \\ &= \frac{1}{2}AB.I_AZ + \frac{1}{2}AC.I_AY - \frac{1}{2}BC.I_AX \\ &= \frac{1}{2}(b+c-a)r_a \\ &= (p-a)r_a \end{aligned}$$

And therefore

$$r_a = \frac{S}{p-a} = \frac{2S}{b+c-a}$$

Totally similarly

$$r_b = \frac{S}{p-b} = \frac{2S}{c+a-b}; \quad r_c = \frac{S}{p-c} = \frac{2S}{a+b-c}$$

And additionally

$$r = \frac{S}{p} = \frac{2S}{a+b+c}$$

3.2.Solution for A. Szoros' Inequality II

Rewrite (II):

$$\left(\frac{R}{2}\right)^2 \geq \frac{4S^2}{27}\sum\frac{1}{(b+c-a)(c+a-b)} \geq \frac{4S^2}{(a+b+c)^2}$$

$$\Leftrightarrow \left(\frac{R}{2}\right)^2 \geq \frac{4S^2(a+b+b)}{27(b+c-a)(c+a-b)(a+b-c)} \geq \frac{4S^2}{(a+b+c)^2}$$

Firstly, I show the back part of this double-inequality, which could be rewritten as

$$(a+b+c)^3 \geq 27(b+c-a)(c+a-b)(a+b-c)$$

Since a, b, and c are three sides of a triangle, there exist three positive numbers x, y, and z such that

$$a = x+y; \quad b = y+z; \quad c = z+x$$

Hence, the above inequality should be transformed to

$$\begin{aligned} &8(x+y+z)^3 \geq 216xyz \\ &\Leftrightarrow (x+y+z)^3 \geq 27xyz \\ &\Leftrightarrow x+y+x \geq 3\sqrt[3]{xyz} \end{aligned}$$

This is true due to the AM-GM inequality of three non-negative numbers.

Return to the front part, where we need to prove

$$\left(\frac{R}{2}\right)^2 \geq \frac{pS^2}{27(p-a)(p-b)(p-c)} = \frac{p^2}{27}$$

$$\Leftrightarrow \frac{R}{2} \geq \frac{p}{3\sqrt{3}} = \frac{a+b+c}{6\sqrt{3}} = \frac{R(\sin A + \sin B + \sin C)}{3\sqrt{3}}$$

$$\Leftrightarrow \sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$$

This is also true in any triangle ABC thanks to the proof of using Jensen's inequality to concave functions. However, here I represent an amazing way to prove this, which was recommended by teacher Tran Phuong. Recall

$$\begin{aligned} T &= \sin A + \sin B + \sin C \\ &= \sin A + \sin B + \sin(A + B) \\ &= \sin A + \sin B + \sin A \cos B + \sin B \cos A \\ &= \frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\sin A + \frac{\sqrt{3}}{2}\sin B\right) + \frac{1}{\sqrt{3}}\left(\sqrt{3}\cos A \sin B + \sqrt{3}\cos B \sin A\right) \end{aligned}$$

Using AM-GM inequality from the geometric mean side, we get

$$T \leq \frac{1}{\sqrt{3}}\left[\frac{3}{4} + (\sin A)^2 + \frac{3}{4} + (\sin B)^2\right] + \frac{1}{2\sqrt{3}}[3(\cos A)^2 + (\sin B)^2 + 3(\cos B)^2 + (\sin A)^2] = \frac{3\sqrt{3}}{2}$$

Both parts of inequality (II) are proven, QED. Equalities hold when $a = b = c$. ■
In Section 5, I would demonstrate more about Tran Phuong's recommendation.

4. A. Szoros' Inequality III

Problem: Given triangle ABC, prove that

$$\frac{4R}{a} \geq \frac{b}{l_c} + \frac{c}{l_b} \geq \frac{8r}{a} \quad (III)$$

4.1.About angle bisectors of a triangle

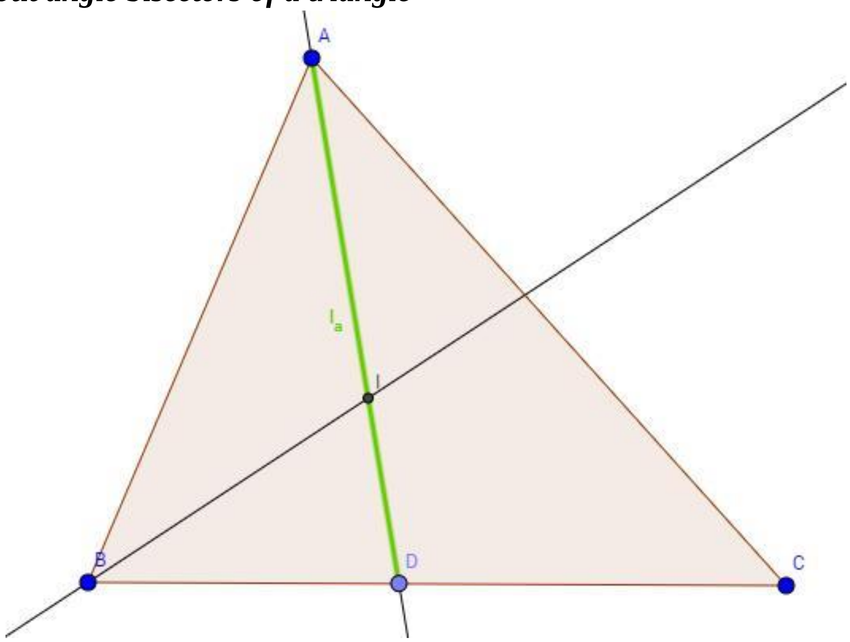

Figure 2. Angle bisector AD and incenter I of triangle ABC

Lemma I. (where l_a is the length of AD)

$$l_a = \frac{2bc\cos\frac{A}{2}}{b+c}$$

Proof. According to the angle bisector's property

$$\frac{BD}{AB} = \frac{CD}{AC} = \frac{BD+CD}{AB+AC} = \frac{BC}{AB+AC} = \frac{a}{b+c}$$

$$\Longrightarrow BD = \frac{ac}{b+c}; \quad CD = \frac{ab}{b+c}$$

Using the law of cosine in triangles ABD and ACD:

$$\begin{cases} BD^2 = AB^2 + AD^2 - 2AB.AD\cos\widehat{BAD} \\ CD^2 = AC^2 + AD^2 - 2AC.AD\cos\widehat{CAD} \end{cases}$$

$$\Leftrightarrow \begin{cases} \left(\frac{ac}{b+c}\right)^2 = c^2 + l_a^2 - 2cl_a\cos\frac{A}{2} \\ \left(\frac{ab}{b+c}\right)^2 = b + l_a^2 - 2bl_a\cos\frac{A}{2} \end{cases}$$

$$\Longrightarrow \frac{a^2(c^2-b^2)}{(b+c)^2} = c^2 - b^2 - 2(c-b)l_a\cos\frac{A}{2}$$

$$\Longrightarrow \frac{a^2}{b+c} = b + c - 2l_a\cos\frac{A}{2}$$

(we might assume that $b \neq c$)

$$\Longrightarrow 2l_a\cos\frac{A}{2} = b + c - \frac{a^2}{b+c} = \frac{b^2+c^2-a^2+2bc}{b+c} = \frac{2bc\cos A + 2bc}{b+c}$$

$$= \frac{2bc(1+\cos A)}{b+c} = \frac{4bc\left(\cos\frac{A}{2}\right)^2}{b+c}$$

$$\Longrightarrow l_a = \frac{2bc\cos\frac{A}{2}}{b+c}$$

If $b = c$, triangle ABC becomes isosceles at vertex A, where $l_a = h_a$, and the Lemma is clearly true since

$$\frac{2bc\cos\frac{A}{2}}{b+c} = \frac{2b^2\cos\frac{A}{2}}{2b} = b\cos\frac{A}{2} = h_a$$

QED. ●

Lemma II. (where l_a is the length of AD)

$$l_a = \frac{2}{b+c}\sqrt{bcp(p-a)}$$

Proof. In Figure 2, I solve for AD in triangle ABD, of which AB, BD, and $\cos B$ are given.

$$AD^2 = AB^2 + BD^2 - 2AB.BD\cos\widehat{ABD}$$

$$\Longrightarrow l_a^2 = c^2 + \left(\frac{ac}{b+c}\right)^2 - 2c\left(\frac{ac}{b+c}\right)\left(\frac{c^2+a^2-b^2}{2ca}\right)$$

$$l_a^2 = c^2 + \left(\frac{ac}{b+c}\right)^2 - \frac{c(c^2+a^2-b^2)}{b+c}$$
$$= \frac{1}{(b+c)^2}[a^2c^2 + c^2(b+c)^2 - c(b+c)(c^2+a^2-b^2)]$$
$$= \frac{1}{(b+c)^2}[a^2c^2 + c(b+c)(bc+c^2-c^2-a^2+b^2)]$$
$$= \frac{1}{(b+c)^2}[a^2c^2 + (bc+c^2)(bc-a^2+b^2)]$$
$$= \frac{1}{(b+c)^2}(a^2c^2 + b^2c^2 - a^2bc + b^3c + bc^3 - a^2c^2 + b^2c^2)$$
$$= \frac{1}{(b+c)^2}(b^3c + 2b^2c^2 + bc^3 - a^2bc)$$
$$= \frac{bc(b^2+2bc+c^2-a^2)}{(b+c)^2}$$
$$= \frac{bc[(b+c)^2-a^2]}{(b+c)^2}$$
$$= \frac{bc(b+c+a)(b+c-a)}{(b+c)^2}$$
$$= \frac{4bcp(p-a)}{(b+c)^2}$$
$$\Longrightarrow\ l_a = \frac{2}{b+c}\sqrt{bcp(p-a)}$$

QED. ●

From the above lemmas, we could obtain

$$\cos\frac{A}{2} = \sqrt{\frac{p(p-a)}{bc}}$$

And in Figure 2, using the law of sine in triangle ABD, we get

$$\frac{BD}{\sin\frac{A}{2}} = \frac{AD}{\sin B}$$

And therefore

$$\sin\frac{A}{2} = \frac{BD\sin B}{AD} = \frac{\left(\frac{ac}{b+c}\right)\frac{b}{2R}}{\frac{2}{b+c}\sqrt{bcp(p-a)}} = \frac{abc}{4R\sqrt{bcp(p-a)}} = \frac{S}{\sqrt{bcp(p-a)}}$$
$$= \sqrt{\frac{(p-b)(p-c)}{bc}}$$

These formulas allow us to find trigonometric functions of $\frac{A}{2}$ with three sides given. Furthermore, I would represent one direct way to find any triangle's incenter in plane Oxy.

Property: Given triangle ABC with incenter I, then

$$a\overrightarrow{IA} + b\overrightarrow{IB} + c\overrightarrow{IC} = \vec{0}$$

Proof: In Figure 2, BI is the angle bisector of $\widehat{ABC}$. Consider triangle ABD:

$$\frac{ID}{BD} = \frac{IA}{AB} \Longrightarrow IA = \frac{AB}{BD} ID = \frac{c}{\frac{ac}{b+c}} ID = \frac{b+c}{a} ID$$

$$\Longrightarrow\ a\overrightarrow{IA} = -(b+c)\overrightarrow{ID}$$

And note that

$$\frac{DB}{c} = \frac{DC}{b} \Longrightarrow b\overrightarrow{DB} = -c\overrightarrow{DC}$$

Hence

$$\begin{aligned}
&a\overrightarrow{IA} + b\overrightarrow{IB} + c\overrightarrow{IC} \\
&= -(b+c)\overrightarrow{ID} + b\overrightarrow{IB} + c\overrightarrow{IC} \\
&= b\left(\overrightarrow{IB} - \overrightarrow{ID}\right) + c\left(\overrightarrow{IC} - \overrightarrow{ID}\right) \\
&= b\overrightarrow{DB} + c\overrightarrow{DC} \\
&= \vec{0}
\end{aligned}$$

QED. ●

4.2.Solution for A. Szoros' Inequality III

Rewrite (III):

$$\begin{aligned}
&\frac{4R}{a} \geq \frac{b}{\frac{2ab\cos\frac{C}{2}}{a+b}} + \frac{c}{\frac{2ac\cos\frac{B}{2}}{a+c}} \geq \frac{8r}{a} \\
&\Leftrightarrow 4R \geq \frac{a+b}{2\cos\frac{C}{2}} + \frac{a+c}{2\cos\frac{B}{2}} \geq 8r \\
&\Leftrightarrow\ 4R \geq \frac{2R(\sin A + \sin B)}{2\cos\frac{C}{2}} + \frac{2R(\sin A + \sin C)}{2\cos\frac{B}{2}} \geq 8r \\
&\Leftrightarrow 4 \geq \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{\sin\frac{A+B}{2}} + \frac{2\sin\frac{A+C}{2}\cos\frac{A-C}{2}}{\sin\frac{A+C}{2}} \geq \frac{8r}{R} \\
&\Leftrightarrow 2 \geq \cos\frac{A-B}{2} + \cos\frac{A-C}{2} \geq \frac{4r}{R}
\end{aligned}$$

The front part is obviously true since $\cos\theta \leq 1,\ \forall\theta \in \mathbb{R}$, so equality holds only when $A = B = C = \frac{\pi}{3}$.

However, to prove the back part of this double-inequality, I am supposed to return to initial steps, with help from Lemma II. Rewrite the back part of (III):

$$\frac{b(a+b)}{2\sqrt{abp(p-c)}}+\frac{c(a+c)}{2\sqrt{acp(p-b)}}\geq\frac{8S}{pa}$$
$$\Leftrightarrow\ \frac{(a+b)\sqrt{ab}}{2\sqrt{p-c}}+\frac{(a+c)\sqrt{ac}}{2\sqrt{p-b}}\geq 8\sqrt{(p-a)(p-b)(p-c)}\quad(*)$$

Using AM-GM inequality, of which

$$a+b\geq 2\sqrt{ab};\quad a+c\geq 2\sqrt{ac}$$
$$LHS(*)\geq\frac{ab}{\sqrt{p-c}}+\frac{ac}{\sqrt{p-b}}\geq\frac{2a\sqrt{bc}}{\sqrt[4]{(p-b)(p-c)}}$$

So it is enough to show that

$$a\sqrt{bc}\geq 4\sqrt{(p-a)(p-b)(p-c)}\sqrt[4]{(p-b)(p-c)}$$
$$\Leftrightarrow a^2bc\geq 16(p-a)(p-b)(p-c)\sqrt{(p-b)(p-c)}\quad(**)$$

Indeed, from geometric mean side to arithmetic mean side:

$$2\sqrt{(p-b)(p-c)}\leq p-b+p-c=a$$
$$2\sqrt{(p-b)(p-c)}\leq a$$
$$2\sqrt{(p-c)(p-a)}\leq b$$
$$2\sqrt{(p-a)(p-b)}\leq c$$
$$\Rightarrow RHS(**)\leq LHS(**)$$

Equality holds when $a=b=c$. QED. ∎

5. More discussions

5.1.The relation between circumscribed radius and inradius of a triangle

From A. Szoros' Inequalities II and III, we obtain $R\geq 2r$. This is also consequently resulted in from many triangle-trigonometric problems, and I would represent one direct way of proving this:

$$R\geq 2r\Leftrightarrow\frac{abc}{4S}\geq\frac{2S}{p}\Leftrightarrow abc\geq\frac{8S^2}{p}=8(p-a)(p-b)(p-c)$$

This is due to a result in last subsection, where

$$2\sqrt{(p-b)(p-c)}\leq a$$
$$2\sqrt{(p-c)(p-a)}\leq b$$
$$2\sqrt{(p-a)(p-b)}\leq c$$

And the back part of (III) leads to a nice problem:

$$\cos\frac{A-B}{2}+\cos\frac{A-C}{2}\geq\frac{4r}{R}$$

Or generally

$$\cos\frac{A-B}{2}+\cos\frac{B-C}{2}+\cos\frac{C-A}{2}\geq\frac{6r}{R}$$

5.2.Proving basic triangle-trigonometric inequalities

In triangle ABC

$$T=\sin A+\sin B+\sin C\leq\frac{3\sqrt{3}}{2}\quad(iv)$$

$$U=\cos A+\cos B+\cos C\leq\frac{3}{2}\quad(v)$$

$$V=\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\leq\frac{3}{2}\quad(vi)$$

$$W=\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\leq\frac{3\sqrt{3}}{2}\quad(vii)$$

Except (v), all (iv), (vi), and (vii) could be proven by using Jensen's inequality to concave functions. But I would represent a method proposed by Tran Phuong to prove these inequalities, just like in subsection 4.2. Moreover, problem (v) could be proven by using the SOS method, a highly convenient way.
Indeed:

$$RHS(v)-LHS(v)=1-\cos A-(\cos B+\cos C)+\frac{1}{2}$$
$$=2\left(\sin\frac{A}{2}\right)^2-2\cos\frac{B+C}{2}\cos\frac{B-C}{2}+\frac{1}{2}\left(\cos\frac{B-C}{2}\right)^2+\frac{1}{2}\left(\sin\frac{B-C}{2}\right)^2$$
$$=2\left(\sin\frac{A}{2}\right)^2-2\sin\frac{A}{2}\cos\frac{B-C}{2}+\frac{1}{2}\left(\cos\frac{B-C}{2}\right)^2+\frac{1}{2}\left(\sin\frac{B-C}{2}\right)^2$$
$$=\frac{1}{2}\left(2\sin\frac{A}{2}-\cos\frac{B-C}{2}\right)^2+\frac{1}{2}\left(\sin\frac{B-C}{2}\right)^2\geq 0$$

QED. Equality holds when $A=B=C=\frac{\pi}{3}$. ●

<u>Tran Phuong's recommendation:</u>
Proof of (v), assuming $C=\max\limits_{(0;\pi)}\{A;B;C\}$:

$$U=\cos A+\cos B+\cos C=\cos A+\cos B-\cos(A+B)$$
$$=1(\cos A+\cos B)-\cos A\cos B+\sin A\sin B$$

Using AM-GM inequality:

$$U\leq\frac{1}{2}[1+(\cos A+\cos B)^2]-\cos A\cos B+\frac{1}{2}[(\sin A)^2+(\sin B)^2]=\frac{3}{2}$$

Proof of (vi):

$$V=\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=\sin\frac{A}{2}+\sin\frac{B}{2}+\cos\frac{A+B}{2}$$
$$=1\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)+\cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}$$

Using AM-GM inequality:

$$V \le \frac{1}{2}\left[1+\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)^2\right]-\sin\frac{A}{2}\sin\frac{B}{2}+\frac{1}{2}\left[\left(\cos\frac{A}{2}\right)^2+\left(\cos\frac{B}{2}\right)^2\right]=\frac{3}{2}$$

Proof of (vii):

$$W = \cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=\cos\frac{A}{2}+\cos\frac{B}{2}+\sin\frac{A+B}{2}$$
$$=\cos\frac{A}{2}+\cos\frac{B}{2}+\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{A}{2}$$
$$=\frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\cos\frac{A}{2}+\frac{\sqrt{3}}{2}\cos\frac{B}{2}\right)+\frac{1}{\sqrt{3}}\left(\sqrt{3}\sin\frac{A}{2}\cos\frac{B}{2}+\sqrt{3}\sin\frac{B}{2}\cos\frac{A}{2}\right)$$

Using AM-GM inequality:

$$W \le \frac{1}{\sqrt{3}}\left[\frac{3}{4}+\left(\cos\frac{A}{2}\right)^2+\frac{3}{4}+\left(\cos\frac{B}{2}\right)^2\right]$$
$$+\frac{1}{2\sqrt{3}}\left[3\left(\sin\frac{A}{2}\right)^2+\left(\cos\frac{A}{2}\right)^2+3\left(\sin\frac{B}{2}\right)^2+\left(\cos\frac{B}{2}\right)^2\right]=\frac{3\sqrt{3}}{2}$$

QED. To all of these problems, equalities hold when triangle ABC is equilateral. ●

5.3.About A. Szoros' Inequality I

$$\sin\frac{A}{2}=\sqrt{\frac{(p-b)(p-c)}{bc}}$$

Using this formula, then (I) becomes

$$\frac{abc}{(p-a)(a+b)(a+c)} \ge \sqrt{\frac{(p-b)(p-c)}{bc}}$$
$$\Leftrightarrow abc\sqrt{bc} \ge (p-a)(a+b)(a+c)\sqrt{(p-b)(p-c)} \quad (viii)$$

Like I mentioned above:

$$2\sqrt{(p-c)(p-a)} \le b$$
$$2\sqrt{(p-a)(p-b)} \le c$$
$$\Rightarrow bc \ge 4(p-a)\sqrt{(p-b)(p-c)}$$

So it would be enough were it possible to show that

$$a\sqrt{bc} \ge \frac{(a+b)(a+c)}{4}$$

However, this is totally not true, since

$$a+b \ge 2\sqrt{ab}; \quad a+c \ge 2\sqrt{ac}$$

Therefore, problem $(viii)$ could be an unpleasant challenge in any contest.

6. Conclusion

Solving Alexandru Szoros' Inequalities about triangle and trigonometric factors brought me three lessons. First, transform expressions from complicated to simple forms. Second, try to find common factors of both sides. And final, basic methods with classical inequalities should be approached initially, there must be a key to clinch the problems.

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