ABOUT CALCULUS OF LIMITS FOR FAMOUS SEQUENCES

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Problem 579 proposed by the great mathematician Traian Lalescu in "Gazeta Matematica" vol. VI (1900-1901) at page 148 requires to compute:

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right); n \geq 2$$

and in vol. VII (1901-1902) at page 159 is proved that $\lim_{n\to\infty} L_n = \frac{1}{e}$.
Problem 2042 proposed by Romeo T. Ianculescu in vol. XIX (1913-1914) at page 160 requires to compute:

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty}\left((n+1)\sqrt[n+1]{n+1} - n\sqrt[n]{n}\right); n \geq 2$$

and in vol. XIX (1913-1914) at page 457 is proven that $\lim_{n\to\infty} I_n = 1$.
In 1989 in GMB/4 at page 139 D. M. Bătinețu Giurgiu propose the problem C:890 with the following enunciation:
If $B_n = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}}; n \geq 2$ compute $\lim_{n\to\infty} B_n$.
Celebrating 115 years since the publication of Lalescu's problem and 103 years since the publication of Ianculescu's problem, we take great pleasure to recall the contribution of some valuable collaborators of Mathematical Gazette or of some prestigious journals from abroad. They published extensions and valuable generalisations. We mention:
D. M. Bătinețu Giurgiu, Maria Bătinețu Giurgiu, Marcel Țena, Mihaly Bencze, Marius Șomodi, Nicolae Mușuroia, Ovidiu Pop, Neculai Stanciu, Nicușor Zlota, Tănase Negoi, Traian Ianculescu, Kee-Wai Lau (Hong Kong), Michel Bataille (France), Anastasios Kotronis (Greece), Moti Levy (Israel), Paolo Perfetti (Italy), Omran Kouba (Siria), Angel Plaza (Spain), P. P. Dalyai (Hungary), Arkady Alt (USA).

Theorem 0.1. *If $r, s \in [0,\infty)$ and $(a_n)_{n\geq 1} \subset [0,\infty)$*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n \cdot n^{s+1}} = a \in (0,\infty) \text{ then:}$$

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}^{r+1}}}{(n+1)^{rs+r+s}} - \frac{\sqrt[n]{a_n^{r+1}}}{n^{rs+r+s}}\right) = \frac{a^{r+1}}{e^{(r+1)(s+1)}} \tag{0.1}$$

Proof.

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n^{s+1}} = \lim_{n\to\infty} \sqrt[n]{\frac{a_n}{n^{n(s+1)}}} =$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^{(n+1)(s+1)}} \cdot \frac{n^{n(s+1)}}{a_n} =$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{a_n \cdot n^{s+1}} \cdot \left(\frac{n}{n+1}\right)^{(n+1)(s+1)} = a \cdot \frac{1}{e^{s+1}} = \frac{a}{e^{s+1}}$$

$$\text{If } u_n = \left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)^{r+1} \cdot \left(\frac{n}{n+1}\right)^{rs+r+s}; n \geq 1$$

$$\text{then } \lim_{n\to\infty} u_n = 1 \text{ and } \lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1.$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \left(\left(\frac{a_{n+1}}{a_n}\right)^{n+1} \cdot \frac{1}{\sqrt[n+1]{a_{n+1}^{r+1}}} \cdot \left(\frac{n}{n+1}\right)^{n(rs+r+s)}\right) =$$

$$= \lim_{n\to\infty} \left(\left(\frac{a_{n+1}}{a_n \cdot n^{s+1}}\right)^{r+1} \cdot \left(\frac{(n+1)^{s+1}}{\sqrt[n+1]{a_{n+1}}}\right)^{r+1} \cdot \left(\frac{n}{n+1}\right)^{n(rs+r+s)+(s+1)+(r+1)}\right) =$$

$$= a^{r+1} \cdot \frac{e^{(r+1)(s+1)}}{a^{r+1}} \cdot e^{rs+r+s} =$$

Hence:

$$\Delta_n = \frac{\sqrt[n+1]{a_{n+1}^{r+1}}}{(n+1)^{rs+r+s}} - \frac{\sqrt[n]{a_n^{n+1}}}{rs+r+s} =$$

$$= \frac{\sqrt[n]{a_n^{r+1}}}{n^{rs+r+s}} \cdot (u_n - 1) = \frac{\sqrt[n]{a_n^{r+1}}}{n^{rs+r+s}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n =$$

$$= \left(\frac{\sqrt[n]{a_n}}{n^{s+1}}\right)^{r+1} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n; n \geq 2 \text{ then:}$$

$$\lim_{n\to\infty} \Delta_n = \left(\frac{a}{e^{s+1}}\right)^{r+1} \cdot 1 \cdot \ln e = \frac{a^{r+1}}{e^{(r+1)(s+1)}}$$

If $a_n = n!; r = 0$ then $s = 0; a = 1$ it follows

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right) = \frac{1}{e}$$

□

Theorem 0.2. *If $r, s \in (0, \infty)$ and $(a_n)_{n\geq 1} \subset (0, \infty); n \in \mathbb{N}^*$ with $\lim_{n\to\infty} \frac{a_{n+1}}{a_n \cdot n^{s+1}} = a \in (0, \infty)$ then:*

$$\lim_{n\to\infty} \left(\frac{(n+1)^{rs+r+1}}{\sqrt[n+1]{a_{n+1}^r}} - \frac{n^{rs+r+1}}{\sqrt[n]{a_n^r}}\right) = \frac{e^{r(s+1)}}{a^r} \tag{0.2}$$

Proof. It is obviously $\lim_{n\to\infty} \frac{n^{s+1}}{\sqrt[n]{a_n}} = \frac{e^{s+1}}{a}$ and if

$$v_n = \left(\frac{n+1}{n}\right)^{rs+r+1} \cdot \left(\frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_{n+1}}}\right)^r =$$

$$= \left(\frac{n+1}{n}\right)^{rs+s+1} \cdot \left(\frac{\sqrt[n]{a_n}}{n^{s+1}} \cdot \frac{(n+1)^{s+1}}{\sqrt[n+1]{a_{n+1}}}\right) \cdot \left(\frac{n}{n+1}\right); n \geq 2$$

$$\text{then } \lim_{n\to\infty} v_n = 1 \text{ and } \lim_{n\to\infty} \frac{v_n - 1}{\ln v_n} = 1 \text{ as well as:}$$

$$\lim_{n\to\infty} v_n^n = \lim_{n\to\infty} \left(\left(\frac{n+1}{n}\right)^{n(rs+r+1)} \cdot \left(\frac{a_n}{a_{n+1}}\right)^r \cdot \sqrt[n+1]{a_{n+1}^r}\right) =$$

$$= e^{rs+r+1} \cdot \lim_{n\to\infty}\left(\Big(\frac{a_n \cdot n^{s+1}}{a_{n+1}}\Big)^r \cdot \Big(\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^{s+1}}\Big)^r \cdot \Big(\frac{n+1}{n}\Big)^{r(s+1)}\right) =$$

$$= e^{rs+r+1} \cdot \frac{1}{a^r} \cdot \frac{a^r}{a^{r(s+1)}} = e \text{ and then}$$

$$B_n = \frac{(n+1)^{rs+r+1}}{\sqrt[n+1]{a_{n+1}^2}} - \frac{n^{rs+r+1}}{\sqrt[n]{a_n^r}} = \frac{n^{rs+r+1}}{\sqrt[n]{a_n^r}}(v_n - 1) =$$

$$= \frac{n^{rs+r+1}}{\sqrt[n]{a_n^r}} \cdot \frac{v_n - 1}{\ln v_n} \cdot \ln v_n =$$

$$= \Big(\frac{n^{s+1}}{\sqrt[n]{a_n}}\Big)^r \cdot \frac{v_n - 1}{\ln v_n} \cdot \ln v_n^n ; n \ge 1$$

$$\lim_{n\to\infty} B_n = \Big(\frac{e^{s+1}}{a}\Big)^r \cdot 1 \cdot \ln e = \frac{e^{r(s+1)}}{a^r}$$

For $a_n = n!; n \in \mathbb{N}^*; s = 0, r = 1, a = 1$ and we obtain Bătinețu Giurgiu's sequence's limit

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}}\right) = e$$

□

Theorem 0.3. *If* $r \in (0,\infty); (a_n)_{n\ge1}; (b_n)_{n\ge1} \subset (0,\infty)$ *and* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a; \lim_{n\to\infty} \frac{b_{n+1}-b_n}{n^r} = b; a, b \in (0,\infty)$ *if:*

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}^{r+1}}}{(n+1)^r} \cdot b_{n+1} - \frac{\sqrt[n]{a_n^{r+1}}}{n^r} \cdot b_n\right) = \frac{a^{r+1} \cdot b}{r+1} \tag{0.3}$$

Proof.

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a$$

$$\lim_{n\to\infty} \frac{b_n}{n^{r+1}} = \lim_{n\to\infty} \frac{b_{n+1} - b_n}{(n+1)^{r+1} - n^{r+1}} =$$

$$= \lim_{n\to\infty} \frac{n^r}{(n+1)^{r+1} - n^{r+1}} \cdot \frac{b_{n+1} - b_n}{n^r} = \frac{b}{r+1}$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n^{r+1}}}{n^{r+1}} \cdot b_n = \lim_{n\to\infty} \left(\sqrt[n]{a_n}\right)^{n+1} \cdot \lim_{n\to\infty} \frac{b_n}{n^{r+1}} = \frac{a^{r+1} \cdot b}{r+1}$$

$$\text{If } w_n = \left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)^{r+1} \cdot \Big(\frac{n}{n+1}\Big)^r \cdot \frac{b_{n+1}}{b_n}; n \ge 2 \text{ then:}$$

$$\lim_{n\to\infty} w_n = \Big(\frac{a}{a}\Big)^{r+1} \cdot 1 \cdot \lim_{n\to\infty}\Big(\frac{b_{n+1}}{(n+1)^{r+1}} \cdot \frac{n^{r+1}}{b_n}\Big) \cdot \Big(\frac{n}{n+1}\Big)^{r+1} = 1$$

$$\lim_{n\to\infty} \frac{w_n - 1}{\ln w_n} = 1$$

$$\lim_{n\to\infty} w_n^n = \lim_{n\to\infty} \Big(\frac{a_{n+1}}{a_n}\Big)^{n+1} \cdot \frac{1}{\sqrt[n+1]{a_{n+1}^{r+1}}} \cdot \Big(\frac{n}{n+1}\Big)^{nr} \cdot \Big(\frac{b_{n+1}}{b_n}\Big)^n =$$

$$= a^{r+1} \cdot \frac{1}{a^{r+1}} \cdot \frac{1}{e^r} \cdot \lim_{n\to\infty}\left(\Big(1 + \frac{b_{n+1} - b_n}{b_n}\Big)^{\frac{b_n}{b_{n+1}-b_n}}\right)^{\frac{b_{n+1}-b_n}{n^n} \cdot \frac{n^{n+1}}{b_n}} =$$

$$= \frac{1}{e^r} \cdot e^{\lim_{n\to\infty} \frac{b_{n+1}-b_n}{n^r} \cdot \lim_{n\to\infty} \frac{n^{r+1}}{b_n}} =$$

$$= \frac{1}{e^r} \cdot e^{b \cdot \frac{r+1}{b}} = \frac{1}{e^r} \cdot e^{r+1} = e$$

$$G_n = \frac{\sqrt[n+1]{a_{n+1}^{r+1}}}{(n+1)^r} \cdot b_{n+1} - \frac{\sqrt[n]{a_n^{r+1}}}{n^r} \cdot b_n =$$

$$= \sqrt[n]{a_n^{r+1}} \cdot \frac{b_n}{n^r} \cdot (w_n - 1) =$$

$$= \sqrt[n]{a_n^{r+1}} \cdot \frac{b_n}{n^r} \cdot \frac{w_n - 1}{\ln w_n} \cdot \ln w_n =$$

$$= \sqrt[n]{a_n^{r+1}} \cdot \frac{b_n}{n^{r+1}} \cdot \frac{w_n - 1}{\ln w_n} \cdot \ln w_n^n; n \geq 2$$

It follows:

$$\lim_{n\to\infty} G_n = a^{r+1} \cdot \frac{b}{r+1} \cdot 1 \cdot \ln e = \frac{a^{r+1} \cdot b}{r+1}$$

For $r = 0; a_n = n; b_n = n; n \geq 2$ it follows $a = b = 1$ and so:

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} \left((n+1) \sqrt[n+1]{n+1} - n \sqrt[n]{n} \right) = 1$$

namely Ianculescu's sequence.

References

[1] Bătinețu M. D., *Sequences.* Albatros, Publishing House, Bucharest, Romania, 1979.

[2] Bătinețu - Giurgiu M. D., Bătinețu - Giurgiu Maria, Bîrchi - Damian I., Semenescu Augustin *Mathematical Analysis - Problems for XII grade.*
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[3] Bătinețu - Giurgiu M. D., *Proposing some sequences.* GM2/1999, pag 46-49.

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